INSTRUCTOR'S EDITION

COLLEGE ALGEBRA

with Applications for Business and the Life Sciences



IT'S ALL ABOUT YOU

Second Edition

Ron Larson
Anne Hodgkins

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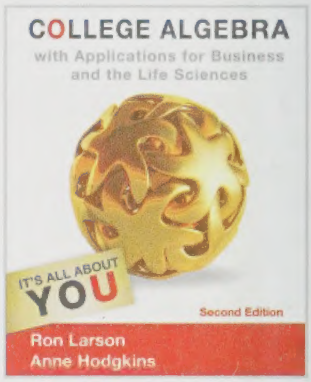
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Applications

BIOLOGY AND LIFE SCIENCES

Antler spread of an elk, 405
Bacteria count, 235, 238, 247, 368, 408
Bacteria growth, 356, 396, 407, 408
Blood oxygen level, 108, 112
Blue oak height, 103
Body length of a baby girl, 113
Body mass index (BMI), 419
Body temperature, 137
Botany, 528
Calories burned by exercise, 500
Carbon dating, 390, 396
Carbon dioxide in the atmosphere, 431
Clinical trial, 19
Cricket chirps, 574
Crop spraying mixture, 441
Diet supplement, 452, 453
Dissections, 582
Flea, consumption of a, 39
Forensics, 79
Forest yield, 385
Fruit production, 499
Galloping speeds of animals, 376
Gardening, 442
Genders of children, 594, 603
Genetics, 602
Gypsy moths, 527
Hair growth rate, 192
Heart rate, 453
Hospital patients, 271
Human height, 137
Human memory model, 325, 333, 364, 366, 385, 405, 406, 408
Intelligence, 207
Kidney donation, 578
Kidney transplants, 261
Lab practical, 583
Lead concentration in human bones, 345
Learning curve, 396, 407
Lung volume, 218
Metabolic rate of a beetle, 113
Morris water maze, 171
Native prairie grasses, 385
Nesting falcons, 453
Nutrition, 450
Optimal area of an archaeological dig site, 330
Oxygen level in a pond, 61
Pest management in a forest, 192
Plant biology lab, 583
Population
  of bears, 408
  of deer, 324, 394, 407
  of an endangered species, 397
  of fish, 333, 394, 397
Radioactive decay, 353, 356, 397
Recycling, 325, 333
Research study, 19
Rocky Mountain spotted fever, 388
Skill retention model, 366
Smokestack emissions, 333
Spread of a virus, 392
Suburban wildlife, 385
Surgery chance of success, 591
Venus flytrap growth, 536
Waterborne disease, 486
Weight of a puppy, 183, 190
Wind chill, 38
Zebrafish embryos, 602

BUSINESS

Advertising expenses, 271, 281, 290, 294
Break-even analysis, 137, 154, 415, 418, 419, 467
Budget variance, 7, 9
Company profits, 146, 155
Consumer and producer surplus, 449, 452, 470
Contract bonuses for football players, 500
Cost, 237
Cost-benefit model, 320
Cost equation, 110
Cost, revenue, and profit, 206
Declining balances depreciation, 37
Defective units, 594, 603
Demand function, 294, 295, 355, 356, 368, 385, 406
Dividends for Coca-Cola, 207
Earnings-dividend ratio
  Wal-Mart Stores, 345
Furniture production, 452
Hotel pricing, 499
Hourly wages, 192
Increasing profit, 143
Inventory
  of kayaks, 452
  levels, 500, 535, 538
  of movie players, 470
Investment income of Aflac, 155
Job applicants, 582, 584, 602
Labor/wage requirements, 500, 512, 535
Making a sale, 592, 594, 595
Market research, 125, 154
Media selection for advertising, 462
Monthly
  cost, 103
  cost for an airline, 124
  sales, 92
Operating cost for a fleet of vans, 135
Optimal
  cost, 260, 330, 459, 462
  profit, 260, 330, 458, 461, 462, 463, 471, 472
  revenue, 259, 330, 462, 471
Payroll mix-up, 594, 603
Point of equilibrium, 427, 430, 444, 468, 472
Price
  of a product, 155
Price-earnings (P/E) ratio
  DISH Network, 239
  Walt Disney, 239
Production
  cost, 102
  daily, 499, 535
  limit, 91, 114
Profit, 155, 156, 229, 237, 238, 260, 261, 281, 296, 331, 345
  IBM, 557

Johnson & Johnson, 206
of a landscaping company, 557
maximum, 218, 219
of a pharmaceutical sales company, 601
Wal-Mart, 566
Purchasing power, 193
Quality control, 9
Raw materials, 510, 536
Revenue, 102, 113, 114, 153, 156, 313
Buffalo Wild Wings, 113
California Pizza Kitchen, 193
Costco Wholesale, 332
DIRECTV, 511
of *Harry Potter* movies, 89
of James Cameron movies, 153
Panera Bread Company, 557
Research in Motion, 357
Symantec Corporation, 219
Revenue per share
Amazon.com, 294, 443
California Pizza Kitchen, 193
Google, 443
Sales
Aéropostale, 190
Apple, 175
Bed Bath & Beyond, 332
China-Biotics, 549
of a coffee shop, 554
of concert tickets, 452, 470
of doughnuts, 137
of DVDs, 295
of e-readers, 501
fourth quarter, 182
of a furniture company, 602
Green Mountain Coffee Roasters, 112
Intuit Corp., 245
La-Z-Boy, 566
Nike, 557
Oracle Corp., 79
Perrigo Company, 407
quarterly, 245
of two restaurants, 238
of shoes, 430
Walgreen Company, 147
WebMD Health Corp., 453
Sales per share
Hershey's, 442
Hormel Foods, 334
Ross Stores, 419
TJX Companies, 419
Stores
AutoZone, 182
BJ's Wholesale Club, 431
CVS, 416
Family Dollar, 171
PetSmart, 536
Safeway, 152
Walgreen Company, 416
Wal-Mart, 398
Straight-line depreciation, 191, 245, 248
Supply and demand, 430
Tax liability, 278
Trademarks issued, 442
Transportation costs, 293
Value of a product, 191, 245
Worker's productivity, 399
Years of service for employees, 590

CHEMISTRY AND PHYSICS

Automobile
aerodynamics, 229
crumple zones, 386
stopping distance, 237, 443
Chemical reaction, 399
Chemistry experiment, 602
Circuit analysis, 510
Comet orbit, A26
Diesel mechanics, 345
Distance between sun and planets, 372
Earthquake magnitudes, 393, 398, 407
Electron microscopes, 28
Escape velocity, 35, 38
Estimating speed from skid marks, 38
Estimating the time of death, 399
Falling object, 97, 101, 111, 114, 207
from an airplane, 557
a dive, a dunk, and a punt, 103
Grand Canyon, 153
on the moon, 154
owl and the mouse, 101
Royal Gorge Bridge, 101
X-Scream tower ride, 111
Fluid flow, A26
Height
of a baseball, 295
of a flare, 155
hot air balloon, 111
of a projectile, 146
Hyperbolic mirror, A18
Maximum height
of a baseball, 255
of a diver, 260
of a shot put, 260
Mixture, 92, 153
acid, 429, 441, 468
fuel, 92, 429
Oxidation-reduction reactions, 583
Parabolic reflector, A10
Path
of a baseball, 200, 205
of a cork, 207
of a salmon, 206
of a soccer ball, 155
Period of a pendulum, 38
pH levels, 393, 399
Planetary motion, A26
Pressure on a scuba diver, 182
Radio waves, 91
Radioactive decay, 396, 406, 408
Refrigeration, 61
Resistors, 147
Satellite antenna, A17

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with Applications for Business and the Life Sciences

Second Edition

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Cover Image: Luis Stortini Sabor
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Library of Congress Control Number: 2011935166

Student Edition:

ISBN-13: 978-1-133-10849-8

ISBN-10: 1-133-10849-0

Brooks/Cole
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Printed in the United States of America
1 2 3 4 5 6 7 15 14 13 12 11

Contents

0 Fundamental Concepts of Algebra 1

0.1 Real Numbers: Order and Absolute Value 2
0.2 The Basic Rules of Algebra 10
0.3 Integer Exponents 20
0.4 Radicals and Rational Exponents 29
Quiz Yourself 39
0.5 Polynomials and Special Products 40
Project: Population 47
0.6 Factoring 48
0.7 Fractional Expressions 55
Algebra Tutor 62
Summary and Study Strategies 64
Review Exercises 66
Test Yourself 68

1 Equations and Inequalities 69

1.1 Linear Equations 70
1.2 Mathematical Modeling 80
1.3 Quadratic Equations 93
1.4 The Quadratic Formula 104
Project: Revenue 113
Quiz Yourself 114
1.5 Other Types of Equations 115
1.6 Linear Inequalities 126
1.7 Other Types of Inequalities 138
Algebra Tutor 148
Summary and Study Strategies 150
Review Exercises 152
Test Yourself 156

2 Functions and Graphs 157

2.1 Graphs of Equations 158
2.2 Lines in the Plane 172
2.3 Linear Modeling and Direct Variation 183
2.4 Functions 195
Project: Expenditures 207
Quiz Yourself 208
2.5 Graphs of Functions 209
2.6 Transformations of Functions 220
2.7 The Algebra of Functions 230
Algebra Tutor 240
Summary and Study Strategies 242
Review Exercises 244
Test Yourself 248

3 Polynomial and Rational Functions 249

3.1 Quadratic Functions and Models 250
3.2 Polynomial Functions of Higher Degree 262
3.3 Polynomial Division 272
3.4 Real Zeros of Polynomial Functions 282
Project: Sales 295
Quiz Yourself 296
3.5 Complex Numbers 297
3.6 The Fundamental Theorem of Algebra 306
3.7 Rational Functions 314
Algebra Tutor 326
Summary and Study Strategies 328
Review Exercises 330
Test Yourself 334

4 Exponential and Logarithmic Functions 335

4.1 Inverse Functions 336
4.2 Exponential Functions 346
Project: Drug Concentration 357
4.3 Logarithmic Functions 358
Quiz Yourself 368
4.4 Properties of Logarithms 369
4.5 Solving Exponential and Logarithmic Equations 377
4.6 Exponential and Logarithmic Models 387
Algebra Tutor 400
Summary and Study Strategies 402
Review Exercises 404
Test Yourself 408

5 Systems of Equations and Inequalities 409

5.1 Solving Systems Using Substitution 410
5.2 Solving Systems Using Elimination 420
5.3 Linear Systems in Three or More Variables 432
Project: Revenues Per Share 443
Quiz Yourself 444
5.4 Systems of Inequalities 445
5.5 Linear Programming 454
Algebra Tutor 464
Summary and Study Strategies 466
Review Exercises 467
Test Yourself 472

6 Matrices and Determinants 473

6.1 Matrices and Linear Systems 474
6.2 Operations with Matrices 487
Project: Sales of e-Readers 501
6.3 The Inverse of a Square Matrix 502
Quiz Yourself 512
6.4 The Determinant of a Square Matrix 513
6.5 Applications of Matrices and Determinants 521
Algebra Tutor 530
Summary and Study Strategies 532
Review Exercises 534
Test Yourself 538

7 Sequences, Series, and Probability 539

7.1 Sequences and Summation Notation 540
7.2 Arithmetic Sequences and Partial Sums 550
7.3 Geometric Sequences and Series 558
Quiz Yourself 567
7.4 The Binomial Theorem 568
Project: Fractals 574
7.5 Counting Principles 575
7.6 Probability 585
Algebra Tutor 596
Summary and Study Strategies 598
Review Exercises 600
Test Yourself 604

Appendices A1

Appendix A: An Introduction to Graphing Utilities A1

Appendix B: Conic Sections A8
B.1 Conic Sections A8
B.2 Conic Sections and Translations A19

Appendix C: Further Concepts in Statistics (Web)*
C.1 Data and Linear Modeling
C.2 Measures of Central Tendency and Dispersion

Appendix D: Mathematical Induction (Web)*

Answers to Selected Exercises A27
Answers to Checkpoints A85
Answers to Tech Tutors A91
Index A92

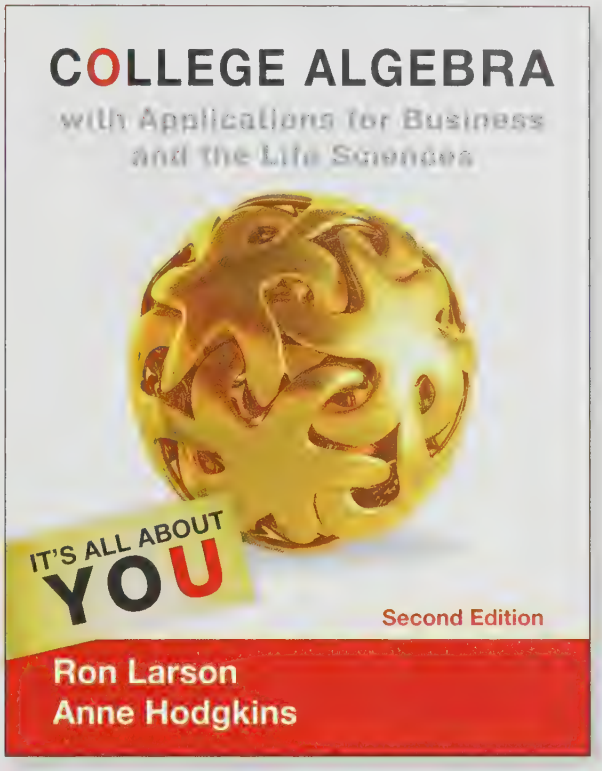
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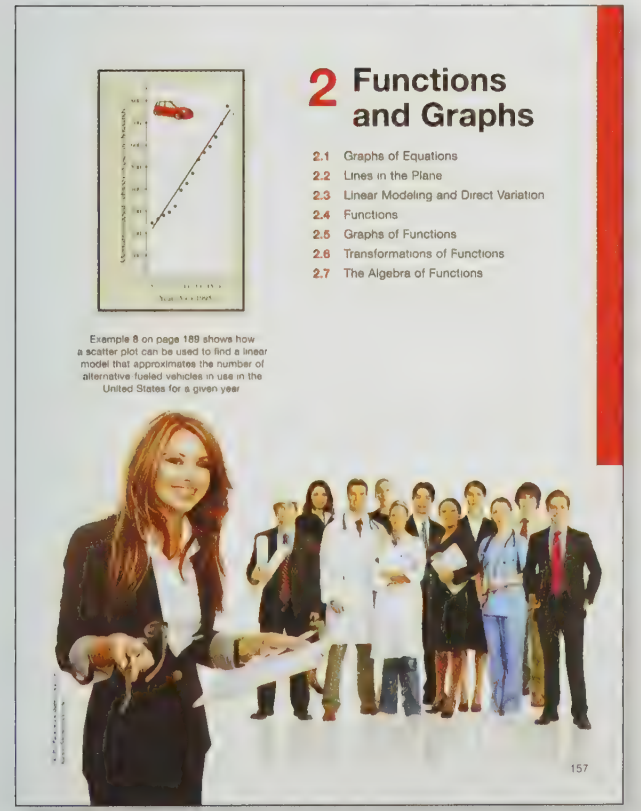
Preface

Welcome to the Second Edition of *College Algebra with Applications for Business and the Life Sciences*! We are always excited about a new edition, but with this edition, we are even more excited. We had a single goal in mind with this revision—to provide you with a book that is both real and relevant. This book has a bright business-oriented design that complements the multitude of business and life sciences applications found throughout.

The theme for the revision is **"IT'S ALL ABOUT YOU."** The pedagogy of the book is rock solid and is based on years of teaching, years of writing, and years of feedback from instructors and students. Please pay special attention to the study aids with a red **U**. These study aids will help you learn algebra, use technology, and prepare for tests. For an overview of these aids, check out ALGEBRA & YOU on page 0.

In each exercise set, quiz, and test, be sure to notice the reference to **CalcChat.com.** At this free site, you can download a step-by-step solution to any odd-numbered exercise. Also, you can talk to a tutor, free, during the hours posted at the site (20 hours a week in the summer, 40 hours a week during the school year).





New To This Edition

NEW Chapter Opener

Each *Chapter Opener* highlights a real-life problem from an example in the chapter, showing a graph related to the data and describing the math concept used to solve the problem.

NEW Section Opener

Each *Section Opener* highlights a real-life problem in the exercises, showing a graph for the situation with a description of how you will use the math of the section to solve the problem.

NEW SUMMARIZE

The *Summarize* feature at the end of each section helps you organize the lesson's key concepts into a concise summary, providing you with a valuable study tool.

NEW ALGEBRA TUTOR

The *Algebra Tutor* appears throughout each chapter and offers algebraic support at point of use. This support is revisited in a two-page algebra review at the end of the chapter, where additional details of example solutions with explanations are provided.

NEW HOW DO YOU SEE IT? Exercise

The *How Do You See It?* exercise in each section presents a real-life problem that you will solve by visual inspection using the concepts learned in the lesson.

REVISED Exercise Sets

The exercise sets have been carefully and extensively examined to ensure they are rigorous, relevant, and cover all topics suggested by our users. The exercises have been reorganized and titled so you can better see the connections between examples and exercises. Multi-step, real-life exercises reinforce problem-solving skills and mastery of concepts by giving you the opportunity to apply the concepts in real-life situations.

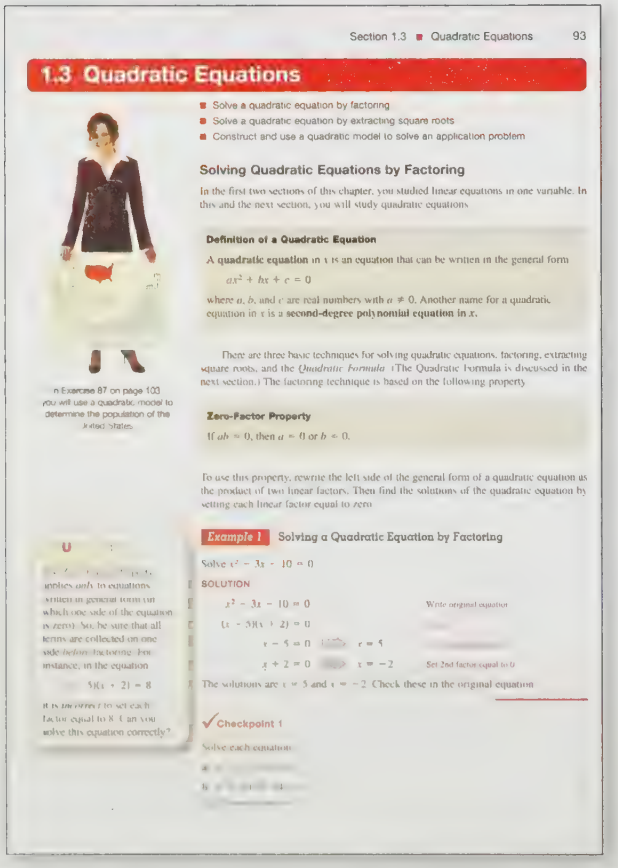
88. **HOW DO YOU SEE IT?** Each figure shows a person or a ball about to go into motion through the air. Match each figure with the position equation that represents the situation. Without calculating, compare the terms of the equations to determine which situation involves the least amount of time in the air. Explain your reasoning.

(i)  (ii)  (iii) 

(a) $s = -16t^2 + 33t$

(b) $s = -16t^2 + 10$

(c) $s = -16t^2 - 20t + 10$



Section 1.3 ■ Quadratic Equations 93

1.3 Quadratic Equations

- Solve a quadratic equation by factoring
- Solve a quadratic equation by extracting square roots
- Construct and use a quadratic model to solve an application problem

Solving Quadratic Equations by Factoring

In the first two sections of this chapter, you studied linear equations in one variable. In this and the next section, you will study quadratic equations.

Definition of a Quadratic Equation

A quadratic equation in x is an equation that can be written in the general form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. Another name for a quadratic equation in x is a second-degree polynomial equation in x.

There are three basic techniques for solving quadratic equations: factoring, extracting square roots, and the *Quadratic Formula*. (The Quadratic Formula is discussed in the next section.) The factoring technique is based on the following property.

Zero-Factor Property

If $ab = 0$, then $a = 0$ or $b = 0$.

To use this property, rewrite the left side of the general form of a quadratic equation as the product of two linear factors. Then find the solutions of the quadratic equation by setting each linear factor equal to zero.

Example 1 Solving a Quadratic Equation by Factoring

Solve $x^2 - 3x - 10 = 0$.

SOLUTION

$x^2 - 3x - 10 = 0$ Write original equation

$(x - 5)(x + 2) = 0$

$x - 5 = 0 \Rightarrow x = 5$

$x + 2 = 0 \Rightarrow x = -2$ Set 2nd factor equal to 0

The solutions are $x = 5$ and $x = -2$. Check these in the original equation.

✓Checkpoint 1

Solve each equation.

In Exercise 87 on page 103 you will use a quadratic model to determine the population of the United States.

applies only to equations written in general form (in which one side of the equation is zero). So, be sure that all terms are collected on one side before factoring. For instance, in the equation

$5(x + 2) = 8$

it is incorrect to set each factor equal to 8. Can you solve this equation correctly?

Calc Chat

For the past several years, an independent website—**CalcChat.com**—has been maintained to provide free solutions to all odd-numbered problems in the text. Thousands of students have visited the site for practice and help with their homework. For this edition, information from **CalcChat.com**, including which solutions students accessed most often, was used to help guide the revision of the exercises.

Table of Contents Changes

We have moved Section 2.8 (Inverse Functions), to the beginning of Chapter 4 (Exponential and Logarithmic Functions) as the new Section 4.1, Section 7.7 (Mathematical Induction) is now Appendix D and has been moved to the student website at *www.cengagebrain.com*.

Trusted Features

Section Objectives

A bulleted list of learning objectives provides you the opportunity to preview what will be presented in the upcoming section.

Definitions and Theorems

All definitions and theorems are highlighted for emphasis and easy recognition.

Checkpoint

Paired with every example, the *Checkpoint* problems encourage immediate practice and check your understanding of the concepts presented in the example. Answers to all *Checkpoint* problems appear at the back of the text to reinforce understanding of the skill sets learned.

Business Capsule

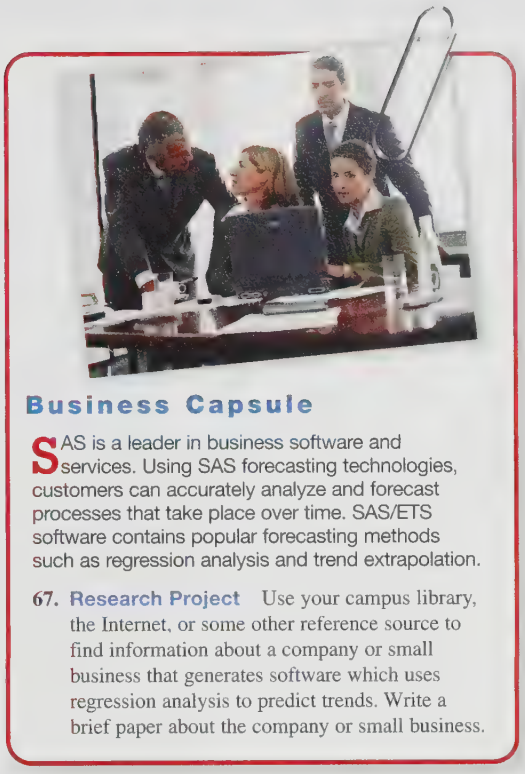
Business Capsules appear at the end of selected sections. These capsules and their accompanying research project highlight business situations related to the mathematical concepts covered in the chapter.

STUDY TIP

These hints and tips can be used to reinforce or expand upon concepts, help you learn how to study mathematics, caution you about common errors, address special cases, or show alternative or additional steps to a solution of an example.

TECH TUTOR

The *Tech Tutor* gives suggestions for effectively using tools such as calculators, graphing calculators, and spreadsheet programs to help deepen your understanding of concepts, ease lengthy calculations, and provide alternate solution methods for verifying answers obtained by hand.

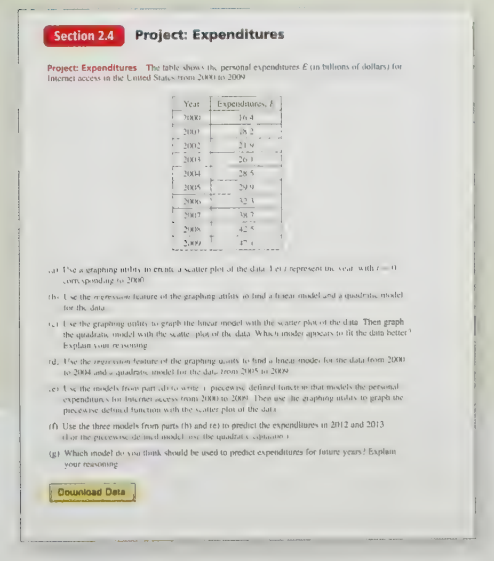


Business Capsule

SAS is a leader in business software and services. Using SAS forecasting technologies, customers can accurately analyze and forecast processes that take place over time. SAS/ETS software contains popular forecasting methods such as regression analysis and trend extrapolation.

67. Research Project Use your campus library, the Internet, or some other reference source to find information about a company or small business that generates software which uses regression analysis to predict trends. Write a brief paper about the company or small business.

93. Project: Expenditures For a project involving the personal expenditures for Internet access in the United States from 2000 to 2009, visit this text's website at *www.cengagebrain.com.* *(Source: Bureau of Economic Analysis)*



Section 2.4 Project: Expenditures

Project: Expenditures The table shows the personal expenditures E (in billions of dollars) for Internet access in the United States from 2000 to 2009.

Year	Expenditures, E
2000	16.4
2001	18.2
2002	21.9
2003	20.3
2004	28.5
2005	29.9
2006	32.1
2007	38.7
2008	42.5
2009	47.4

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year with $t = 0$ corresponding to 2000.

(b) Use the regression feature of the graphing utility to find a linear model and a quadratic model for the data.

(c) Use the graphing utility to graph the linear model with the scatter plot of the data. Then graph the quadratic model with the scatter plot of the data. Which model appears to fit the data better? Explain your reasoning.

(d) Use the regression feature of the graphing utility to find a linear model for the data from 2000 to 2004 and a quadratic model for the data from 2005 to 2009.

(e) Use the models from part (d) to write a piecewise defined function that models the personal expenditures for Internet access from 2000 to 2009. Then use the graphing utility to graph the piecewise defined function with the scatter plot of the data.

(f) Use the three models from parts (b) and (e) to predict the expenditures in 2012 and 2013 (if the piecewise defined model uses the quadratic equation).

(g) Which model do you think should be used to predict expenditures for future years? Explain your reasoning.

Download Data

SKILLS WARM UP

The *Skills Warm Up* appears at the beginning of the exercise set for each section. These problems help you to review previously learned skills that you will use in solving the section exercises.

Project

The projects at the end of selected sections involve in-depth applied exercises in which you will work with large, real-life data sets, often creating or analyzing models. These projects are offered online at *www.cengagebrain.com.*

Instructor Resources

Print

Complete Solutions Manual
ISBN-13: 978-1-133-62844-6
The *Complete Solutions Manual* provides worked-out solutions for all exercises in the text, including Checkpoints, Quiz Yourself, Test Yourself, and Tech Tutors.

Media

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The *Student Solutions Manual* provides complete worked-out solutions to all odd-numbered exercises in the text. In addition, the solutions of all Checkpoint, Quiz Yourself, Test Yourself, and Tech Tutor exercises are included.

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Acknowledgements

We would like to thank our colleagues who have helped us develop this program. Their encouragement, criticisms, and suggestions have been invaluable to us.

Reviewers

Victor Akatsa, *Chicago State University*

Victor Cifarelli, *UNC Charlotte*

Diane Fournier, *St. Cloud Technical and Community College*

Javier Garza, *Tarleton State University*

Jane Golden, *Hillsborough Community College*

Libin Mou, *Bradley University*

Dale Quinn, *Coastal Carolina University*

Kia Shafe, *Saddleback College*

Judith Wood, *College of Central Florida*

We would also like to thank the following reviewers, who have given us many useful insights to this and previous editions.

Rosalie Abraham, *Florida Community College at Jacksonville;* Judith A. Ahrens, *Pellissippi State Technical Community College;* Sandra Beken, *Horry-Georgetown Technical College;* Diane Benjamin, *University of Wisconsin–Platteville;* Dona Boccio, *Queensborough Community College;* Michael Brook, *University of Delaware;* Tim Chappell, *Metropolitan Community College–Penn Valley;* Kent Craghead, *Colby Community College;* Carol Edwards, *St. Louis Community College at Florissant Valley;* Warrene Ferry, *Jones County Junior College;* Thomas L. Fitzkee, *Francis Marion University;* David Frank, *University of Minnesota;* Michael Frantz, *University of La Verne;* Nick Geller, *Collin College;* Carolyn H. Goldberg, *Niagara County Community College;* Linda Herndon, *OSB, Benedictine College;* Ruth E. Hoffman, *Toccoa Falls College;* Carl Hughes, *Fayetteville State University;* Buddy A. Johns, *Wichita State University;* Annie Jones, *Calhoun State Community College;* Steven Z. Kahn, *Anne Arundel Community College;* Claire Krukenberg, *Eastern Illinois University;* John Kubicek, *Southwest Missouri State University;* Charles G. Laws, *Cleveland State Community College;* Eileen Lee, *Framingham State College;* John A. Lewallen, *Southeastern Louisiana University;* Gael Mericle, *Minnesota State University Mankato;* Michael Montano, *Riverside Community College;* Sue Neal, *Wichita State University;* Terrie L. Nichols, *Cuyamaca College;* Mark Omodt, *Anoka-Ramsey Community College;* Shahrokh Parvini, *San Diego Mesa College;* Jim Rutherfoord, *Chattahoochee Technical College;* G. Bryan Stewart, *Tarrant County College;* Jacqueline Stone, *University of Maryland;* David Surowski, *Kansas State University;* Pamela K. Trim, *Southwest Tennessee Community College;* Laurie Varecka, *University of Arizona;* Jamie Whitehead, *Texarkana College*

In addition, we would like to thank the staff at Larson Texts, Inc. who assisted with proofreading the manuscript, preparing and proofreading the art package, and checking and typesetting the supplements.

On a personal level, we would like to thank our families, especially Deanna Gilbert Larson and Jay N. Torok, for their love, patience, and support. Also, a special thanks goes to R. Scott O'Neil.

If you have suggestions for improving this text, please feel free to write to us. Over the past few years we have received many useful comments from both instructors and students, and we value these comments very highly.

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ALGEBRA & YOU

Every feature in this text is designed to help you learn algebra. Whenever you see a red **U**, pay special attention to the study aid. These study aids represent years of experience in teaching students *just like you.* Ron Larson

STUDY TIP

In Example 3(c), note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

The *Study Tips* occur at point of use throughout the text. They represent **common questions** that students ask me, **insights** into understanding concepts, and **alternative ways to look at concepts**. For instance, the *Study Tip* at the left provides insight into evaluating functions using function notation.

TECH TUTOR



You can write a program for a graphing utility to solve equations using the Quadratic Formula. Use a program to solve the equation in Example 2.

The *Tech Tutors* give suggestions on how you can use various types of technology to help understand the material. This includes **graphing calculators**, **computer graphing programs**, and **spreadsheet programs** such as Excel. For instance, the *Tech Tutor* at the left points out that some calculators and some computer programs are capable of solving equations using the Quadratic Formula.

ALGEBRA TUTOR

For help in solving equations similar to the one in Example 2, see the review of solving equations on page 148.

Throughout years of teaching, I have found that the greatest stumbling block to success in mathematics is a weakness in algebra. Each time you see an *Algebra Tutor*, please read it carefully. Then, flip ahead to the referenced page and give yourself a chance to enjoy a brief **algebra refresher**. It will be time well spent.

HOW DO YOU SEE IT?

The *How Do You See It?* question in each exercise set helps you **visually summarize concepts** without messy computations.

SUMMARIZE

The *Summarize* outline at the end of each section asks you to write each learning objective in **your own words**.

SKILLS WARM UP

The *Skills Warm Up* exercises that precede each exercise set will help you **review previously learned skills**.

SUMMARY AND STUDY STRATEGIES

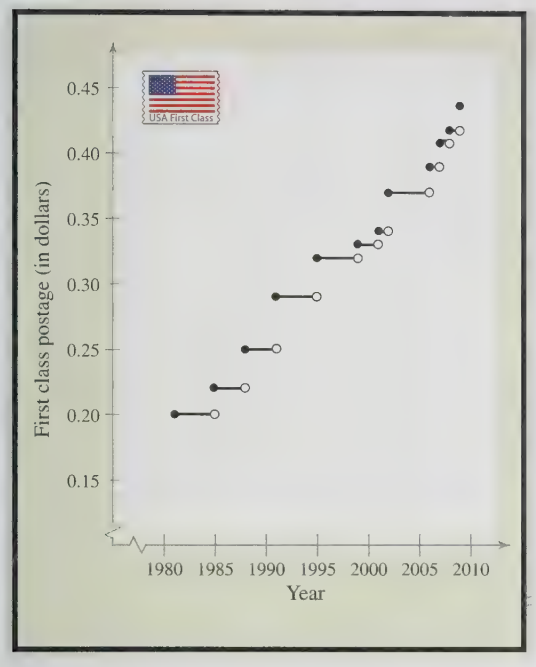
The *Summary and Study Strategies,* coupled with the Review Exercises are designed to help you organize your thoughts as you **prepare for a chapter test**.

QUIZ YOURSELF

The *Quiz Yourself* occurs midway in each chapter. Take each of these quizzes as you would **take a quiz in class**.

TEST YOURSELF

The *Test Yourself* occurs at the end of each chapter. All questions are answered so you can **check your progress**.



Example 9 on page 26 shows how the compound interest formula can be used to determine the average annual rate of inflation for a first class postage stamp over a 26-year period.

0 Fundamental Concepts of Algebra

0.1 Real Numbers: Order and Absolute Value
0.2 The Basic Rules of Algebra
0.3 Integer Exponents
0.4 Radicals and Rational Exponents
0.5 Polynomials and Special Products
0.6 Factoring
0.7 Fractional Expressions



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0.1 Real Numbers: Order and Absolute Value



In Exercise 81 on page 9, you can use a number line to determine the range of temperatures for a city in a given month.

- Classify real numbers as natural numbers, integers, rational numbers, or irrational numbers.
- Use the real number line and inequalities to order real numbers and to denote subsets of real numbers.
- Find the absolute value of a number and use absolute value to find the distance between two numbers.

Real Numbers

The formal term that is used in mathematics to refer to a collection of objects is the word **set.** For instance, the set

$$\{1, 2, 3\}$$

contains the three numbers 1, 2, and 3. Note that a pair of braces { } is used to enclose the members of the set. In this text, a pair of braces will always indicate the members of a set. Parentheses ( ) and brackets [ ] are used to represent other ideas.

The set of numbers that is used in arithmetic is the set of **real numbers.** The term *real* distinguishes real numbers from *imaginary* numbers. (You will learn about imaginary numbers in Chapter 3.)

A set A is called a **subset** of a set B if every member of A is also a member of B. Here are some examples.

- $\{1, 2, 3\}$ is a subset of $\{1, 2, 3, 4\}$.
- $\{0, 4\}$ is a subset of $\{0, 1, 2, 3, 4\}$.
- $\{0, 5\}$ is *not* a subset of $\{0, 1, 2, 3, 4\}$ because 5 is not a member of $\{0, 1, 2, 3, 4\}$.

One of the most commonly used subsets of real numbers is the set of **natural numbers** or **positive integers**

$$\{1, 2, 3, 4, \ldots\}.$$ Set of positive integers

Note that the three dots indicate that the pattern continues. For instance, the set also contains the numbers 5, 6, 7, and so on.

Positive integers can be used to describe many quantities that you encounter in everyday life. Here are some examples.

- You are taking four classes this term.
- You are paying \$700 per month for rent.
- You are working 20 hours per week.

But even in everyday life, positive integers cannot describe some concepts accurately. For instance, you could have a zero balance in your checking account, or the temperature could be $-10°$ (10 degrees below zero). To describe such quantities, you need to expand the set of positive integers to include **zero** and the **negative integers.** The expanded set is called the set of **integers,** which can be written as shown.

Zero

$$\{\underbrace{\ldots, -3, -2, -1}_{\text{Negative integers}},\ 0,\ \underbrace{1, 2, 3, \ldots}_{\text{Positive integers}}\}$$

The set of integers is a subset of the set of real numbers. This means that every integer is a real number.

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Even with the set of integers, there are still many quantities in everyday life that you cannot describe accurately. The costs of many items are not in whole dollar amounts, but in parts of dollars, such as \$1.19 or \$39.98. You might work $8\frac{1}{2}$ hours, or you might miss the first *half* of a movie. To describe such quantities, the set of integers is expanded to include **fractions.** The expanded set is called the set of **rational numbers.** Formally, a real number is called **rational** if it can be written as the ratio p/q of two integers, where $q \neq 0$. (The symbol $\neq$ means **not equal to.**) Here are some examples of rational numbers.

$$2 = \frac{2}{1} \qquad 0.333\ldots = \frac{1}{3}$$

$$0.125 = \frac{1}{8} \qquad 1.126126\ldots = \frac{125}{111}$$

STUDY TIP

Make sure you understand that not all fractions are rational numbers. For instance, the fraction $\frac{\sqrt{2}}{3}$ is not a rational number.

Real numbers that cannot be written as the ratio of two integers are called **irrational.** For instance, the numbers

$$\sqrt{2} = 1.4142135\ldots$$

and

$$\pi = 3.1415926\ldots$$

are irrational numbers. The decimal representation of a rational number is either *terminating* or *repeating*. For instance, the decimal representation of

$$\tfrac{1}{4} = 0.25 \qquad \text{Terminating decimal}$$

is terminating, and the decimal representation of

$$\tfrac{4}{11} = 0.363636\ldots \qquad \text{Repeating decimal}$$

$$= 0.\overline{36}$$

is repeating. (The line over "36" indicates which digits repeat.)

The decimal representation of an irrational number neither terminates nor repeats. When you perform calculations using decimal representations of nonterminating decimals, you usually use a decimal approximation that has been **rounded** to a certain number of decimal places. For instance, rounded to four decimal places, the decimal approximations of $\frac{2}{3}$ and π are

$$\frac{2}{3} \approx 0.6667$$

and

$$\pi \approx 3.1416.$$

The symbol $\approx$ means **approximately equal to.**

The Venn diagram in Figure 0.1 shows the relationships between the real numbers and several commonly used subsets of the real numbers.

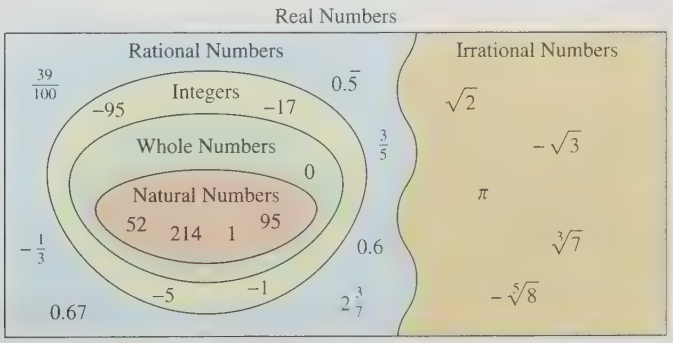
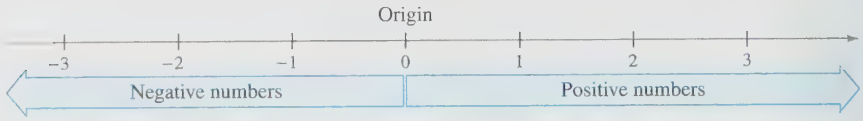


FIGURE 0.1

The Real Number Line and Ordering

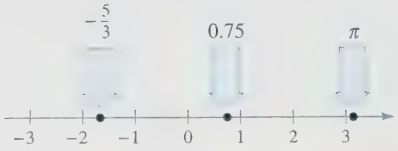
The **real number line** can be used to represent the real numbers. It consists of a horizontal line with a point (the **origin**) labeled as 0 (zero). Points to the left of zero are associated with **negative numbers,** and points to the right of zero are associated with **positive numbers,** as shown in Figure 0.2. The real number zero is neither positive nor negative. So, when you want to talk about real numbers that might be positive *or* zero, you can use the term **nonnegative real numbers.**



The Real Number Line

FIGURE 0.2

Each point on the real number line corresponds to exactly one real number, and each real number corresponds to exactly one point on the real number line, as shown in Figure 0.3. The number associated with a point on the real number line is the **coordinate** of the point.



Every real number corresponds to a point on the real number line.

FIGURE 0.3

The real number line provides you with a way of comparing any two real numbers. For instance, if you choose any two (different) numbers on the real number line, one of the numbers must be to the left of the other number. The number to the left is **less than** the number to the right, and the number to the right is **greater than** the number to the left.

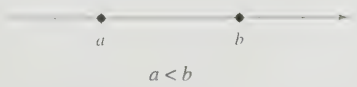
Definition of Order on the Real Number Line

If the real number a lies to the left of the real number b on the real number line, a is **less than** b, which is denoted by

$$a < b$$

as shown in Figure 0.4. This relationship can also be described by saying that b is **greater than** a and writing $b > a$.

The inequality $a \leq b$ means that a is **less than or equal to** b. This relationship can also be described by saying that b is **greater than or equal to** a and writing $b \geq a$.



a is to the left of b.

FIGURE 0.4

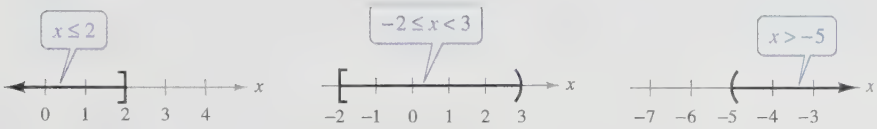
The symbols $<$, $>$, $\leq$, and $\geq$ are called **inequality symbols.** Inequalities are useful in denoting subsets of real numbers, as shown in Examples 1 and 2.

Example 1 Interpreting Inequalities

a. The inequality $x \leq 2$ denotes all real numbers that are less than or equal to 2, as shown in Figure 0.5(a).

b. The inequality $-2 \leq x < 3$ means that $x \geq -2$ *and* $x < 3$. This **double inequality** denotes all real numbers between -2 and 3, including -2 but *not* including 3, as shown in Figure 0.5(b).

c. The inequality $x > -5$ denotes all real numbers that are greater than -5, as shown in Figure 0.5(c).



(a) (b) (c)

FIGURE 0.5

✓Checkpoint 1

Give a verbal description of the subset of real numbers represented by $x \geq 7$.

In Figure 0.5, notice that a bracket is used to *include* the endpoint of an interval and a parenthesis is used to *exclude* the endpoint.

Example 2 Inequalities and Subsets of Real Numbers

a. "c is nonnegative" means that c is greater than or equal to zero, which you can write as $c \geq 0$.

b. "b is at most 5" can be written as $b \leq 5$.

c. "d is negative" can be written as $d < 0$, and "d is greater than -3" can be written as $-3 < d$. Combining these two inequalities produces $-3 < d < 0$.

d. "x is positive" can be written as $0 < x$, and "x is not more than 6" can be written as $x \leq 6$. Combining these two inequalities produces $0 < x \leq 6$.

✓Checkpoint 2

Use inequality notation to describe each subset of real numbers.

a. x is at least 5.

b. y is greater than 4, but no more than 11.

c. z is greater than or equal to -1 and less than or equal to 10.

The following property of real numbers is called the **Law of Trichotomy.** As the "tri" in its name suggests, this law states that for any two real numbers a and b, precisely one of *three* relationships is possible.

$a < b$, $a = b$, or $a > b$ Law of Trichotomy

Absolute Value and Distance

The **absolute value** of a real number is its *magnitude*, or its value disregarding its sign. For instance, the absolute value of -3, written $|-3|$, has the value of 3.

Be sure you see from the definition that the absolute value of a real number is never negative. It is either positive or zero. For instance, if $a = -5$, then $|-5| = -(-5) = 5$. Moreover, 0 is the only real number whose absolute value is zero. That is, $|0| = 0$.

Definition of Absolute Value

Let a be a real number. The **absolute value** of a, denoted by $|a|$, is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Example 3 Finding Absolute Value

a. $|-7| = 7$ **b.** $|-4.8| = 4.8$ **c.** $-|-9| = -(9) = -9$

Checkpoint 3

Evaluate $|-12|$.

Example 4 Comparing Real Numbers

Place the correct symbol ($<$, $>$, or $=$) between the two real numbers.

a. $|-4| \quad |4|$ **b.** $|-5| \quad 3$ **c.** $-|-1| \quad |-1|$

SOLUTION

a. $|-4| = |4|$, because both are equal to 4.

b. $|-5| > 3$, because $|-5| = 5$ and 5 is greater than 3.

c. $-|-1| < |-1|$, because $-|-1| = -1$ and $|-1| = 1$.

Checkpoint 4

Place the correct symbol ($<$, $>$, or $=$) between the two real numbers.

a. $-|-6| \quad -|6|$ **b.** $-|5| \quad |-5|$

Properties of Absolute Value

Let a and b be real numbers. Then the following properties are true.

1. $|a| \geq 0$
2. $|-a| = |a|$
3. $|ab| = |a|\,|b|$
4. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \; b \neq 0$



The distance between -3 and 4 is 7.

FIGURE 0.6

Absolute value can be used to define the distance between two numbers on the real number line. To see how this is done, consider the numbers -3 and 4, as shown in Figure 0.6. To find the distance between these two numbers, subtract either number from the other and then take the absolute value of the difference.

$$\text{(Distance between } -3 \text{ and } 4) = |-3 - 4| = |-7| = 7$$

Distance Between Two Numbers

Let a and b be real numbers. The **distance between a and b** is given by

$$\text{Distance} = |b - a| = |a - b|.$$

Example 5 Distance Between Two Numbers

a. The distance between 2 and -6 is $|2 - (-6)| = |8| = 8$.

b. "The distance between x and 2 is at least 3" can be written as $|x - 2| \geq 3$.

Checkpoint 5

Find the distance between -5 and 3.

Example 6 Budget Variance

For each type of expense, a home health care company wants the absolute value of the difference between the actual and budgeted amounts to be less than or equal to \$500 *and* less than or equal to 5% of the budgeted amount. By letting a represent the actual expenses and b the budgeted expenses, these restrictions can be written as

$$|a - b| \leq 500 \quad \text{and} \quad |a - b| \leq 0.05b.$$

For travel, office supplies, and wages, the company budgeted \$12,500, \$750, and \$84,600. The actual amounts paid for these expenses were \$12,872.56, \$704.15, and \$85,143.95. Are these amounts within budget restrictions?

SOLUTION Create a table.

	Budgeted Expense, b	*Actual Expense, a*	$\lvert a - b \rvert$	$0.05b$
Travel	\$12,500	\$12,872.56	\$372.56	\$625.00
Office supplies	\$750	\$704.15	\$45.85	\$37.50
Wages	\$84,600	\$85,143.95	\$543.95	\$4230.00

From this table, you can see that travel expenses pass both tests, so they are within budget restrictions. Office supply expenses pass the first test but fail the second test, so they are *not* within budget restrictions. Wage expenses fail the first test but pass the second test, so they are *not* within budget restrictions.

Checkpoint 6

In Example 6, the company budgeted \$28,000 for medical supplies, but actually paid \$30,100. Is this within budget restrictions?



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SUMMARIZE (Section 0.1)

1. Describe the commonly used subsets of the real numbers *(pages 2 and 3)*.
2. Describe how to use the real number line to denote subsets of real numbers *(pages 4 and 5)*. For examples of using the real number line to denote subsets of real numbers, see Examples 1 and 2.
3. State the absolute value of a real number *(page 6)*. For examples of using absolute value, see Examples 3, 4, 5, and 6.

Exercises 0.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Classifying Real Numbers In Exercises 1–6, determine which numbers in the set are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers.

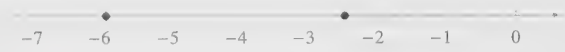
1. $\{-9, \frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0.1\}$
2. $\{\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}\}$
3. $\{12, -13, 1, \sqrt{4}, \sqrt{6}, \frac{3}{2}\}$
4. $\{3, -1, \frac{1}{3}, \frac{6}{3}, \frac{1}{2}\sqrt{2}, -7.5\}$
5. $\{\frac{8}{2}, -\frac{8}{3}, \sqrt{10}, -4, 9, 14.2\}$
6. $\{25, -17, \frac{12}{5}, \sqrt{9}, \sqrt{8}, -\sqrt{8}\}$

Writing Fractions as Decimals In Exercises 7–10, use a calculator to find the decimal form of the rational number. If the number is a nonterminating decimal, then write the repeating pattern.

7. $\frac{2}{3}$
8. $\frac{9}{40}$
9. $\frac{14}{111}$
10. $\frac{49}{160}$

Identifying Points on the Real Number Line In Exercises 11 and 12, approximate the two plotted numbers and place the correct symbol (< or >) between them.

11. 

12. 

Comparing Numbers on the Real Number Line In Exercises 13–18, plot the two real numbers on the real number line and place the appropriate inequality symbol (< or >) between them.

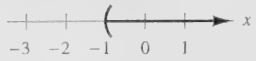
13. $-6, 7$
14. $-4, -8$
15. $1, -3.5$
16. $4.3, 1$
17. $\frac{5}{6}, \frac{2}{3}$
18. $-\frac{8}{7}, -\frac{3}{7}$

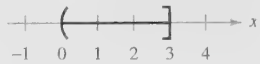
Using a Calculator to Order Numbers In Exercises 19–22, use a calculator to order the numbers from least to greatest.

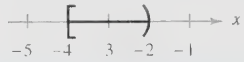
19. $\frac{7}{2}, 2\sqrt{3}, 3.45, \frac{204}{60}, \frac{31}{9}$
20. $\frac{559}{500}, 1.12, \frac{\sqrt{5}}{2}, \frac{115}{99}, \frac{\pi}{3}$
21. $\frac{7071}{5000}, \frac{584}{413}, \sqrt{2}, \frac{47}{33}, \frac{127}{90}$
22. $\frac{26}{15}, \sqrt{3}, 1.7320, \frac{381}{220}, \sqrt{10} - \sqrt{2}$

Writing an Inequality In Exercises 23–26, write an inequality that describes the graph.

23. 

24. 

25. 

26. 

Interpreting Inequalities In Exercises 27–36, give a verbal description of the subset of real numbers that is represented by the inequality, and sketch the subset on the real number line. *See Example 1.*

27. $x < 0$
28. $x < 2$
29. $x \leq 5$
30. $x \geq -2$
31. $x > 3.5$
32. $x \geq 4.5$
33. $-2 < x < 2$
34. $0 \leq x \leq 5$
35. $-1 \leq x < 0$
36. $0 < x \leq 6$

Inequalities and Subsets of Real Numbers In Exercises 37–44, use inequality notation to describe the subset of real numbers. *See Example 2.*

37. x is positive.
38. t is no more than 20.
39. y is greater than 5 and less than or equal to 12.
40. m is at least -3 and at most 8.
41. The graduate assistant's age A is at least 21.
42. The yield Y is no more than 42 bushels per acre.
43. In the last decade, the annual rate of inflation r was at least -0.5%, but no more than 3.8%.
44. A bank may charge up to \$4.00 for an ATM transaction.

Finding Absolute Value In Exercises 45–54, evaluate the expression. *See Example 3.*

45. $|-10|$
46. $|0|$
47. $-\frac{3}{2} - \left|-\frac{3}{2}\right|$
48. $-\left|\frac{1}{3}\right| - \left|-\frac{2}{3}\right|$
49. $-0.8|-2|$
50. $-6|-0.2|$
51. $\frac{-5}{|-5|}$
52. $\frac{|-4|}{-4}$
53. $|3 - \pi|$
54. $|4 - \pi|$

Comparing Real Numbers In Exercises 55–60, place the correct symbol ($<$, $>$, or $=$) between the two real numbers. *See Example 4.*

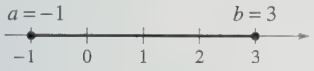
55. $|-7| \quad |7|$ **56.** $-5 \quad -|5|$

57. $|-3| \quad -|-3|$ **58.** $-|-6| \quad |-6|$

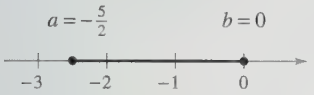
59. $-|-2| \quad -|2|$ **60.** $-(-2) \quad -2$

The Distance Between Two Numbers In Exercises 61–70, find the distance between a and b. *See Example 5.*

61.



62.



63. $a = -4, b = -3$ **64.** $a = -1, b = 11$

65. $a = -7, b = 0$ **66.** $a = 3, b = 9$

67. $a = 126, b = 75$ **68.** $a = -126, b = -75$

69. $a = 2.22, b = -1.45$ **70.** $a = 9.34, b = -5.65$

Using Absolute Value Notation In Exercises 71–78, use absolute value notation to describe the sentence. *See Example 5.*

71. The distance between z and 6 is greater than 1.

72. The distance between x and $\frac{9}{2}$ is no more than 3.

73. The distance between x and -10 is at least 6.

74. The distance between z and 0 is less than 8.

75. y is at least six units from 0.

76. x is less than eight units from 0.

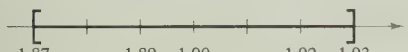
77. x is more than five units from m.

78. y is at most two units from a.

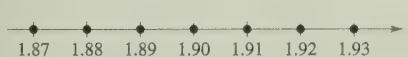
79. Travel While traveling on the Overseas Highway in the Florida Keys, you pass mile marker 108 at Jewfish Creek, and then mile marker 27 at Ramrod Key. How far do you travel between the mile markers?

80. HOW DO YOU SEE IT? Match each description with its graph. [The graphs are labeled (i) and (ii).] Which types of real numbers shown in the Venn diagram on page 3 may be included in a range of prices? a range of lengths? Explain.

(i)



(ii)



(a) An item is priced within \$0.03 of \$1.90.

(b) The distance between the prongs of an electric plug may not differ from 1.9 centimeters by more than 0.03 centimeter.

Temperature In Exercises 81 and 82, the record January temperatures (in degrees Fahrenheit) for a city are given. Find the distance between the numbers to determine the range of temperatures for January.

81. Milwaukee, Wisconsin: lowest: $-26°$F
highest: $63°$F

82. Fairbanks, Alaska: lowest: $-61°$F
highest: $52°$F

Budget Variance In Exercises 83–86, the accounting department of an Internet start-up company checks whether various actual expenses differ from the budgeted expenses by more than \$500 or by more than 5%. Complete the missing parts of the table. Then determine whether the actual expense passes the "budget variance test." *See Example 6.*

	Budgeted Expense, b	*Actual Expense, a*	$\lvert a - b \rvert$	$0.05b$
83.	\$30,000	\$29,123.45		
84.	\$125,500	\$126,347.85		
85.	\$12,000	\$11,735.68		
86.	\$8300	\$8632.59		

Quality Control In Exercises 87–92, the quality control inspector for a tire factory tests the rim diameters of various tires. The inspector rejects any tire that differs by more than 0.02 inch or by more than 0.12% of the expected diameter measure. Complete the missing parts of the table. Then determine whether the tire passes the test.

	Expected Diameter, b	*Actual Diameter, a*	$\lvert a - b \rvert$	$0.0012b$
87.	14 in.	13.998 in.		
88.	15 in.	15.012 in.		
89.	16 in.	15.973 in.		
90.	18 in.	18.027 in.		
91.	13 in.	13.022 in.		
92.	17 in.	16.992 in.		

93. Think About It Describe the real number values of u and v for which $|u + v|$ is *greater than*, *less than*, and *equal to* $|u| + |v|$.

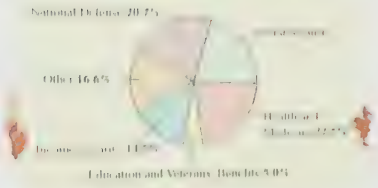
94. Think About It Is the set of nonnegative real numbers the same as the set of positive real numbers? Explain.

95. Writing For what real numbers a is $|a| = -a$? Explain.

96. Writing Describe the differences among the sets of natural numbers, integers, rational numbers, and irrational numbers.

0.2 The Basic Rules of Algebra







In Exercise 69 on page 19, you will use a circle graph to analyze the expenses of the federal government.

- Identify the terms of, and evaluate, an algebraic expression.
- Identify basic rules of algebra and perform operations on real numbers.
- Develop an understanding of the properties of equality.
- Use a calculator to evaluate an expression.

Algebraic Expressions

One of the basic characteristics of algebra is the use of letters (or combinations of letters) to represent numbers. The letters used to represent numbers are called **variables,** and combinations of letters and numbers are called **algebraic expressions.** Some examples of algebraic expressions are

$$5x, \quad 2x - 3, \quad \frac{4}{x^2 + 2}, \quad \text{and} \quad 7x + y.$$

Algebraic Expression

A collection of letters (called **variables**) and real numbers (called **constants**) that are combined using the operations of addition, subtraction, multiplication, and division is an **algebraic expression.** (Other operations can also be used to form an algebraic expression.)

The **terms** of an algebraic expression are those parts that are separated by addition. For example, the algebraic expression

$$x^2 - 5x + 8$$

has three terms: x^2, $-5x$, and 8. Note that $-5x$, rather than $5x$, is a term, because

$$x^2 - 5x + 8 = x^2 + (-5x) + 8.$$

The terms x^2 and $-5x$ are the **variable terms** of the expression, and 8 is the **constant term** of the expression. The numerical factor of a variable term is the **coefficient** of the variable term. For instance, the coefficient of the variable term $-5x$ is -5, and the coefficient of the variable term x^2 is 1.

Example 1 Identifying the Terms of an Algebraic Expression

Algebraic Expression	*Terms*
a. $4x - 3$	$4x, -3$
b. $2x + 4y - 5$	$2x, 4y, -5$
c. $-7 + 6p^3 - p^2$	$-7, 6p^3, -p^2$

✓ Checkpoint 1

Identify the terms of each algebraic expression.

a. $8 - 15x$

b. $4x^2 - 3y + 7$

c. $z^4 - 10z^3 + 1$

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ALGEBRA TUTOR

For help in evaluating the expressions in Example 2, see the *Chapter 0 Algebra Tutor* on page 62.

Example 2 Symbols of Grouping

a. $7 - 3(4 - 2) = 7 - 3(2)$
$= 7 - 6$
$= 1$

b. $(4 - 5) - (3 - 6) = (-1) - (-3)$
$= -1 + 3$
$= 2$

Checkpoint 2

Simplify the expression $5(7 - 3) + 9$.

The **Substitution Principle** states, "If $a = b$, then a can be replaced by b in any expression involving a." You use this principle to **evaluate** an algebraic expression by substituting numerical values for each of the variables in the expression. In the first evaluation shown below, 3 is substituted for x in the expression $-3x + 5$.

Expression	*Value of Variable*	*Substitution*	*Value of Expression*
$-3x + 5$	$x = 3$	$-3(3) + 5$	$-9 + 5 = -4$
$3x^2 + 2x - 1$	$x = -1$	$3(-1)^2 + 2(-1) - 1$	$3 - 2 - 1 = 0$
$-2x(x + 4)$	$x = -2$	$-2(-2)(-2 + 4)$	$-2(-2)(2) = 8$
$\frac{1}{x - 2}$	$x = 2$	$\frac{1}{2 - 2}$	$\frac{1}{0}$ is undefined.

TECH TUTOR



To evaluate the expression $3 + 4x$ for the x-values 2 and 5, use the *last entry* feature of a graphing utility.

1. Evaluate $3 + 4 \cdot 2$.
2. Press [2nd] [ENTRY] (recalls previous expression to the home screen).
3. Cursor to 2, replace 2 with 5, and press [ENTER].

Example 3 Evaluating Algebraic Expressions

Evaluate each algebraic expression at $x = -2$ and $y = 3$.

a. $4y - 2x$ **b.** $5 + x^2$ **c.** $5 - y^2$

SOLUTION

a. When $x = -2$ and $y = 3$, the expression $4y - 2x$ has a value of
$4(3) - 2(-2) = 12 + 4$
$= 16.$

b. When $x = -2$, the expression $5 + x^2$ has a value of
$5 + (-2)^2 = 5 + 4$
$= 9.$

c. When $y = 3$, the expression $5 - y^2$ has a value of
$5 - (3)^2 = 5 - 9$
$= -4.$

Checkpoint 3

Evaluate each algebraic expression at $x = 4$ and $y = -5$.

a. $7x - y$ **b.** $3y + x^2$ **c.** $x - y^2$

Basic Rules of Algebra

The four basic arithmetic operations are **addition, multiplication, subtraction,** and **division,** denoted by the symbols $+$, $\times$ or $\cdot$, $-$, and $\div$, respectively. Of these, addition and multiplication are considered to be the two primary arithmetic operations. Subtraction and division are defined as the inverse operations of addition and multiplication, as follows.

Subtraction: Add the opposite.

$a - b = a + (-b)$

Division: Multiply by the reciprocal.

If $b \neq 0$, then $a \div b = a\left(\frac{1}{b}\right) = \frac{a}{b}$.

In these definitions, $-b$ is called the **additive inverse** (or opposite) of b, and $1/b$ is called the **multiplicative inverse** (or reciprocal) of b. In place of $a \div b$, you can use the fractional form a/b. In this form, a is called the **numerator** of the fraction and b is called the **denominator.**

STUDY TIP

Remember that the basic rules of algebra are true for variables and algebraic expressions as well as for real numbers.

Basic Rules of Algebra

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property	*Example*
Commutative Property of Addition	
$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication	
$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition	
$(a + b) + c = a + (b + c)$	$(-x + 5) + x^2 = -x + (5 + x^2)$
Associative Property of Multiplication	
$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Property	
$a(b + c) = ab + ac$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$
$(a + b)c = ac + bc$	$(y + 8)y = y \cdot y + 8 \cdot y$
Additive Identity Property	
$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property	
$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property	
$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property	
$a \cdot \frac{1}{a} = 1, \quad a \neq 0$	$(x^2 + 4)\left(\frac{1}{x^2 + 4}\right) = 1$

Because subtraction is defined as "adding the opposite," the Distributive Property is also true for subtraction. For instance, the "subtraction form" of $a(b + c) = ab + ac$ is

$$\begin{aligned} a(b - c) &= a[b + (-c)] \\ &= ab + a(-c) \\ &= ab - ac. \end{aligned}$$

Example 4 Identifying the Basic Rules of Algebra

Identify the rule of algebra illustrated by each statement.

a. $(4x^2)5 = 5(4x^2)$

b. $(2y^3 + y) - (2y^3 + y) = 0$

c. $(4 + x^2) + 3x^2 = 4 + (x^2 + 3x^2)$

d. $(x - 5)7 + (x - 5)x = (x - 5)(7 + x)$

e. $2x \cdot \frac{1}{2x} = 1, \quad x \neq 0$

SOLUTION

a. This statement illustrates the Commutative Property of Multiplication.

b. This statement illustrates the Additive Inverse Property.

c. This statement illustrates the Associative Property of Addition.

d. This statement illustrates the Distributive Property in reverse order.

$$ab + ac = a(b + c) \qquad \text{Distributive Property}$$
$$(x - 5)7 + (x - 5)x = (x - 5)(7 + x)$$

e. This statement illustrates the Multiplicative Inverse Property.

STUDY TIP

Note that in Example 4(e), it is important that x be a nonzero number. If x was allowed to be zero, then you would have 0 in the denominator, and the reciprocal of zero is undefined.

✓Checkpoint 4

Identify the rule of algebra illustrated by each statement.

a. $3x^2 \cdot 1 = 3x^2$

b. $x^2 + 5 = 5 + x^2$

The following three lists summarize the basic properties of negation, zero, and fractions. When studying these lists, do not only *memorize* a verbal description of each property, but try to gain an understanding of why each property is valid.

Properties of Negation

Let a and b be real numbers, variables, or algebraic expressions.

Property	*Example*
1. $(-1)a = -a$	$(-1)7 = -7$
2. $-(-a) = a$	$-(-6) = 6$
3. $(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4. $(-a)(-b) = ab$	$(-2)(-6) = 2 \cdot 6$
5. $-(a + b) = (-a) + (-b)$	$-(3 + 8) = (-3) + (-8)$

Be sure you see the difference between the opposite of a number and a negative number. If a is negative, then its opposite, $-a$, is positive. For instance, if $a = -5$, then $-a = -(-5) = 5$.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions. Then the following properties are true.

1. $a + 0 = a$ and $a - 0 = a$
2. $a \cdot 0 = 0$
3. $\frac{0}{a} = 0, \quad a \neq 0$
4. $\frac{a}{0}$ is undefined.
5. Zero-Factor Property: If $ab = 0$, then $a = 0$ or $b = 0$.

The "or" in the Zero-Factor Property includes the possibility that both factors are zero. This is called an **inclusive or,** and it is the way the word "or" is always used in mathematics.



Properties of Fractions

Let a, b, c, and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$. Then the following properties are true.

1. *Equivalent fractions:* $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
2. *Rules of signs:* $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ and $\frac{-a}{-b} = \frac{a}{b}$
3. *Generate equivalent fractions:* $\frac{a}{b} = \frac{ac}{bc}, \quad c \neq 0$
4. *Add or subtract with like denominators:* $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$
5. *Add or subtract with unlike denominators:* $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$
6. *Multiply fractions:* $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
7. *Divide fractions:* $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \quad c \neq 0$

In Property 1 (equivalent fractions), the phrase "if and only if" implies the two statements below.

- If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.
- If $ad = bc$, where $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$.

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Example 5 Properties of Zero and Properties of Fractions

a. $x - \frac{0}{5} = x - 0 = x$ Properties 3 and 1 of zero

b. $\frac{x}{5} = \frac{3 \cdot x}{3 \cdot 5} = \frac{3x}{15}$ Generate equivalent fractions.

c. $\frac{x}{3} + \frac{2x}{5} = \frac{x \cdot 5 + 3 \cdot 2x}{15}$ Property 5 of fractions

$= \frac{11x}{15}$ Simplify.

d. $\frac{7}{x} \div \frac{3}{2} = \frac{7}{x} \cdot \frac{2}{3} = \frac{14}{3x}$ Divide fractions.

✓Checkpoint 5

Simplify the expression $\frac{x}{4} + \frac{2x}{3}$.

If a, b, and c are integers such that $ab = c$, then a and b are **factors** or **divisors** of c. For example, 2 and 3 are factors of 6 because $2 \cdot 3 = 6$. A **prime number** is a positive integer that has exactly two factors: itself and 1. For example, 2, 3, 5, 7, and 11 are prime numbers, whereas 1, 4, 6, 8, 9, and 10 are not. The numbers 4, 6, 8, 9, and 10 are **composite** because they can be written as the products of two or more prime numbers. The number 1 is neither prime nor composite. The **Fundamental Theorem of Arithmetic** states that every positive integer greater than 1 is a prime number or can be written as the product of prime numbers in precisely one way (disregarding order). For instance, the *prime factorization* of 24 is

$$24 = 2 \cdot 2 \cdot 2 \cdot 3.$$

When you are adding or subtracting fractions that have unlike denominators, you can use Property 4 of fractions by rewriting both of the fractions so that they have the same denominator. This is called the **least common denominator** method.

STUDY TIP

To find the LCD, first factor the denominators.

$15 = 3 \cdot 5$

$9 = 3^2$

$5 = 5$

The LCD is the product of the prime factors, with each factor given the highest power of its occurrence in any denominator. So, the LCD is

$3^2 \cdot 5 = 45.$

Example 6 Adding and Subtracting Fractions

Evaluate $\frac{2}{15} - \frac{5}{9} + \frac{4}{5}$.

SOLUTION Begin by factoring the denominators to find the least common denominator (LCD). Use the LCD, 45, to rewrite the fractions and simplify.

$$\frac{2}{15} - \frac{5}{9} + \frac{4}{5} = \frac{2 \cdot 3}{15 \cdot 3} - \frac{5 \cdot 5}{9 \cdot 5} + \frac{4 \cdot 9}{5 \cdot 9}$$

$$= \frac{6 - 25 + 36}{45}$$

$$= \frac{17}{45}$$

✓Checkpoint 6

Evaluate $\frac{3}{4} + \frac{2}{3} - \frac{1}{2}$.

Equations and Calculators

An **equation** is a statement of equality between two expressions. So, the statement

$$a + b = c + d$$

means that the expressions

$$a + b \quad \text{and} \quad c + d$$

represent the same number. For instance, because $1 + 4$ and $3 + 2$ both represent the number 5, you can write

$$1 + 4 = 3 + 2.$$

Three important properties of equality follow.

Properties of Equality

Let a, b, and c be real numbers, variables, or algebraic expressions.

1. Reflexive: $a = a$
2. Symmetric: If $a = b$, then $b = a$.
3. Transitive: If $a = b$ and $b = c$, then $a = c$.

In algebra, you often rewrite expressions by making substitutions that are permitted under the Substitution Principle. Two important consequences of the Substitution Principle are the following rules.

1. If $a = b$, then $a + c = b + c$.
2. If $a = b$, then $ac = bc$.

The first rule allows you to add the same number to each side of an equation. The second allows you to multiply each side of an equation by the same number. The converses of these two rules are also true and are listed below.

1. If $a + c = b + c$, then $a = b$.
2. If $ac = bc$ and $c \neq 0$, then $a = b$.

So, you can also subtract the same number from each side of an equation as well as divide each side of an equation by the same nonzero number.

The table below shows keystrokes for several similar functions on a standard scientific calculator and a graphing calculator. These keystrokes may not be the same as those for your calculator. Consult your user's guide for specific keystrokes.

Graphing Calculator	*Scientific Calculator*
ENTER	=
(−)	+/−
^	y^x
x^{-1}	1/x

For example, you can evaluate the expressions 10^2 and 13^3 on a graphing calculator or a scientific calculator as shown.

Expression	*Graphing Calculator*	*Scientific Calculator*
10^2	10 x^2 ENTER	10 x^2 =
13^3	13 ^ 3 ENTER	13 y^x 3 =

TECH TUTOR

Be sure you see the difference between the change sign key [+/−] or [(−)] and the subtraction key [−], as used in Example 7(a).

Example 7 Using a Calculator

Scientific Calculator

	Expression	*Keystrokes*	*Display*
a.	$-12^2 - 100$	12 [x²] [+/−] [−] 100 [=]	−244
b.	$24 \div 2^3$	24 [÷] 2 [yˣ] 3 [=]	3
c.	$3(10 - 4^2) \div 2$	3 [×] [(] 10 [−] 4 [x²] [)] [÷] 2 [=]	−9
d.	37% of 40	.37 [×] 40 [=]	14.8

Graphing Calculator

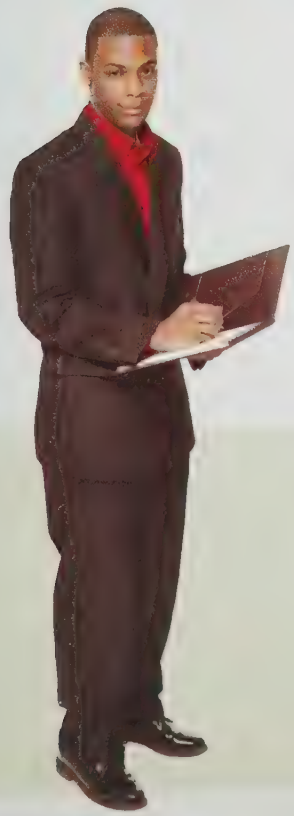
	Expression	*Keystrokes*	*Display*
a.	$-12^2 - 100$	[(−)] 12 [x²] [−] 100 [ENTER]	−244
b.	$24 \div 2^3$	24 [÷] 2 [^] 3 [ENTER]	3
c.	$3(10 - 4^2) \div 2$	3 [(] 10 [−] 4 [x²] [)] [÷] 2 [ENTER]	−9
d.	37% of 40	.37 [×] 40 [ENTER]	14.8

✓ Checkpoint 7

Write the keystrokes you can use to evaluate $6(8^3 - 481)$ on a graphing calculator or a scientific calculator. ■

When rounding decimals, look at the *decision digit* (the digit at the right of the last digit you want to keep). Round up when the decision digit is 5 or greater, and round down when it is 4 or less.

Example 8 Rounding Decimal Numbers



	Number	*Rounded to Three Decimal Places*	
a.	$\sqrt{2} = 1.4142135\ldots$	1.414	Round down because $2 < 4$.
b.	$\frac{7}{9} = 0.7777777\ldots$	0.778	Round up because $7 > 5$.

✓ Checkpoint 8

Use a calculator to evaluate $4\left(\frac{2}{3} + \frac{4}{5}\right)$. Then round the result to two decimal places. ■

SUMMARIZE (Section 0.2)

1. State the definition of an algebraic expression *(page 10)*. For an example of evaluating an algebraic expression, see Example 3.
2. State a basic rule of algebra, a property of negation, a property of zero, and a property of fractions *(pages 12–14)*. For examples that use these properties, see Examples 4, 5, and 6.
3. State the properties of equality *(page 16)*.

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SKILLS WARM UP 0.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.1.

In Exercises 1–4, place the correct inequality symbol (< or >) between the two real numbers.

1. $-4 \quad -2$ **2.** $0 \quad -3$ **3.** $-\frac{11}{9} \quad -\frac{13}{9}$ **4.** $-\frac{12}{33} \quad \frac{11}{33}$

In Exercises 5–8, find the distance between the two real numbers.

5. 4, 6 **6.** $-2, 2$ **7.** $0, -5$ **8.** $-1, 3$

Exercises 0.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Identifying the Terms of an Algebraic Expression In Exercises 1–6, identify the terms of the algebraic expression. *See Example 1.*

1. $7x + 4$ **2.** $3x - 6$

3. $x^2 - 4x + 8$ **4.** $4x^3 + x - 5$

5. $2x^2 - 9x + 13$ **6.** $3x^4 + 2x^3 + x^2 - 1$

Symbols of Grouping In Exercises 7–10, simplify the expression. *See Example 2.*

7. $(2 - 7) + 3$ **8.** $-3 - (5 - 2)$

9. $(4 - 7)(-2)$ **10.** $-5(-2 - 6)$

Evaluating Algebraic Expressions In Exercises 11–16, evaluate the expression at each value of x. (If not possible, state the reason.) *See Example 3.*

	Expression	*Values*	
11.	$4x - 6$	(a) $x = -1$	(b) $x = 0$
12.	$5 - 3x$	(a) $x = -3$	(b) $x = 2$
13.	$x^2 - 3x + 4$	(a) $x = -2$	(b) $x = 2$
14.	$-x^3 + 2x - 1$	(a) $x = 0$	(b) $x = 2$
15.	$\frac{x}{x - 2}$	(a) $x = -2$	(b) $x = 2$
16.	$\frac{x + 3}{x - 3}$	(a) $x = 3$	(b) $x = -3$

Evaluating Algebraic Expressions In Exercises 17–22, evaluate the expression at $x = 3$, $y = -2$, and $z = 4$. *See Example 3.*

17. $x + 3y + z$ **18.** $6z + 5x - 3y$

19. $x^2 - 5y + 4z$ **20.** $z^2 + 6y - x$

21. $\frac{x - y}{5z}$ **22.** $\frac{4z - 2y}{20x}$

Identifying the Basic Rules of Algebra In Exercises 23–38, identify the rule(s) of algebra illustrated by the statement. *See Example 4.*

23. $3 + 4 = 4 + 3$ **24.** $x + 9 = 9 + x$

25. $-15 + 15 = 0$ **26.** $(x + 2) - (x + 2) = 0$

27. $2(x + 3) = 2x + 6$ **28.** $(5 + 9)(6) = 5(6) + 9(6)$

29. $2\left(\frac{1}{2}\right) = 1$

30. $\frac{1}{h + 6}(h + 6) = 1, \quad h \neq -6$

31. $h + 0 = h$ **32.** $(z - 2) + 0 = z - 2$

33. $57 \cdot 1 = 57$ **34.** $1 \cdot (1 + x) = 1 + x$

35. $6 + (7 + 8) = (6 + 7) + 8$

36. $x + (y + 10) = (x + y) + 10$

37. $x(3y) = (x \cdot 3)y = (3x)y$

38. $\frac{1}{7}(7 \cdot 12) = \left(\frac{1}{7} \cdot 7\right)12 = 1 \cdot 12 = 12$

Writing the Prime Factorization In Exercises 39–42, write the prime factorization of the number.

39. 48 **40.** 24

41. 240 **42.** 150

Properties of Zero and Properties of Fractions In Exercises 43–48, simplify the expression. *See Example 5.*

43. $\frac{0}{7} + 3x$ **44.** $\frac{0}{12}(x + 1)$

45. $\frac{x}{4} + \frac{3x}{5}$ **46.** $\frac{x}{8} + \frac{5}{36}$

47. $\frac{8}{x} \div \frac{2}{3}$ **48.** $\frac{3x}{4} \div \frac{9}{2}$

Operations with Fractions In Exercises 49–56, perform the indicated operation(s). (Write fractional answers in simplest form.) *See Examples 5 and 6.*

49. $\frac{2 - 79}{-11}$ **50.** $\frac{27 - 35}{4}$

51. $\frac{5}{8} + \frac{1}{4} - \frac{5}{6}$ **52.** $\frac{10}{11} + \frac{6}{33} - \frac{13}{66}$

53. $\frac{2}{5} \cdot \frac{7}{8}$ **54.** $\left(-\frac{2}{3}\right) \cdot \frac{5}{8} \cdot \frac{3}{4}$

55. $\left(\frac{2}{3} \div 8\right) + \left(3 \cdot \frac{3}{12}\right)$

56. $\left(\frac{3}{5} \div 3\right) - \left(6 \cdot \frac{4}{8}\right)$

Using a Calculator In Exercises 57–64, use a calculator to evaluate the expression. (Round to two decimal places.) *See Examples 7 and 8.*

57. $3\left(-\frac{5}{12}+\frac{3}{8}\right)$

58. $2\left(-7+\frac{1}{6}\right)$

59. $\dfrac{11.46-5.37}{3.91}$

60. $\dfrac{-8.31+4.83}{7.65}$

61. $-108 \div 3^3$

62. $-80 \div 2^5$

63. $5(18-2^3) \div 10$

64. $-6^2-[7+(-2)^3]$

Using a Calculator In Exercises 65–68, use a calculator to solve. *See Example 7.*

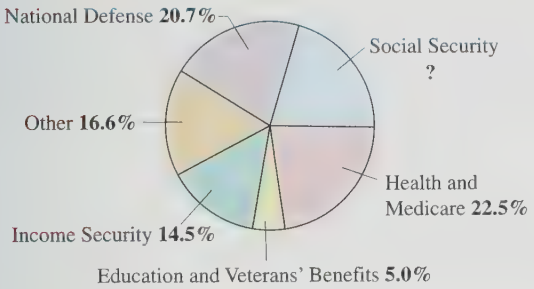
65. 35% of 68

66. 35% of 820

67. 125% of 37

68. 147% of 44

69. Federal Government Expenses The circle graph shows the types of expenses of the federal government in 2008. *(Source: U.S. Office of Management and Budget)*



(a) What percent of the total expenses was spent on Social Security?

(b) The total 2008 expenses were $2,982,900,000,000. Find the amount spent for each category in the circle graph. (Round to the nearest billion dollars.)

70. Research Study The percent of people in a research study who have a particular health risk is 39.5%. The total number of people in the study is 12,857. How many people have the health risk?

71. Clinical Trial The percent of patients in a clinical trial of a cancer treatment showing a decrease in tumor size is 49.2%. There are 3445 patients in the trial. How many patients show a decrease in tumor size?

72. Calculator Keystrokes Write the expression that corresponds to each set of keystrokes.

(a) 5 × (2.7 − 9.4) = Scientific

5 (2.7 − 9.4) ENTER Graphing

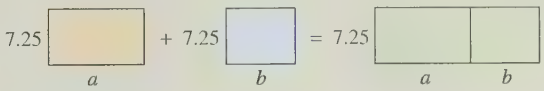
(b) 2 × (4 +/− + 2) = Scientific

2 ((−) 4 + 2) ENTER Graphing

73. Pet Spending The data below show pet spending in the United States in 2009. Find the percent of total pet spending for each category. Then use a spreadsheet software program to make a circle graph for the percent data. *(Source: American Pet Products Association)*

Total pet spending (2009):	**$45.53 billion**
Food:	$17.56 billion
Vet care:	$12.04 billion
Supplies/OTC medicine:	$10.41 billion
Live animal purchases:	$2.16 billion
Grooming and boarding:	$3.36 billion

74. HOW DO YOU SEE IT? You earn $7.25 per hour for each of two work study jobs. You work a hours as an office assistant and b hours as an events coordinator. The area model represents the situation.



(a) What quantity does the area of the gold rectangle represent? the blue rectangle? the green rectangle?

(b) What rule of algebra does the model illustrate?



Business Capsule

PetSmart, the largest pet store chain in the U.S. and Canada, has grown by offering pet lodging services in 180 of their 1187 stores. PetsHotels provide supervised play areas with toys and slides, blankets, TV, and snacks. Special fee services (grooming, training, and phoning pet parents) are also provided. Sales for these services have increased an average of 19% each year since PetsHotels began in 2004.

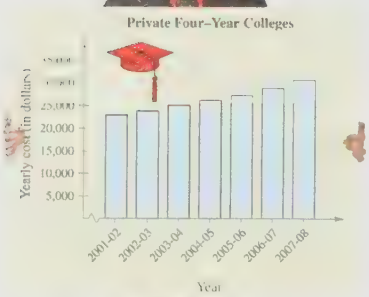
75. Research Project Use your campus library, the Internet, or some other reference source to find information on a "special services" company experiencing strong growth, and write a brief report about the company.

The symbol indicates an exercise in which you are instructed to use graphing technology or a spreadsheet software program. The solutions of other exercises may also be facilitated by use of appropriate technology.

AP Photo/Gerry Broome

0.3 Integer Exponents







In Exercise 83 on page 28, you will use the compound interest formula to find the average rate of inflation for the yearly costs of attending a private four-year college in the United States.

- Use properties of exponents.
- Use scientific notation to represent real numbers and use a calculator to evaluate exponential expressions.
- Use interest formulas to solve application problems.

Properties of Exponents

Repeated multiplication of a real number by itself can be written in **exponential form.** Here are some examples.

Repeated Multiplication	*Exponential Form*
$7 \cdot 7$	7^2
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

Exponential Notation

Let a be a real number, a variable, or an algebraic expression, and let n be a positive integer. Then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** The expression a^n is read as "a to the nth **power**" or simply "a to the nth."

When multiplying exponential expressions with the same base, *add* exponents.

$$a^m \cdot a^n = a^{m+n}$$ Add exponents when multiplying.

For instance, to multiply 2^2 and 2^3, you can write

$$2^2 \cdot 2^3 = \overbrace{(2 \cdot 2)}^{\text{Two factors}} \cdot \overbrace{(2 \cdot 2 \cdot 2)}^{\text{Three factors}} = \overbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}^{\text{Five factors}}$$
$$= 2^{2+3} = 2^5.$$

On the other hand, when dividing exponential expressions with the same base, *subtract* exponents. That is,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0.$$ Subtract exponents when dividing.

For instance, to divide 3^4 by 3^2, you can write

$$\frac{3^4}{3^2} = \frac{\overbrace{3 \cdot 3 \cdot 3 \cdot 3}^{\text{Four factors}}}{\underbrace{3 \cdot 3}_{\text{Two factors}}} = \frac{\cancel{3}^{1} \cdot \cancel{3}^{1} \cdot 3 \cdot 3}{\cancel{3}_{1} \cdot \cancel{3}_{1}} = \overbrace{3 \cdot 3}^{\text{Two factors}}$$
$$= 3^{4-2} = 3^2.$$

These and other properties of exponents are summarized on the following page.

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Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (Assume all denominators and bases are nonzero.)

Property	*Example*	
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6$	Product of Powers
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$	Quotient of Powers
3. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3 x^3 = 125x^3$	Power of a Product
4. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$	Power of a Quotient
5. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12}$	Power of a Power
6. $a^{-n} = \dfrac{1}{a^n}$	$y^{-4} = \dfrac{1}{y^4}$	Definition of negative exponent
7. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$	Definition of zero exponent
8. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n, \quad a \neq 0, b \neq 0$	$\left(\dfrac{3}{2}\right)^{-3} = \left(\dfrac{2}{3}\right)^3$	
9. $\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert 2^2 \rvert = \lvert 2 \rvert^2 = 2^2$	

Notice that these properties of exponents apply for *all* integers m and n, not just positive integers. For instance, by the Quotient of Powers Property,

$$\frac{3^4}{3^{-5}} = 3^{4-(-5)} = 3^{4+5} = 3^9.$$

STUDY TIP

It is important to recognize the difference between exponential forms such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. Similarly, in $(5x)^3$, the parentheses indicate that the exponent applies to the 5 as well as to the x, whereas in $5x^3 = 5(x^3)$, the exponent applies only to the x.

Example 1 **Using Properties of Exponents**

a. $3^{-1} \cdot 3^4 = 3^{-1+4} = 3^3 = 27$

b. $\dfrac{5^6}{5^4} = 5^{6-4} = 5^2 = 25$

c. $5\left(\dfrac{2}{5}\right)^3 = 5 \cdot \dfrac{2^3}{5^3} = 5 \cdot 5^{-3} \cdot 2^3 = 5^{-2} \cdot 2^3 = \dfrac{2^3}{5^2} = \dfrac{8}{25}$

d. $(-5 \cdot 2^3)^2 = (-5)^2 \cdot (2^3)^2 = 25 \cdot 2^6 = 25 \cdot 64 = 1600$

e. $(-3ab^4)(4ab^{-3}) = -3(4)(a)(a)(b^4)(b^{-3}) = -12a^2b$

f. $3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0$

g. $\left(\dfrac{5x^3}{y}\right)^2 = \dfrac{5^2(x^3)^2}{y^2} = \dfrac{25x^6}{y^2}$

Checkpoint 1

Evaluate each expression.

a. $(2^4)^2$

b. $8^{-1} \cdot 4^3 \cdot 4^2$

c. $(-6)^0 + 3^{-1}$

Study Tip

Rarely in algebra is there only one way to solve a problem. Don't be concerned if the steps you use to solve a problem are not exactly the same as the steps presented here. The important thing is to use steps that you understand *and* that, of course, are justified by the rules of algebra. For instance, you might prefer the following steps to simplify Example 2(d).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^{2} = \frac{y^2}{9x^4}$$

Example 2 Rewriting with Positive Exponents

a. $x^{-1} = \frac{1}{x}$ Definition of negative exponent

b. $\frac{1}{3x^{-2}} = \frac{1(x^2)}{3}$ The exponent -2 applies only to x.

$= \frac{x^2}{3}$ Simplify.

c. $\frac{12a^3b^{-4}}{4a^{-2}b} = \frac{12a^3 \cdot a^2}{4b \cdot b^4}$ Definition of negative exponent

$= \frac{3a^5}{b^5}$ Product of Powers Property

d. $\left(\frac{3x^2}{y}\right)^{-2} = \frac{3^{-2}(x^2)^{-2}}{y^{-2}}$ Power of a Quotient and Power of a Product Properties

$= \frac{3^{-2}x^{-4}}{y^{-2}}$ Power of a Power Property

$= \frac{y^2}{3^2x^4}$ Definition of negative exponent

$= \frac{y^2}{9x^4}$ Simplify.

Checkpoint 2

Rewrite $\left(\frac{3}{x^{-2}z^4}\right)^{-3}$ with positive exponents and simplify.

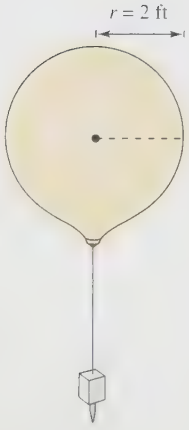


FIGURE 0.7

Example 3 Ratio of Volume to Surface Area

The volume V and surface area S of a sphere are given by

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2$$

where r is the radius of the sphere. A spherical weather balloon has a radius of 2 feet, as shown in Figure 0.7. Find the ratio of the volume to the surface area.

SOLUTION To find the ratio, write the quotient V/S and simplify.

$$\frac{V}{S} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2}$$

$$= \frac{\frac{4}{3}\pi 2^3}{4\pi 2^2}$$

$$= \frac{4\pi \cdot 2^3}{3 \cdot 4\pi \cdot 2^2}$$

$$= \frac{1}{3}(2) = \frac{2}{3}$$

Checkpoint 3

Evaluate $\frac{\frac{5}{7}x^7}{25x^5}$ at $x = 7$.

Scientific Notation and Calculators

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, a drop of water contains more than 33 billion billion molecules—that is, 33 followed by 18 zeros.

$$33{,}000{,}000{,}000{,}000{,}000{,}000$$

It is convenient to write such numbers in **scientific notation.** This notation has the form

$$c \times 10^n$$

where $1 \leq c < 10$ and n is an integer. So, the number of molecules in a drop of water can be written in scientific notation as

$$3.3 \times 10{,}000{,}000{,}000{,}000{,}000{,}000 = 3.3 \times 10^{19}.$$

The *positive* exponent 19 indicates that the number is *large* (10 or more) and that the decimal point has been moved 19 places. A *negative* exponent in scientific notation indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.000000000000000000000000000009.$$

28 decimal places

Example 4 Converting to Scientific Notation

a. $0.0000572 = 5.72 \times 10^{-5}$ Number is less than 1.

b. $149{,}400{,}000 = 1.494 \times 10^{8}$ Number is greater than 10.

c. $32.675 = 3.2675 \times 10^{1}$ Number is greater than 10.

✓Checkpoint 4

Write each number in scientific notation.

a. 6,318,000 **b.** 0.00345

Example 5 Converting to Decimal Notation

a. $3.125 \times 10^{2} = 312.5$ Number is greater than 10.

b. $3.73 \times 10^{-6} = 0.00000373$ Number is less than 1.

c. $7.91 \times 10^{5} = 791{,}000$ Number is greater than 10.

✓Checkpoint 5

Write each number in decimal notation.

a. 4.28×10^{5} **b.** 2.5×10^{-3}

Most calculators automatically use scientific notation when showing large (or small) numbers that exceed the display range. Try multiplying $86{,}500{,}000 \times 6000$. If your calculator follows standard conventions, then its display should be

5.19 11 or 5.19E11 .

This means that $c = 5.19$ and the exponent of 10 is $n = 11$, which implies that the number is 5.19×10^{11}. To *enter* numbers in scientific notation, your calculator should have an exponential entry key labeled EXP or EE.

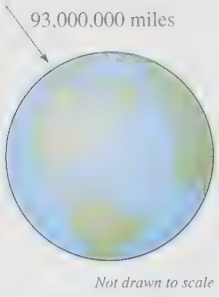


FIGURE 0.8

Example 6 The Speed of Light

The distance between Earth and the sun is approximately 93 million miles, as shown in Figure 0.8. How many minutes does it take for light to travel from the sun to Earth? Use the fact that light travels at a rate of approximately 186,000 miles per second.

SOLUTION Use the formula *distance* = (*rate*)(*time*) to find the time.

$$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{93 \text{ million miles}}{186{,}000 \text{ miles per second}}$$

$$= \frac{9.3 \times 10^7 \text{ miles}}{1.86 \times 10^5 \text{ miles/second}}$$

$$= 5 \times 10^2 \text{ seconds}$$

$$= 500 \text{ seconds}$$

$$\approx 8.33 \text{ minutes}$$

Note that to convert 500 seconds to 8.33 minutes, you divide by 60 because there are 60 seconds in one minute.

Checkpoint 6

The distance between Mars and the sun is approximately 142 million miles. How many minutes does it take for light to travel from the sun to Mars?

One of the most useful features of a calculator is its ability to evaluate exponential expressions. Consult your user's guide for specific keystrokes.

Example 7 Using a Calculator to Raise a Number to a Power

Scientific Calculator

	Expression	*Keystrokes*	*Display*
a.	$13^4 + 5$	13 [y^x] 4 [+] 5 [=]	28566
b.	$3^{-2} + 4^{-1}$	3 [y^x] 2 [+/−] [+] 4 [y^x] 1 [+/−] [=]	0.361111111
c.	$\dfrac{7^5 + 1}{7^5 - 1}$	[(] 7 [y^x] 5 [+] 1 [)] [÷] [(] 7 [y^x] 5 [−] 1 [)] [=]	1.000119005

Graphing Calculator

	Expression	*Keystrokes*	*Display*
a.	$13^4 + 5$	13 [^] 4 [+] 5 [ENTER]	28566
b.	$3^{-2} + 4^{-1}$	3 [^] [(−)] 2 [+] 4 [^] [(−)] 1 [ENTER]	.3611111111
c.	$\dfrac{7^5 + 1}{7^5 - 1}$	[(] 7 [^] 5 [+] 1 [)] [÷] [(] 7 [^] 5 [−] 1 [)] [ENTER]	1.000119005

TECH TUTOR

Make sure you include parentheses as needed when entering expressions in your calculator. Notice the use of parentheses in Example 7(c).

Checkpoint 7

Use a calculator to evaluate each expression.

a. $5^3 - 2^4$

b. $\dfrac{4^4 - 6}{2^5 + 18}$

Applications

One of the most familiar investment problems is finding the balance in a savings account. The formulas below can be used to find the balance in a savings account that earns simple interest or compound interest. You will learn more about interest in Chapter 4.

Balance in an Account

The balance A in an account that earns an annual interest rate r (in decimal form) for t years is given by one of the following.

$A = P(1 + rt)$ Simple interest

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compound interest

In both formulas, P is the principal (or the initial deposit). In the formula for compound interest, n is the number of compoundings *per year*. Make sure you convert all units of time t to years. For instance, 6 months $= \frac{1}{2}$ year. So, $t = \frac{1}{2}$.



Example 8 Finding the Balance in an Account

A family receives a tax refund of $6000 and wants to invest it for 5 years. Which savings plan will earn more money?

a. 2% simple annual interest

b. 1.5% annual interest compounded quarterly

SOLUTION

a. The balance after 5 years is

$$\begin{aligned} A &= P(1 + rt) \\ &= 6000[1 + 0.02(5)] \\ &= \$6600. \end{aligned}$$

b. The balance after 5 years is

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ &= 6000\left(1 + \frac{0.015}{4}\right)^{(4)(5)} \\ &\approx \$6466.40. \end{aligned}$$

Savings plan (a) will earn

$$6600 - 6000 = \$600$$

and savings plan (b) will earn

$$6466.40 - 6000 = \$466.40.$$

So, plan (a) will earn more money.

Checkpoint 8

In Example 8, how much money would the family earn in a savings plan with 3.4% annual interest compounded monthly?

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The compound interest formula can also be used to determine the rate of inflation. To apply the formula, you must know the costs of an item in two different years, as demonstrated in Example 9.

Example 9 Finding the Rate of Inflation

In 1984, the cost of a first class postage stamp was \$0.20. By 2010, the cost had increased to \$0.44, as shown in Figure 0.9. Find the average annual rate of inflation for a first class postage stamp over this 26-year period. *(Source: U.S. Postal Service)*

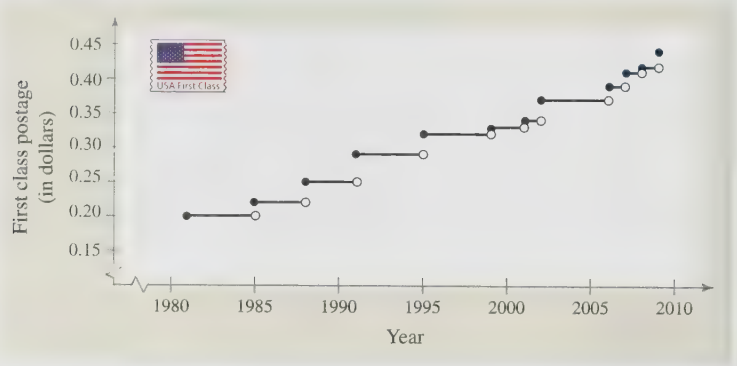


FIGURE 0.9

SOLUTION To find the average annual rate of inflation, use the formula for compound interest with *annual* compounding. So, find the value of r that will make the following equation true.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$0.44 = 0.20(1 + r)^{26}$$

You can begin by guessing that the average annual rate of inflation was 4%. Entering $r = 0.04$ in the formula, you find that $0.20(1 + 0.04)^{26} \approx 0.5545$. Because this result is more than 0.44, try some smaller values of r. Finally, you can discover that

$$0.20(1 + 0.031)^{26} \approx 0.44.$$

So, the average annual rate of inflation for a first class postage stamp from 1984 to 2010 was about 3.1%.



David Gilder/Shutterstock.com

✓ Checkpoint 9

In a certain year, the registration fee for the Medical College Admission Test (MCAT) was \$210. Four years later, the fee was \$235. What was the average annual rate of inflation over this four-year period? ■

SUMMARIZE (Section 0.3)

1. Make a list of the properties of exponents *(page 21)*. For examples that use these properties, see Examples 1, 2, and 3.
2. State the definition of scientific notation *(page 23)*. For examples of scientific notation, see Examples 4, 5, and 6.
3. State the formulas for simple and compound interest *(page 25)*. For applications of these formulas, see Examples 8 and 9.

SKILLS WARM UP 0.3 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.2.

In Exercises 1–10, perform the indicated operation(s) and simplify.

1. $\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)$
2. $\left(\frac{1}{4}\right)(5)(4)$
3. $3\left(-\frac{2}{7}\right) + 11\left(-\frac{2}{7}\right)$
4. $11\left(-\frac{1}{4}\right) + \frac{5}{4}$
5. $\frac{1}{2} \div (-2)$
6. $-\frac{1}{3} \div \left(-\frac{1}{3}\right)$
7. $\frac{1}{7} + \frac{1}{3} - \frac{1}{21}$
8. $\frac{1}{3} + \frac{1}{2} - \frac{5}{6}$
9. $\frac{1}{12} - \left(\frac{1}{3} + \frac{1}{8}\right)$
10. $\left(\frac{1}{2} - \frac{1}{3}\right) \div \frac{1}{6}$

Exercises 0.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Using Properties of Exponents In Exercises 1–14, evaluate the expression. Write fractional answers in simplest form. *See Example 1.*

1. $2^2 \cdot 2^4$
2. $3 \cdot 3^5$
3. $\frac{2^6}{2^3}$
4. $\frac{5^7}{5^5}$
5. $(3^3)^2$
6. $(2^5)^3$
7. -3^4
8. $(-3)^4$
9. $(2^3 \cdot 3^2)^2$
10. $(-3 \cdot 4^2)^3$
11. $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$
12. $\left(\frac{-5}{4}\right)^3\left(\frac{4}{5}\right)^2$
13. 3^0
14. $(-2)^0$

Using Properties of Exponents In Exercises 15–36, simplify the expression. *See Example 1.*

15. $(-5z)^3$
16. $(-2w)^5$
17. $(8x^4)(2x^3)$
18. $5x^4(x^2)$
19. $10(x^2)^2$
20. $2(4x^4)^3$
21. $(-z)^3(3z^4)$
22. $(6y^2)(2y^3)^3$
23. $\frac{25y^8}{15y^4}$
24. $\frac{10x^9}{4x^6}$
25. $\left(\frac{4}{y}\right)^3\left(\frac{3}{y}\right)^4$
26. $\left(\frac{5}{z}\right)^2\left(\frac{2}{z}\right)^3$
27. $\frac{7x^2}{x^3}$
28. $\frac{5z^5}{z^7}$
29. $\frac{x^2 \cdot x^n}{x^3 \cdot x^n}$
30. $\frac{x^n \cdot x^{2n}}{x^{3n}}$
31. $3^n \cdot 3^{2n}$
32. $2^m \cdot 2^{3m}$
33. $\frac{15(x+3)^3}{9(x+3)^2}$
34. $\frac{24(x-2)^5}{8(x-2)^4}$
35. $(2x^5)^0, \quad x \neq 0$
36. $(x+5)^0, \quad x \neq -5$

Rewriting with Positive Exponents In Exercises 37–50, rewrite the expression with positive exponents and simplify. *See Examples 1 and 2.*

37. $8 \cdot (-2)^{-2} \cdot 4^{-1}$
38. $6 \cdot 2^{-3} \cdot 3^{-1}$
39. $\left(\frac{1}{2}\right)^{-3}$
40. $\left(\frac{2}{3}\right)^{-3}$
41. $5^{-1} - 2^{-1}$
42. $4^{-1} - 2^{-2}$
43. $4x^{-2}$
44. $-7x^{-1}$
45. $\frac{9a^2b^{-2}}{3a^{-1}b}$
46. $\frac{10x^4y^{-4}}{5x^2y^{-2}}$
47. $\left(\frac{x}{10}\right)^{-1}$
48. $\left(\frac{y}{5}\right)^{-2}$
49. $\left(\frac{x^{-3}y^4}{5}\right)^{-3}$
50. $\left(\frac{2z^2}{y}\right)^{-2}$

Evaluating Expressions with Exponents In Exercises 51–54, evaluate the expression for the indicated value of x. *See Example 3.*

	Expression	*Value*		*Expression*	*Value*
51.	$\frac{x^4}{2x^2}$	$x = -6$	52.	$4x^{-3}$	$x = 2$
53.	$7x^{-2}$	$x = 4$	54.	$8x^0 - (8x)^0$	$x = -7$

55. **Discovery** Using your calculator, find the values of 10^3, 10^2, 10^1, 10^0, 10^{-1}, and 10^{-2}. What do you notice?

56. **Error Analysis** Describe the error.

$10^{-5} = 0.000001$

Converting to Scientific Notation In Exercises 57–62, write the number in scientific notation. *See Example 4.*

57. **Land Area of Earth:** 57,300,000 square miles
58. **Water Area of Earth:** 139,500,000 square miles
59. **Light Year:** 9,460,000,000,000 kilometers
60. **Mass of a Bacterium:** 0.0000000000000003 gram
61. **Thickness of a Soap Bubble:** 0.0000001 meter
62. **One Micron (millionth of a meter):** 0.00003937 inch

Converting to Decimal Notation In Exercises 63–66, write the number in decimal notation. *See Example 5.*

63. Number of Air Sacs in the Lungs: 3.5×10^8

64. Temperature of the Core of the Sun: 1.5×10^7 degrees Celsius

65. Charge of an Electron: 1.602×10^{-19} coulomb

66. Width of a Human Hair: 9.0×10^{-5} meter

Operations with Scientific Notation In Exercises 67 and 68, evaluate each expression without using a calculator. *See Example 6.*

67. (a) $(1.2 \times 10^7)(5 \times 10^{-3})$ (b) $\dfrac{6.0 \times 10^8}{3.0 \times 10^{-3}}$

68. (a) $(9.8 \times 10^{-2})(3 \times 10^7)$ (b) $\dfrac{9.0 \times 10^5}{4.5 \times 10^{-2}}$

Operations with Scientific Notation In Exercises 69 and 70, write each number in scientific notation. Perform the operation and write your answer in scientific notation.

69. (a) 48,000,000,000(250,000,000)

(b) $\dfrac{0.000000012}{0.0000064}$

70. (a) 0.00000034(0.00000006) (b) $\dfrac{18{,}000{,}000{,}000}{2{,}400{,}000}$

Using a Calculator to Raise a Number to a Power In Exercises 71–76, use a calculator to evaluate the expression. *See Example 7.*

71. $11^5 + 49$

72. $17^3 + 5^3$

73. $2^{-3} + 5^{-1}$

74. $3^{-4} + 64^{-1}$

75. $\dfrac{4^4 + 1}{4^4 - 1}$

76. $\dfrac{5^4 - 1}{5^4 + 1}$

Using a Calculator In Exercises 77–80, use a calculator to evaluate each expression. Write your answer in scientific notation. (Round to three decimal places.)

77. (a) 0.000345(8,900,000,000)

(b) $\dfrac{67{,}000{,}000 + 93{,}000{,}000}{0.0052}$

78. (a) 0.000045(9,200,000)

(b) $\dfrac{0.0000928 - 0.0000021}{0.0061}$

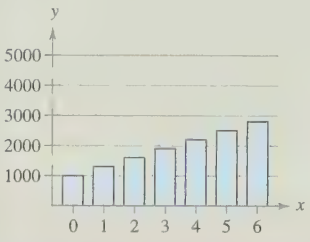
79. (a) $(9.3 \times 10^6)^3(6.1 \times 10^{-4})$ (b) $\dfrac{(2.414 \times 10^4)^6}{(1.68 \times 10^5)^5}$

80. (a) $(1.2 \times 10^2)^2(5.3 \times 10^{-5})$ (b) $\dfrac{(3.28 \times 10^{-6})^{10}}{(5.34 \times 10^{-3})^{22}}$

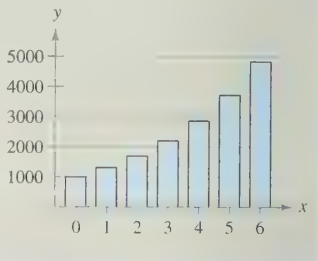
81. Balance in an Account You deposit \$10,000 in an account with an annual interest rate of 3.75%. Determine the balance in the account after 12 years when the interest is compounded (a) daily ($n = 365$), (b) weekly, (c) monthly, and (d) quarterly. How is the balance affected by the type of compounding?

82. HOW DO YOU SEE IT? Each bar graph represents the value y of a \$1000 investment after x years. One investment earns simple interest and the other earns compound interest. Which type of interest does each graph show? Explain.

(a)



(b)



83. College Costs The average yearly cost of tuition, fees, and room and board at private four-year colleges in the United States was \$22,896 for the academic year 2001/2002 and \$30,778 for the academic year 2007/2008. Find the average annual rate of inflation over this seven-year period. *(Source: U.S. National Center for Education Statistics)*

84. Becoming a Millionaire The compound interest formula can be rewritten as

$$P = \frac{A}{(1 + r/n)^{nt}}.$$

Find the principal P that you need to invest on the day your child is born at 4.5% annual interest compounded quarterly to make your child a millionaire on his or her 21st birthday.

85. The Speed of Light The closest distance from Earth to Neptune is 2.68 billion miles. At this distance, how long does it take for the image of Neptune to reach Earth, given that light travels about 186,000 miles per second?

86. Electron Microscopes Electron microscopes provide greater magnification than traditional light microscopes by using focused beams of electrons instead of visible light. It is the extremely short wavelengths of the electron beams that make electron microscopes so powerful. The wavelength λ (in meters) of any object in motion is given by

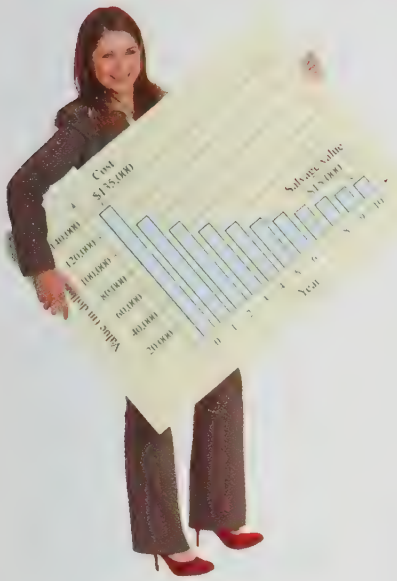
$$\lambda = \frac{6.626 \times 10^{-34}}{mv}$$

where m is the mass (in kilograms) of the object and v is its velocity (in meters per second). Find the wavelength of an electron with a mass of 9.11×10^{-31} kilogram and a velocity of 5.9×10^6 meters per second.

(Submitted by Brian McIntyre, Senior Laboratory Engineer for the Optics Electron Microscopy Facility at the University of Rochester.)

0.4 Radicals and Rational Exponents

- Evaluate expressions involving radicals and develop an understanding of the properties of radicals.
- Simplify a radical and rationalize a denominator.
- Use properties of rational exponents and use a calculator to evaluate a radical.



In Exercise 100 on page 38, you will use a bar graph to find the annual depreciation rate of an ultrasound machine.

Radicals and Properties of Radicals

A **square root** of a number is defined as one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a **cube root** of a number is one of its three equal factors. Here are some examples.

Number	*Equal Factors*	*Root*
$25 = (-5)^2$	$(-5)(-5)$	-5 (square root)
$-64 = (-4)^3$	$(-4)(-4)(-4)$	-4 (cube root)
$81 = 3^4$	$3 \cdot 3 \cdot 3 \cdot 3$	3 (fourth root)

Definition of *n*th Root of a Number

Let a and b be real numbers and let n be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** If

$$n = 2$$

the root is a **square root,** and if

$$n = 3$$

the root is a **cube root.**

From this definition, you can see that some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The following definition distinguishes between these two roots.

Principal *n*th Root of a Number

Let a be a real number that has at least one real nth root. **The principal *n*th root of *a*** is the nth root that has the same sign as a, and it is denoted by the **radical symbol**

$$\sqrt[n]{a}.$$ Principal nth root

The positive integer n is the **index** (the plural of index is indexes or indices) of the radical, and the number a is the **radicand.**

index → $\sqrt[n]{a}$ ← radicand

If $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$.

Poznyakov/Shutterstock.com

Example 1 **Evaluating Expressions Involving Radicals**

a. The principal square root of 121 is $\sqrt{121} = 11$ because $11^2 = 121$.

b. The principal cube root of $\frac{125}{64}$ is $\sqrt[3]{\frac{125}{64}} = \frac{5}{4}$ because $\left(\frac{5}{4}\right)^3 = \frac{5^3}{4^3} = \frac{125}{64}$.

c. The principal fifth root of -32 is $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

d. $-\sqrt{49} = -7$ because $7^2 = 49$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81.

Checkpoint 1

Evaluate each expression.

a. $\sqrt{\frac{1}{49}}$

b. $\sqrt[4]{625}$

c. $-\sqrt[3]{-8}$

From Example 1, you can make the following generalizations about nth roots of a real number.

1. If a is a positive real number and n is a positive *even* integer, then a has exactly two real nth roots, which are denoted by $\sqrt[n]{a}$ and $-\sqrt[n]{a}$.
2. If a is any real number and n is an *odd* integer, then a has only one (real) nth root. It is the principal nth root and is denoted by $\sqrt[n]{a}$.
3. If a is negative and n is an *even* integer, then a has no (real) nth root.

Integers such as 1, 4, 9, 16, 49, and 81 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers. Then the following properties are true.

Property	*Example*
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \quad b \neq 0$	$\frac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\frac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

A common special case of Property 6 is $\sqrt{a^2} = |a|$.

Simplifying Radicals

Radical Expressions in Simplest Form

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called *rationalizing the denominator*).
3. The index of the radical has been reduced as far as possible.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

Example 2 Simplifying Even Roots

a. $\sqrt[4]{48} = \sqrt[4]{16 \cdot 3}$ Find largest fourth-power factor.

$= \sqrt[4]{2^4 \cdot 3}$ Rewrite.

$= 2\sqrt[4]{3}$ Find fourth root.

b. $\sqrt{75x^3} = \sqrt{25x^2 \cdot 3x}$ Find largest square factor.

$= \sqrt{(5x)^2 \cdot 3x}$ Rewrite.

$= 5x\sqrt{3x}, \quad x \geq 0$ Find root of perfect square.

c. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

d. $\sqrt{\frac{7x^2}{9}} = \frac{\sqrt{7x^2}}{\sqrt{9}} = \frac{\sqrt{7}|x|}{3}$

STUDY TIP

In Examples 2(c) and 2(d), note that the absolute value symbols are included in the answer because $\sqrt[4]{x^4} = |x|$ and $\sqrt{x^2} = |x|$.

Checkpoint 2

Simplify $\sqrt{18x^5}$.

Example 3 Simplifying Odd Roots

a. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3}$ Find largest cube factor.

$= \sqrt[3]{2^3 \cdot 3}$ Rewrite.

$= 2\sqrt[3]{3}$ Find root of perfect cube.

b. $\sqrt[5]{32a^{11}} = \sqrt[5]{32a^{10} \cdot a}$ Find largest fifth-power factor.

$= \sqrt[5]{(2a^2)^5 \cdot a}$ Rewrite.

$= 2a^2\sqrt[5]{a}$ Find fifth root.

c. $\sqrt[3]{-40x^6} = \sqrt[3]{(-8x^6) \cdot 5}$ Find largest cube factor.

$= \sqrt[3]{(-2x^2)^3 \cdot 5}$ Rewrite.

$= -2x^2\sqrt[3]{5}$ Find root of perfect cube.

Checkpoint 3

Simplify $\sqrt[3]{54x^4}$.

Some fractions have radicals in the denominator. To **rationalize a denominator** of the form $a + b\sqrt{m}$, multiply the numerator and denominator by the **conjugate** $a - b\sqrt{m}$.

$$a + b\sqrt{m} \quad \text{and} \quad a - b\sqrt{m} \qquad \text{Conjugates}$$

When $a = 0$, the rationalizing factor of $\sqrt{m}$ is itself, $\sqrt{m}$.

Example 4 Rationalizing Single-Term Denominators

a. To rationalize the denominator of the following fraction, multiply *both* the numerator and the denominator by $\sqrt{3}$ to obtain

$$\frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{5\sqrt{3}}{2\sqrt{3^2}}$$
$$= \frac{5\sqrt{3}}{2(3)} = \frac{5\sqrt{3}}{6}.$$

b. To rationalize the denominator of the following fraction, multiply *both* the numerator and the denominator by $\sqrt[3]{5^2}$. Note how this eliminates the radical from the denominator by producing a perfect *cube* in the radicand.

$$\frac{2}{\sqrt[3]{5}} = \frac{2}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$$
$$= \frac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}}$$
$$= \frac{2\sqrt[3]{25}}{5}$$

✓Checkpoint 4

Simplify $\frac{1}{\sqrt[3]{4}}$ by rationalizing the denominator. ■

Example 5 Rationalizing a Denominator with Two Terms

$$\frac{2}{3 + \sqrt{7}} = \frac{2}{3 + \sqrt{7}} \cdot \frac{3 - \sqrt{7}}{3 - \sqrt{7}} \qquad \text{Multiply numerator and denominator by conjugate.}$$
$$= \frac{2(3 - \sqrt{7})}{3^2 - (\sqrt{7})^2} \qquad \text{Multiply fractions.}$$
$$= \frac{2(3 - \sqrt{7})}{9 - 7} \qquad \text{Simplify.}$$
$$= \frac{2(3 - \sqrt{7})}{2} \qquad \text{Divide out common factors.}$$
$$= 3 - \sqrt{7} \qquad \text{Simplify.}$$

STUDY TIP

Don't confuse an expression such as $\sqrt{2} + \sqrt{7}$ with $\sqrt{2 + 7}$. In general, $\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$.

✓Checkpoint 5

Simplify $\frac{10}{4 - \sqrt{11}}$ by rationalizing the denominator. ■

Rational Exponents and Calculators

The definition below shows how radicals are used to define **rational exponents.** Until now, work with exponents has been restricted to integer exponents.

STUDY TIP

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken. It doesn't matter which operation is performed first, provided the nth root exists.

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$$

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined to be $a^{1/n} = \sqrt[n]{a}$.

If m is a positive integer that has no common factor with n, then $a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m$ and $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$.

The properties of exponents mentioned in Section 0.3 also apply to rational exponents (provided the roots indicated by the denominators exist). Some of those properties are relisted here, with different examples.

Properties of Exponents

Let r and s be rational numbers, and let a and b be real numbers, variables, or algebraic expressions. If the roots indicated by the rational exponents exist, then the following properties are true.

Property	*Example*
1. $a^r a^s = a^{r+s}$	$4^{1/2}(4^{1/3}) = 4^{5/6}$
2. $\dfrac{a^r}{a^s} = a^{r-s}, \quad a \neq 0$	$\dfrac{x^2}{x^{1/2}} = x^{2-(1/2)} = x^{3/2}, x \neq 0$
3. $(ab)^r = a^r b^r$	$(2x)^{1/2} = 2^{1/2}(x^{1/2})$
4. $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}, \quad b \neq 0$	$\left(\dfrac{x}{3}\right)^{1/3} = \dfrac{x^{1/3}}{3^{1/3}}$
5. $(a^r)^s = a^{rs}$	$(x^3)^{1/3} = x$
6. $a^{-r} = \dfrac{1}{a^r}, \quad a \neq 0$	$4^{-1/2} = \dfrac{1}{4^{1/2}} = \dfrac{1}{2}$
7. $\left(\dfrac{a}{b}\right)^{-r} = \left(\dfrac{b}{a}\right)^r, \quad a \neq 0, \quad b \neq 0$	$\left(\dfrac{x}{4}\right)^{-1/2} = \left(\dfrac{4}{x}\right)^{1/2} = \dfrac{2}{x^{1/2}}$

STUDY TIP

Rational exponents can be tricky. Remember, the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. For instance, $(-8)^{5/6}$ is not defined because $\sqrt[6]{-8}$ is not a real number. And yet, $(-8)^{2/3}$ is defined because $\sqrt[3]{-8} = -2$.

Example 6 Simplifying with Rational Exponents

a. $(27)^{1/3} = \sqrt[3]{27}$

$= 3$

b. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

c. $(-5x^{2/3})(3x^{-1/3}) = -15x^{(2/3)-(1/3)}$

$= -15x^{1/3}, \quad x \neq 0$

Checkpoint 6

Simplify $(3^{1/2})(3^{3/2})$.

Example 7 Reducing the Index of a Radical

a. $\sqrt[6]{a^4} = a^{4/6} = a^{2/3} = \sqrt[3]{a^2}$

b.
$$\sqrt[3]{\sqrt{125}} = (125^{1/2})^{1/3} \quad \text{Rewrite with rational exponents.}$$
$$= (125)^{1/6} \quad \text{Multiply exponents.}$$
$$= (5^3)^{1/6} \quad \text{Rewrite base as perfect cube.}$$
$$= 5^{3/6} \quad \text{Multiply exponents.}$$
$$= 5^{1/2} \quad \text{Reduce exponent.}$$
$$= \sqrt{5} \quad \text{Rewrite as radical.}$$

✓Checkpoint 7

Use rational exponents to reduce the index of the radical $\sqrt[6]{2^3}$. ■

Radical expressions can be combined (added or subtracted) if they are **like radicals**—that is, if they have the same index and radicand. For instance, $2\sqrt{3x}$ and $\frac{1}{2}\sqrt{3x}$ are like radicals, but $\sqrt[3]{3x}$ and $2\sqrt{3x}$ are not like radicals.

Example 8 Simplifying by Combining Like Radicals

a.
$$2\sqrt{48} + 3\sqrt{27} = 2\sqrt{16 \cdot 3} + 3\sqrt{9 \cdot 3} \quad \text{Find square factors.}$$
$$= 2 \cdot 4\sqrt{3} + 3 \cdot 3\sqrt{3} \quad \text{Find square roots.}$$
$$= 8\sqrt{3} + 9\sqrt{3} \quad \text{Simplify.}$$
$$= 17\sqrt{3} \quad \text{Combine like terms.}$$

b.
$$\sqrt[3]{16x} - \sqrt[3]{54x} = \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot 2x} \quad \text{Find cube factors.}$$
$$= 2\sqrt[3]{2x} - 3\sqrt[3]{2x} \quad \text{Find cube roots.}$$
$$= -\sqrt[3]{2x} \quad \text{Combine like terms.}$$

✓Checkpoint 8

Simplify the expression $\sqrt{25x} + \sqrt{x}$. ■

You can use a calculator to evaluate radicals by using the square root key $\boxed{\sqrt{\ }}$, the cube root key $\boxed{\sqrt[3]{\ }}$, or the xth root key $\boxed{\sqrt[x]{\ }}$. You can also use the exponential key $\boxed{\wedge}$ or $\boxed{y^x}$. To use these keys, first convert the radical to exponential form.

Example 9 Evaluating a Cube Root with a Calculator

Two ways to evaluate $\sqrt[3]{25}$ using a calculator are shown below.

$\boxed{25}\ \boxed{y^x}\ \boxed{(}\ \boxed{1}\ \boxed{\div}\ \boxed{3}\ \boxed{)}\ \boxed{=}$ Exponential key

$\boxed{\sqrt[3]{\ }}\ \boxed{25}\ \boxed{)}\ \boxed{\text{ENTER}}$ Cube root key

Most calculators will display 2.924017738. So, $\sqrt[3]{25} \approx 2.924$.

✓Checkpoint 9

Use a calculator to evaluate $\sqrt[3]{18}$. Round to three decimal places. ■

TECH TUTOR

Make sure you notice that there may be more than one way to evaluate an expression using a graphing calculator. For instance, you can also use the cube root key to evaluate $\sqrt[3]{-4}$ in Example 10(a).

Example 10 Evaluating Radicals with a Calculator

a. Use the following keystroke sequence to evaluate $\sqrt[3]{-4}$.

4 [+/−] [y^x] [(] 1 [÷] 3 [)] [=] Scientific

[(−)] 4 [^] [(] 1 [÷] 3 [)] [ENTER] Graphing

The display is -1.587401052. So, $\sqrt[3]{-4} \approx -1.587$.

b. Use the following keystroke sequence to evaluate $(1.4)^{-2/5}$.

1.4 [y^x] [(] 2 [÷] 5 [+/−] [)] [=] Scientific

1.4 [^] [(] [(−)] 2 [÷] 5 [)] [ENTER] Graphing

The display is 0.874075175. So, $(1.4)^{-2/5} \approx 0.874$.

Checkpoint 10

Use a calculator to evaluate $6.7^{3/4}$. Round to three decimal places.

Example 11 Escape Velocity

A rocket, launched vertically from Earth, has a velocity of 10,000 meters per second. The *escape velocity*, or the minimum velocity necessary for the rocket to escape the gravitational field of Earth, is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.37 \times 10^{6}}} \text{ meters per second.}$$

Will the rocket escape Earth's gravitational field?

SOLUTION The escape velocity is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.37 \times 10^{6}}} \approx 11{,}190.7 \text{ meters per second.}$$

The velocity of 10,000 meters per second is less than the escape velocity of 11,190.7 meters per second. So, the rocket will not escape Earth's gravitational field.



Checkpoint 11

Will an object traveling at 10,000 meters per second exceed the escape velocity of Venus, which is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(4.87 \times 10^{24})}{6.05 \times 10^{6}}}$$

meters per second?

SUMMARIZE (Section 0.4)

1. State the definitions of the nth root and principal nth root of a number *(page 29)*. For an example of finding roots of numbers, see Example 1.
2. State what is meant by a radical expression in simplest form *(page 31)*. For examples of simplifying radical expressions, see Examples 2, 3, 4, and 5.
3. State the definition of rational exponents *(page 33)*. For examples of simplifying expressions with rational exponents, see Examples 6 and 7.

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SKILLS WARM UP 0.4 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.3.

In Exercises 1–10, simplify the expression.

1. $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$
2. $3(-4)^2$
3. $(-2x)^3$
4. $(-2x^3)(-3x^4)$
5. $(7x^5)(4x)$
6. $(5x^4)(25x^2)^{-1}$
7. $\frac{12z^6}{4z^2}$
8. $\left(\frac{2x}{5}\right)^2\left(\frac{2x}{5}\right)^{-4}$
9. $\left(\frac{3y^2}{x}\right)^0, \quad x \neq 0, \ y \neq 0$
10. $[(x+2)^2(x+2)^3]^2$

Exercises 0.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding the *n*th Roots of a Number In Exercises 1–10, use a radical symbol to denote and evaluate the principal *n*th root (if it exists) of the number. Then find any other real *n*th roots of the number. *See Example 1.*

	Number	*n*		*Number*	*n*
1.	4	2	2.	1	2
3.	−8	3	4.	8	3
5.	27	3	6.	−27	3
7.	−1	4	8.	−9	2
9.	64	6	10.	81	4

Evaluating an Expression Involving Radicals In Exercises 11–20, evaluate the expression. *See Example 1.*

11. $\sqrt{9}$
12. $\sqrt[3]{64}$
13. $-\sqrt[3]{-27}$
14. $\sqrt[3]{0}$
15. $\frac{14}{\sqrt{49}}$
16. $\frac{\sqrt[4]{81}}{6}$
17. $\sqrt[4]{-16}$
18. $\sqrt[6]{-1}$
19. $\sqrt{\frac{-27}{125}}$
20. $\sqrt{\frac{16}{81}}$

Simplifying a Radical Expression In Exercises 21–30, simplify the expression. *See Examples 2 and 3.*

21. $\sqrt[3]{-54}$
22. $\sqrt[4]{162}$
23. $\left(\sqrt[3]{-125}\right)^3$
24. $\sqrt[4]{562^4}$
25. $\sqrt[3]{16x^5}$
26. $\sqrt[5]{96x^5}$
27. $\sqrt{(4x^3)^2}$
28. $\sqrt[4]{(3x^2)^4}$
29. $\sqrt{\frac{z^2}{25}}$
30. $\sqrt[5]{\frac{64}{y^5}}$

Evaluating an Expression Involving Radicals In Exercises 31–36, evaluate the expression at $x = 2$, $y = 3$, and $z = 5$.

31. $\sqrt{2xy^4z^2}$
32. $\sqrt{3x^2yz^6}$
33. $\sqrt[4]{16x^{-4}y^8z^4}$
34. $\sqrt[5]{243x^5y^{-5}z^{15}}$
35. $\sqrt[3]{z} \cdot \sqrt[3]{z^2}$
36. $\sqrt[5]{y^7} \cdot \sqrt[5]{y^3}$

Rationalizing a Denominator In Exercises 37–44, rewrite the expression by rationalizing the denominator. Simplify your answer. *See Examples 4 and 5.*

37. $\frac{1}{\sqrt{5}}$
38. $\frac{5}{\sqrt{10}}$
39. $\frac{8}{\sqrt[3]{2}}$
40. $\frac{5}{\sqrt[3]{(5x)^2}}$
41. $\frac{2x}{5 - \sqrt{3}}$
42. $\frac{5x}{\sqrt{14} - 2}$
43. $\frac{3}{\sqrt{5} + \sqrt{6}}$
44. $\frac{5}{2\sqrt{10} - 5}$

Converting Between Forms In Exercises 45–54, fill in the missing form.

	Radical Form	*Rational Exponent Form*
45.	$\sqrt{9} = 3$	
46.	$\sqrt[3]{125} = 5$	
47.		$32^{1/5} = 2$
48.		$-(144^{1/2}) = -12$
49.	$\sqrt[3]{-216} = -6$	
50.	$\sqrt[5]{-243} = -3$	
51.	$\left(\sqrt[4]{81}\right)^3 = 27$	
52.	$\sqrt[4]{81^3} = 27$	
53.		$125^{2/3} = 25$
54.		$16^{5/4} = 32$

Simplifying with Rational Exponents In Exercises 55–70, simplify the expression. *See Example 6.*

55. $16^{1/2}$
56. $125^{1/3}$
57. $\left(\frac{16}{81}\right)^{-3/4}$
58. $\left(\frac{9}{4}\right)^{-1/2}$
59. $\left(-\frac{1}{64}\right)^{-1/3}$
60. $\left(-\frac{125}{27}\right)^{-1/3}$
61. $5^{1/2} \cdot 5^{3/2}$
62. $4^{1/3} \cdot 4^{5/3}$
63. $\frac{2^{3/2}}{2}$
64. $\frac{5^{1/2}}{5}$
65. $\frac{x^2}{x^{1/2} \cdot x^{3/2}}$
66. $\frac{x \cdot x^{1/2}}{x^{3/2}}$
67. $(x^6x^3)^{1/3}$
68. $(x^3x^{12})^{1/5}$
69. $(16x^8y^4)^{3/4}$
70. $(27x^6y^9)^{2/3}$

Reducing the Index of a Radical In Exercises 71–76, use rational exponents to reduce the index of the radical. *See Example 7.*

71. $\sqrt[4]{y^2}$
72. $\sqrt[9]{x^3}$
73. $\sqrt{\sqrt{32}}$
74. $\sqrt{\sqrt{8}}$
75. $\sqrt[8]{(x+1)^6}$
76. $\sqrt[6]{(x+2)^4}$

Simplifying by Combining Like Radicals In Exercises 77–82, simplify the expression. *See Example 8.*

77. $5\sqrt{x} - 3\sqrt{x}$
78. $3\sqrt{x+1} + 10\sqrt{x+1}$
79. $5\sqrt{50} + 3\sqrt{8}$
80. $2\sqrt{27} - \sqrt{75}$
81. $2\sqrt{4y} - 2\sqrt{9y}$
82. $2\sqrt{108x} + \sqrt{147x}$

Evaluating Radicals with a Calculator In Exercises 83–90, use a calculator to approximate the number. (Round to three decimal places.) *See Examples 9 and 10.*

83. $\sqrt[3]{45}$
84. $\sqrt{57}$
85. $6.7^{2/5}$
86. $18.4^{1.1}$
87. $(0.38)^{-0.6}$
88. $(5.25)^{-1/2}$
89. $\frac{3-\sqrt{5}}{2}$
90. $\frac{-4+\sqrt{12}}{4}$

91. Calculator Write the keystrokes used to evaluate

$$\frac{4-\sqrt{7}}{3}$$

in one step on your calculator.

92. Calculator Write the keystrokes you can use to evaluate

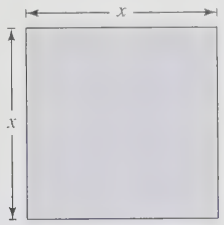
$$\sqrt[3]{(-5)^5}$$

in one step on your calculator.

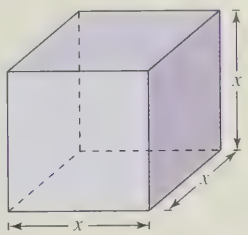
Comparing Radical Expressions In Exercises 93–96, complete the statement with <, =, or >.

93. $\sqrt{5} + \sqrt{3}$ ____ $\sqrt{5+3}$
94. $\sqrt{3} - \sqrt{2}$ ____ $\sqrt{3-2}$
95. $\sqrt{3} \cdot \sqrt[4]{3}$ ____ $\sqrt[8]{3}$
96. $\sqrt{\frac{3}{11}}$ ____ $\frac{\sqrt{3}}{\sqrt{11}}$

97. Geometry Find the dimensions of a square classroom that has 1100 square feet of floor space (see figure).



98. HOW DO YOU SEE IT? Package A is a cube with a volume of 500 cubic inches. Package B is a cube with a volume of 250 cubic inches. Is the length x of a side of package A greater than, less than, or equal to twice the length of a side of package B? Explain.



Declining Balances Depreciation In Exercises 99 and 100, find the annual depreciation rate r by using the declining balances formula

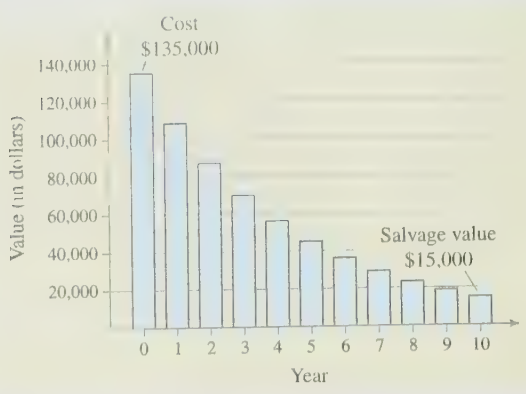
$$r = 1 - \left(\frac{S}{C}\right)^{1/n}$$

where n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars).

99. A mammography machine with an original cost of $25,000 is depreciated over an eight-year period, as shown in the bar graph.



100. An ultrasound machine with an original cost of \$135,000 is depreciated over a 10-year period, as shown in the bar graph.



101. Period of a Pendulum The period T (in seconds) of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L is the length (in feet) of the pendulum. Find the period of a pendulum whose length is 4 feet.

102. Period of a Pendulum Use the formula given in Exercise 101 to find the period of a pendulum whose length is 2.5 feet.

103. Escape Velocity The escape velocity (in meters per second) on the moon is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(7.36 \times 10^{22})}{1.74 \times 10^{6}}}.$$

A rocket, launched vertically from the moon, has a velocity of 2000 meters per second. Will the rocket escape the moon's gravitational field?

104. Erosion A stream of water moving at the rate of v feet per second can carry particles of size

$0.03\sqrt{v}$ inches.

Find the size of the largest particle that can be carried by a stream flowing at the rate of $\frac{1}{2}$ foot per second.

Estimating Speed A formula used to help determine the speed of a car from its skid marks is

$$S = \sqrt{30Df}$$

where S is the least possible speed (in miles per hour) of the car before its brakes are applied, D is the length of the car's skid marks (in feet), and f is the drag factor of the road surface. In Exercises 105 and 106, find the least possible speed of the car for the given conditions.

105. Skid marks: 60 feet, drag factor: 0.90

106. Skid marks: 100 feet, drag factor: 0.75

Wind Chill A wind chill temperature is a measure of how cold it *feels* outside. The wind chill temperature W (in degrees Fahrenheit) is given by

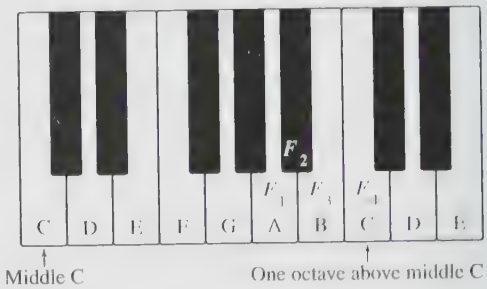
$$W = 35.75 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

where T is the actual temperature (in degrees Fahrenheit) and v is the wind speed (in miles per hour). In Exercises 107 and 108, find the wind chill temperature for the given conditions. *(Source: NOAA's National Weather Service)*

107. Actual temperature: 30°F, wind speed: 20 mph

108. Actual temperature: 10°F, wind speed: 10 mph

Notes on a Musical Scale In Exercises 109–111, find the frequency of the indicated note on a piano (see figure). The musical note A above middle C has a frequency of 440 vibrations per second. If we denote this frequency by F_1, then the frequency of the next higher note is given by $F_2 = F_1 \cdot 2^{1/12}$. Similarly, the frequency of the next note is given by $F_3 = F_2 \cdot 2^{1/12}$.



109. Find the frequency of the musical note B above middle C.

110. Find the frequency of the musical note C that is one octave above middle C.

111. Which note would you expect to have a higher frequency? Explain your reasoning.

(a) Musical note E one octave above the E above middle C

(b) Musical note D one octave above the D above middle C

112. Calculator Experiment Enter any positive real number in your calculator and repeatedly take the square root. What real number does the display appear to be approaching?

113. Calculator Experiment Square the real number $2/\sqrt{5}$ and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?

114. Think About It How can you show that $a^0 = 1$, $a \neq 0$? (*Hint:* Use the property of exponents $a^m/a^n = a^{m-n}$.)

115. Think About It Is it true that $\sqrt{4x^2} = 2x$ for every real number x? Explain.

116. Think About It Explain why $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$.

QUIZ YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, place the correct symbol ($<$, $>$, or $=$) between the two real numbers.

1. $-|-7| \quad |-7|$ **2.** $-(-3) \quad |-3|$

3. Determine which numbers in the set

$$\left\{-6, \sqrt{11}, \sqrt{1}, \frac{3}{4}, -\frac{1}{2}\right\}$$

are natural numbers, integers, rational numbers, and irrational numbers.

4. Use inequality notation to describe the subset of real numbers: x is positive or x is equal to zero.

5. Find the distance between the numbers -6 and 5.

6. Identify the terms of the algebraic expression $3x^2 - 7x + 2$.

In Exercises 7–10, perform the indicated operation(s). (Write fractional answers in simplest form.)

7. $-4 - 0 \cdot 7$ **8.** $\dfrac{31 - 5}{-2}$

9. $\dfrac{2}{3} \cdot \dfrac{5}{4} \cdot \dfrac{3}{7}$ **10.** $\dfrac{11}{15} \div \dfrac{3}{5}$

In Exercises 11 and 12, rewrite the expression with positive exponents and simplify.

11. $(-x)^3(2x^4)$

12. $\dfrac{5y^7}{15y^{-3}}$

13. A flea consumes 0.000014 liter of blood in one day. Write the number of liters in scientific notation.

14. You deposit \$5000 in an account with an annual interest rate of 2.2%, compounded quarterly. Find the balance in the account after 5 years.

In Exercises 15 and 16, evaluate the expression.

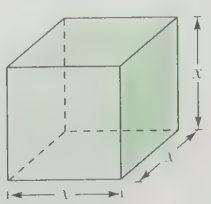
15. $\dfrac{-\sqrt[4]{81}}{3}$

16. $\left(\sqrt[3]{-64}\right)^3$

In Exercises 17–19, simplify the expression.

17. $3^{1/2} \cdot 3^{3/2}$ **18.** $\sqrt[3]{81} - 4\sqrt[3]{3}$ **19.** $\sqrt[10]{12^5}$

20. Find the dimensions of a cube that has a volume of 10,648 cubic centimeters.



0.5 Polynomials and Special Products

- Write a polynomial in standard form and perform operations with polynomials.
- Use special products to multiply polynomials.
- Use polynomials to solve application problems.



In Exercise 64 on page 47, you will find a polynomial that represents the total number of square feet in a floor plan.

Polynomials and Operations with Polynomials

One of the simplest and most common types of algebraic expressions is a **polynomial.** Here are some examples.

$$2x + 5, \quad 3x^4 - 7x^2 + 2x + 4, \quad 5x^2y^2 - xy + 3$$

The first two are *polynomials in x* and the third is a *polynomial in x and y*. The terms of a polynomial in x have the form ax^k, where a is the **coefficient** and k is the **degree** of the term. Because a polynomial is defined as an algebraic sum, the coefficients take on the signs between the terms. For instance, the polynomial

$$2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$$

has coefficients 2, -5, 0, and 1.

Definition of a Polynomial in *x*

Let $a_n, \ldots, a_2, a_1, a_0$ be real numbers and let n be a *nonnegative integer.* A **polynomial in *x*** is an expression of the form

$$a_nx^n + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree** n, and the number a_n is the **leading coefficient.** The number a_0 is the **constant term.** The constant term is considered to have a degree of zero.

Note in the definition of a polynomial in x that the polynomial is written in descending powers of x. This is called the **standard form** of a polynomial.

Example 1 Rewriting a Polynomial in Standard Form

Polynomial	*Standard Form*	*Degree*
a. $4x^2 - 5x^3 - 2 + 3x$	$-5x^3 + 4x^2 + 3x - 2$	3
b. $4 - 9x^2$	$-9x^2 + 4$	2
c. 8	$8 \ (8 = 8x^0)$	0

Checkpoint 1

Rewrite the polynomial

$$7 - 9x^2 + 3x$$

in standard form and state its degree. ■

Another way to describe a polynomial is that it is the sum of *monomials.* A **monomial** is an expression of the form ax^k, where a is a real number and k is a nonnegative integer. The sum of two monomials is called a **binomial.** The sum of three monomials is called a **trinomial.**

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A polynomial that has all zero coefficients is called the **zero polynomial,** denoted by 0. This particular polynomial is not considered to have a degree. For a polynomial in more than one variable, the *degree of a term* is the sum of the powers of the variables in the term. The *degree of the polynomial* is the highest degree of all its terms. For instance, the polynomial $5x^3y - x^2y^2 + 2xy - 5$ has two terms of degree 4, one term of degree 2, and one term of degree 0. The degree of the polynomial is 4.

Example 2 Identifying a Polynomial and Its Degree

a. $-2x^3 + x^2 + 3x - 2$ is a polynomial of degree 3.

b. $\sqrt{x^2 - 3x}$ is not a polynomial because the radical sign indicates a noninteger power of x.

c. $x^2 + 5x^{-1}$ is not a polynomial because of the negative exponent.

✓Checkpoint 2

Determine whether the expression

$$\frac{2x + 5}{x}$$

is a polynomial. If it is, state its degree.

You can **add** and **subtract** polynomials in much the same way that you add and subtract real numbers—you simply add or subtract the *like terms* (terms having the same variables raised to the same powers) by adding or subtracting their coefficients. For instance, $-3x^2$ and $5x^2$ are like terms and their sum is given by

$$-3x^2 + 5x^2 = (-3 + 5)x^2 = 2x^2.$$

ALGEBRA TUTOR

For help in finding the sums and differences in Example 3, see the *Chapter 0 Algebra Tutor* on page 63.

Example 3 Sums and Differences of Polynomials

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

$= (5x^3 + x^3) + (2x^2 - 7x^2) - x + (8 - 3)$ Group like terms.

$= 6x^3 - 5x^2 - x + 5$ Combine like terms.

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$ Distributive Property

$= (7x^4 - 3x^4) + (4x^2 - x^2) + (-4x - 3x) + 2$ Group like terms.

$= 4x^4 + 3x^2 - 7x + 2$ Combine like terms.

✓Checkpoint 3

Find the sum $(2x^2 + x + 3) + (4x + 1)$ and write the resulting polynomial in standard form.

A common mistake is to fail to change the sign of *each* term inside parentheses preceded by a minus sign. Here is an example.

$-(3x^4 - 4x^2 + 3x) = -3x^4 + 4x^2 - 3x$ Correct

~~$-(3x^4 - 4x^2 + 3x) = -3x^4 - 4x^2 + 3x$~~ Common mistake

To find the **product** of two polynomials, you can use the left and right Distributive Properties. For example, treat $(5x + 7)$ as a single quantity and multiply $(3x - 2)$ by $(5x + 7)$, as shown.

$$\begin{aligned}(3x - 2)(5x + 7) &= 3x(5x + 7) - 2(5x + 7)\\ &= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7)\\ &= 15x^2 + 21x - 10x - 14\end{aligned}$$

Product of First terms	Product of Outer terms	Product of Inner terms	Product of Last terms

$$= 15x^2 + 11x - 14$$

You can use the four special products shown in the boxes above to write the product of two binomials in the FOIL form in just one step. This is called the **FOIL Method.**

Example 4 Using the FOIL Method

Use the FOIL Method to find the product of $(2x - 4)$ and $(x + 5)$.

SOLUTION

$$\begin{aligned}(2x - 4)(x + 5) &= \overset{\text{F}}{2x^2} + \overset{\text{O}}{10x} - \overset{\text{I}}{4x} - \overset{\text{L}}{20}\\ &= 2x^2 + 6x - 20\end{aligned}$$

✓Checkpoint 4

Find each product.

a. $(x + 5)(x + 9)$

b. $(3x + 1)(x - 1)$

When multiplying two polynomials, be sure to use the Distributive Property to multiply *each* term of one polynomial by *each* term of the other. The vertical format is a convenient way to multiply two polynomials.

Example 5 Using a Vertical Format to Multiply Polynomials

Multiply $(x^2 - 2x + 2)$ by $(x^2 + 2x + 2)$.

SOLUTION

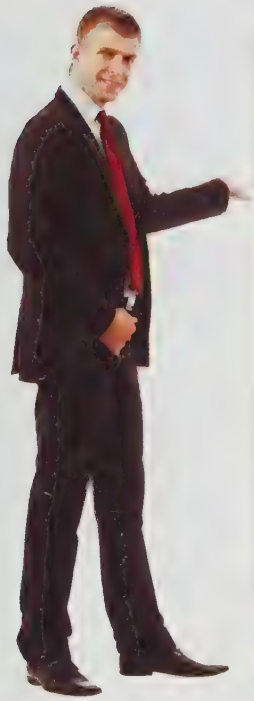
$$\begin{array}{rl}
x^2 - 2x + 2 & \text{Standard form}\\
x^2 + 2x + 2 & \text{Standard form}\\
\hline
x^4 - 2x^3 + 2x^2 & \Leftarrow\ x^2(x^2 - 2x + 2)\\
2x^3 - 4x^2 + 4x & \Leftarrow\ 2x(x^2 - 2x + 2)\\
2x^2 - 4x + 4 & \Leftarrow\ 2(x^2 - 2x + 2)\\
\hline
x^4 + 0x^3 + 0x^2 + 0x + 4 = x^4 + 4 & \text{Combine like terms.}
\end{array}$$

So, $(x^2 - 2x + 2)(x^2 + 2x + 2) = x^4 + 4$.

✓Checkpoint 5

Multiply $(x^2 + x + 4)$ by $(x^2 - 3x + 1)$.

Special Products



Special Products

Let u and v be real numbers, variables, or algebraic expressions.

Special Product	*Example*
Sum and Difference of Two Terms	
$(u + v)(u - v) = u^2 - v^2$	$(x + 4)(x - 4) = x^2 - 16$
Square of a Binomial	
$(u + v)^2 = u^2 + 2uv + v^2$	$(x + 3)^2 = x^2 + 6x + 9$
$(u - v)^2 = u^2 - 2uv + v^2$	$(3x - 2)^2 = 9x^2 - 12x + 4$
Cube of a Binomial	
$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$	$(x + 2)^3 = x^3 + 6x^2 + 12x + 8$
$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$

Example 6 Sum and Difference of Two Terms

$$\begin{aligned}(5x + 9)(5x - 9) &= (5x)^2 - 9^2 \\ &= 25x^2 - 81\end{aligned}$$

Checkpoint 6

Find the product $(3 - x)(3 + x)$.

Example 7 Square and Cube of a Binomial

a. $(6x - 5)^2 = (6x)^2 - 2(6x)(5) + 5^2$

$= 36x^2 - 60x + 25$

b. $(3x + 2)^3 = (3x)^3 + 3(3x)^2(2) + 3(3x)(2)^2 + 2^3$

$= 27x^3 + 54x^2 + 36x + 8$

Checkpoint 7

Find each product.

a. $(x + 4)^2$ **b.** $(x - 3)^3$

Example 8 The Product of Two Trinomials

$$\begin{aligned}(x + y - 2)(x + y + 2) &= [(x + y) - 2][(x + y) + 2] \\ &= (x + y)^2 - 2^2 \\ &= x^2 + 2xy + y^2 - 4\end{aligned}$$

Checkpoint 8

Find the product $(x + 5 - y)(x + 5 + y)$.

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Applications

Example 9 A Savings Plan

At the same time each year for five consecutive years, you deposit money in an account that earns 7% interest, compounded annually. The deposit amounts are \$1500, \$1800, \$2400, \$2600, and \$3000. After the last deposit, is there enough money to pay a \$12,000 college tuition bill?

SOLUTION Using the formula for compound interest, for *each* deposit you have

$$\begin{aligned}\text{Balance} &= P\left(1 + \frac{r}{n}\right)^{nt} \\ &= P(1 + 0.07)^t \\ &= P(1.07)^t.\end{aligned}$$

For the first deposit, $P = 1500$ and $t = 4$. For the second deposit, $P = 1800$ and $t = 3$, and so on. The balances for the five deposits are as follows.

Date	*Deposit*	*Time in Account*	*Balance in Account*
First Year	\$1500	4 years	$1500(1.07)^4$
Second Year	\$1800	3 years	$1800(1.07)^3$
Third Year	\$2400	2 years	$2400(1.07)^2$
Fourth Year	\$2600	1 year	$2600(1.07)$
Fifth Year	\$3000	0 years	3000

By adding these five balances, you can find the total balance in the account to be

$$1500(1.07)^4 + 1800(1.07)^3 + 2400(1.07)^2 + 2600(1.07) + 3000.$$

Note that this expression is in polynomial form. By evaluating the expression, you can find the balance to be \$12,701.03, as shown in Figure 0.10.

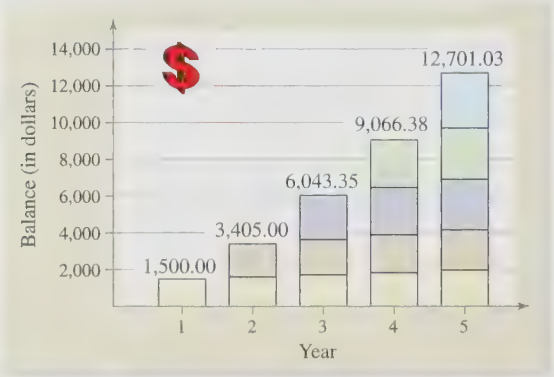


FIGURE 0.10

After the fifth deposit, there is enough money in the account to pay the college tuition bill.

Checkpoint 9

In Example 9, suppose the account earns 5% interest. What is the balance in the account after the last deposit?

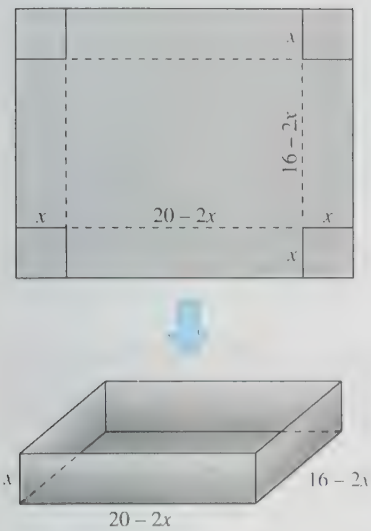


FIGURE 0.11

Example 10 Geometry: Volume of a Box

An open box is made by cutting squares from the corners of a piece of metal that measures 16 inches by 20 inches and turning up the sides, as shown in Figure 0.11. The sides of the cut-out squares are all x inches long, so the box is x inches tall. Write an expression for the volume of the box. Then find the volume when $x = 1$, $x = 2$, and $x = 3$ inches.

SOLUTION

Verbal Model: Volume = Length · Width · Height

Labels:
Height = x (inches)
Width = $16 - 2x$ (inches)
Length = $20 - 2x$ (inches)

Equation:
$$\begin{aligned} \text{Volume} &= (20 - 2x)(16 - 2x)(x) \\ &= (320 - 72x + 4x^2)(x) \\ &= 320x - 72x^2 + 4x^3 \end{aligned}$$

When $x = 1$ inch, the volume of the box is

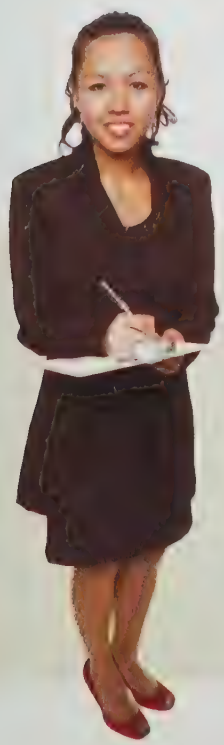
$$\begin{aligned} \text{Volume} &= 320(1) - 72(1)^2 + 4(1)^3 \\ &= 252 \text{ cubic inches.} \end{aligned}$$

When $x = 2$ inches, the volume of the box is

$$\begin{aligned} \text{Volume} &= 320(2) - 72(2)^2 + 4(2)^3 \\ &= 384 \text{ cubic inches.} \end{aligned}$$

When $x = 3$ inches, the volume of the box is

$$\begin{aligned} \text{Volume} &= 320(3) - 72(3)^2 + 4(3)^3 \\ &= 420 \text{ cubic inches.} \end{aligned}$$



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✓ Checkpoint 10

In Example 10, suppose the original piece of metal is 10 inches by 12 inches. Write an expression for the volume of the box. Then find the volume when $x = 2$ and $x = 3$ inches.

SUMMARIZE (Section 0.5)

1. State the definition of a polynomial in x and explain what is meant by the standard form of a polynomial *(page 40)*. For an example of rewriting polynomials in standard form, see Example 1.
2. Describe how to add and subtract polynomials *(page 41)*. For examples of adding and subtracting polynomials, see Example 3.
3. Describe the FOIL Method *(page 42)*. For an example that uses the FOIL Method, see Example 4.
4. Explain how to find the square and cube of a binomial *(page 43)*. For an example of squaring and cubing a binomial, see Example 7.

SKILLS WARM UP 0.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.3.

In Exercises 1–10, perform the indicated operation(s).

1. $(7x^2)(6x)$
2. $(5x^3)(-9x^2)$
3. $(10z^3)(-2z^{-1})$
4. $(-3x^4)(5x^{-5})$
5. $(-3x^2)^3$
6. $-3(x^2)^3$
7. $\dfrac{27z^5}{12z^2}$
8. $\dfrac{32x^4}{20x}$
9. $\left(\dfrac{2x}{3}\right)^{-2}$
10. $\left(\dfrac{x^{-1}}{y^2}\right)^{-1}$

Exercises 0.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Rewriting a Polynomial in Standard Form In Exercises 1–6, rewrite the polynomial in standard form. Then identify the degree and leading coefficient of the polynomial. *See Example 1.*

1. $3x + 4x^2 + 2$
2. $x^2 - 4 - 3x^4$
3. $-8 + x^7$
4. $23 - x^3$
5. $1 - x + 6x^4 - 2x^5$
6. $7 + 8x$

Identifying a Polynomial and Its Degree In Exercises 7–12, determine whether the algebraic expression is a polynomial. If it is, write the polynomial in standard form and state its degree. *See Example 2.*

7. $2x - 3x^3 + 8$
8. $2x^3 + x - 3x^{-1}$
9. $\dfrac{3x + 4}{x}$
10. $\dfrac{2x^2 + 5x - 3}{3}$
11. $w^2 - w^4 + 2w^3$
12. $\sqrt{y^2 - y^4}$

Evaluating a Polynomial In Exercises 13–16, evaluate the polynomial for each value of x.

13. $4x + 5$ (a) $x = -2$ (b) $x = -1$ (c) $x = 0$ (d) $x = 3$
14. $-x^2 + 3$ (a) $x = -3$ (b) $x = -2$ (c) $x = 0$ (d) $x = 1$
15. $-2x^2 + 3x + 4$ (a) $x = -2$ (b) $x = -1$ (c) $x = 0$ (d) $x = 1$
16. $x^3 - 4x^2 + x$ (a) $x = -1$ (b) $x = 0$ (c) $x = 1$ (d) $x = 2$

A Sum or Difference of Polynomials In Exercises 17–22, find the sum or difference and write the resulting polynomial in standard form. *See Example 3.*

17. $(6x + 5) - (8x + 15)$
18. $(3x^2 + 1) - (2x^2 - 2x + 3)$
19. $-(x^3 + 5) + (3x^3 - 4x)$
20. $-(5x^2 - 1) + (-3x^2 + 5)$
21. $(15x^2 - 6) - (-8x^3 - 14x^2 - 17)$
22. $(15x^4 - 18x - 19) - (13x^4 - 5x + 15)$

Multiplying Polynomials In Exercises 23–52, find the product. *See Examples 4, 5, 6, 7, and 8.*

23. $3x(x^2 - 2x + 1)$
24. $z^2(2z^2 + 3z + 1)$
25. $-3x(-x)(3x - 7)$
26. $(2 - x^2)(-2x)(4x)$
27. $(x + 3)(x + 4)$
28. $(x - 5)(x + 10)$
29. $(3x - 5)(2x + 1)$
30. $(7x - 2)(4x - 3)$
31. $(x^2 + 2x)(x^2 + x - 3)$
32. $(3x - 4)(x^2 - 2x + 1)$
33. $(-x^2 + x - 5)(3x^2 + 4x + 1)$
34. $(x^2 + 3x + 2)(2x^2 - x + 4)$
35. $(x + 5)(x - 5)$
36. $(3x + 2)(3x - 2)$
37. $(x + 6)^2$
38. $(3x - 2)^2$
39. $(2x - 5y)^2$
40. $(5 - 8x)^2$
41. $[(x - 3) + y]^2$
42. $[(x + 1) - y]^2$
43. $(x + 1)^3$
44. $(x - 2)^3$
45. $(2x - y)^3$
46. $(3x + 2y)^3$
47. $(3y^2 - 1)(3y^2 + 1)$
48. $(3x^2 - 4y^2)(3x^2 + 4y^2)$
49. $(m - 3 + n)(m - 3 - n)$
50. $(x + y + 1)(x + y - 1)$
51. $(x^2 - x + 1)(x^2 + x + 1)$
52. $(x^2 + 3x - 2)(x^2 - 3x - 2)$

Operations with Polynomials In Exercises 53–56, perform the indicated operation and write the result in standard form.

53. Find the product of $(3x - 7)$ and $(8x + 1)$.
54. Subtract $(3x^2 - 2)$ from $(5x^2 - 7x + 4)$.
55. Find the sum of $(4x^3 + 6)$ and $(6x^2 - 8x - 11)$.
56. Multiply $(x^2 - 2x - 4)$ by $(x + 3)$.

57. **Error Analysis** Describe and correct the error.

$(x - 3)^2 = x^2 + 9$

58. Error Analysis Describe and correct the error.

$x^2 - 6(x + 5) = x^2 - 6x + 30$

59. Compound Interest After 3 years, an investment of \$1000 earning an interest rate r compounded annually will be worth $1000(1 + r)^3$ dollars. Write this expression as a polynomial in standard form.

60. Compound Interest After 2 years, an investment of \$800 earning an interest rate r compounded annually will be worth $800(1 + r)^2$ dollars. Write this expression as a polynomial in standard form.

61. Savings Plan At the same time each year for five consecutive years, you deposit money in an account that earns annually compounded interest. The deposits are \$1200, \$1700, \$1500, \$2200, and \$3000. Is there enough money in the account after the last deposit to pay a \$10,000 college tuition bill when the rate of annually compounded interest is 4%? 3%? 2%?

62. Savings Plan You have an investment that pays an annual dividend. Each January for six consecutive years, you reinvest this dividend in an account that earns 4% interest, compounded annually. The dividends are shown in the table. Is there enough money in the account after the sixth deposit for a \$7000 down payment on a car?

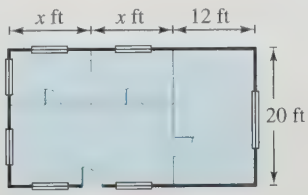
Year	Dividend
1	\$920
2	\$1000
3	\$780
4	\$1310
5	\$1020
6	\$1200

63. Federal Student Aid The total amount (in millions of dollars) of federal student aid disbursed each year from 2000 through 2007 can be approximated by

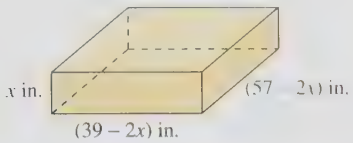
$-164.14x^2 + 6777.8x + 43{,}161$

where x represents the year, with $x = 0$ corresponding to 2000. Evaluate the polynomial at $x = 6$ and $x = 7$. Explain the meaning of your results in the context of the situation. *(Source: U.S. Department of Education)*

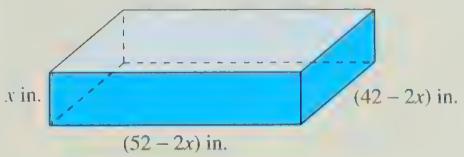
64. Geometry Find a polynomial that represents the total number of square feet in the floor plan.



65. Geometry A box has a length of $(57 - 2x)$ inches, a width of $(39 - 2x)$ inches, and a height of x inches. Find the volume when $x = 4$, $x = 6$, and $x = 10$ inches. Which x-value gives the greatest volume?



66. HOW DO YOU SEE IT? An open box has a length of $(52 - 2x)$ inches, a width of $(42 - 2x)$ inches, and a height of x inches, as shown.

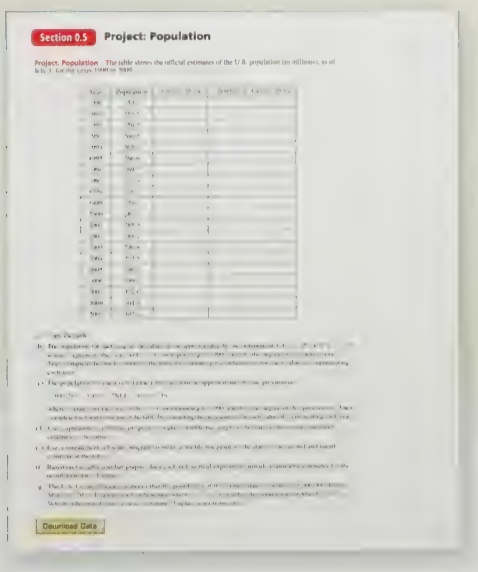


(a) Describe a way that the box could have been made from a rectangular piece of cardboard. Give the original dimensions of the cardboard.

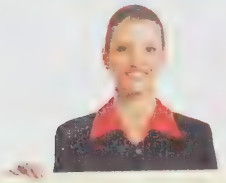
(b) What degree is the polynomial that represents the volume of the box? Explain.

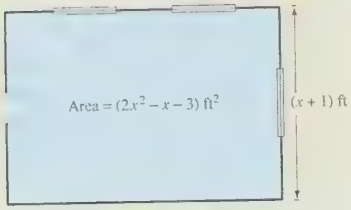
(c) Describe a procedure for finding the value of x (to the nearest tenth of an inch) that yields the maximum possible volume of the box.

67. Project: Population For a project involving the population of the United States from 1990 to 2009, visit this text's website at *www.cengagebrain.com.* *(Source: U.S. Census Bureau)*



0.6 Factoring







In Exercise 85 on page 54, you will factor a polynomial expression for the floor space of a room to find an expression for the length of the room.

- Factor a polynomial by factoring out common factors.
- Factor a polynomial in a special form.
- Factor a trinomial as the product of two binomials.
- Factor a polynomial by grouping.

Common Factors

The process of writing a polynomial as a product is called **factoring.** It is an important tool for solving equations and reducing fractional expressions.

A polynomial that cannot be factored using integer coefficients is called **prime** or **irreducible over the integers.** For instance, the polynomial

$$x^2 - 3$$

is irreducible over the integers. [Over the *real numbers*, it can be factored as

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}).]$$

A polynomial is **completely factored** when each of its factors is prime. For instance,

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4) \qquad \text{Completely factored}$$

is completely factored, but

$$x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4) \qquad \text{Not completely factored}$$

is not completely factored. Its complete factorization is

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. To factor such a polynomial, you can use the Distributive Property in the *reverse* direction.

$$ab + ac = a(b + c) \qquad a \text{ is a common factor.}$$

Example 1 Factoring Out Common Factors

Factor each expression.

a. $6x^3 - 4x$

b. $(x - 2)(2x) + (x - 2)(3)$

SOLUTION

a. Each term of this polynomial has $2x$ as a common factor.

$$6x^3 - 4x = 2x(3x^2) - 2x(2)$$
$$= 2x(3x^2 - 2)$$

b. The binomial factor $(x - 2)$ is common to both terms.

$$(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3)$$

✓Checkpoint 1

Factor the expression

$$(x + 1)^2 + 2x(x + 1).$$

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Factoring Special Polynomial Forms

Factoring Special Polynomial Forms

Factored Form	*Example*
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x - 1)(9x^2 + 3x + 1)$

STUDY TIP

In Example 2, note that the first step in factoring a polynomial is to check for common factors. Once the common factor is factored out, it is often possible to recognize patterns that were not obvious at first glance.

Example 2 Factoring Out a Common Factor First

Factor the expression $3 - 12x^2$.

SOLUTION

$$3 - 12x^2 = 3(1 - 4x^2) \qquad \text{3 is a common factor.}$$
$$= 3[1^2 - (2x)^2] \qquad \text{Difference of two squares}$$
$$= 3(1 + 2x)(1 - 2x) \qquad \text{Completely factored}$$

Checkpoint 2

Factor the expression $x^3 - x$.

Example 3 Factoring the Difference of Two Squares

a. $(x + 2)^2 - y^2 = [(x + 2) + y][(x + 2) - y]$
$= (x + 2 + y)(x + 2 - y)$
$= (x + y + 2)(x - y + 2)$

b. The polynomial $16x^4 - 81$ can be factored by applying the difference of two squares formula twice.

$$16x^4 - 81 = (4x^2)^2 - 9^2$$
$$= (4x^2 + 9)(4x^2 - 9) \qquad \text{First application}$$
$$= (4x^2 + 9)[(2x)^2 - 3^2]$$
$$= (4x^2 + 9)(2x + 3)(2x - 3) \qquad \text{Second application}$$

Checkpoint 3

Factor the expression $100 - 4y^2$.

A perfect square trinomial is the square of a binomial, and it has the following form. Note that the first and last terms of a perfect square trinomial are squares and the middle term is twice the product of u and v.

$$u^2 + 2uv + v^2 = (u + v)^2 \quad \text{or} \quad u^2 - 2uv + v^2 = (u - v)^2$$

Same sign Same sign

Example 4 **Factoring Perfect Square Trinomials**

a. $16x^2 + 8x + 1 = (4x)^2 + 2(4x)(1) + 1^2$
$= (4x + 1)^2$

b. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2$
$= (x - 5)^2$

Checkpoint 4

Factor each expression.

a. $4x^2 - 4x + 1$

b. $x^2 - 12x + 36$

The next two formulas show that sums and differences of cubes factor easily. Pay special attention to the signs of the terms.

Like signs Like signs

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2) \qquad u^3 - v^3 = (u - v)(u^2 + uv + v^2)$$

Unlike signs Unlike signs

Example 5 **Factoring the Sum and Difference of Cubes**

Factor each expression.

a. $x^3 - 27$

b. $3x^3 + 192$

SOLUTION

a. $x^3 - 27 = x^3 - 3^3$ Rewrite 27 as 3^3.
$= (x - 3)(x^2 + 3x + 9)$ Factor.

b. $3x^3 + 192 = 3(x^3 + 64)$ 3 is a common factor.
$= 3(x^3 + 4^3)$ Rewrite 64 as 4^3.
$= 3(x + 4)(x^2 - 4x + 16)$ Factor.

Checkpoint 5

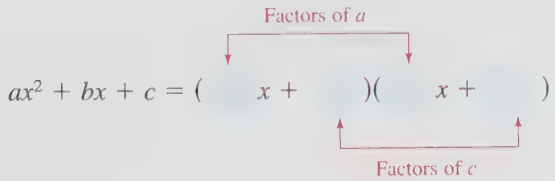
Factor each expression.

a. $y^3 + 1$

b. $5z^3 - 40$

Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.



The goal is to find a combination of factors of a and c such that the outer and inner products add up to the middle term bx. For instance, for the trinomial $6x^2 + 17x + 5$ you can write

$$(2x + 5)(3x + 1) = 6x^2 + \underset{\text{O}}{2x} + \underset{\text{I}}{15x} + 5$$
$$= 6x^2 + 17x + 5.$$

Note that the outer (O) and inner (I) products add up to $17x$.

Example 6 Factoring a Trinomial: Leading Coefficient Is 1

Factor the trinomial $x^2 - 7x + 12$.

SOLUTION For this trinomial, you have $a = 1$, $b = -7$, and $c = 12$. Because b is negative and c is positive, both factors of 12 must be negative. That is, $12 = (-2)(-6)$, $12 = (-1)(-12)$, or $12 = (-3)(-4)$. So, the possible factorizations of $x^2 - 7x + 12$ are

$$(x - 2)(x - 6), \quad (x - 1)(x - 12), \quad \text{and} \quad (x - 3)(x - 4).$$

Testing the middle term, you find that $x^2 - 7x + 12 = (x - 3)(x - 4)$.

✓Checkpoint 6

Factor the trinomial $x^2 + x - 6$.

Example 7 Factoring a Trinomial: Leading Coefficient Is Not 1

Factor the trinomial $2x^2 + x - 15$.

SOLUTION For this trinomial, you have $a = 2$ and $c = -15$, which means that the factors of -15 must have unlike signs. The eight possible factorizations are:

$$(2x - 1)(x + 15) \quad (2x + 1)(x - 15) \quad (2x - 3)(x + 5) \quad (2x + 3)(x - 5)$$
$$(2x - 5)(x + 3) \quad (2x + 5)(x - 3) \quad (2x - 15)(x + 1) \quad (2x + 15)(x - 1)$$

Testing the middle term, you find that $2x^2 + x - 15 = (2x - 5)(x + 3)$.

✓Checkpoint 7

Factor the trinomial $2x^2 - 5x + 3$.

Factoring by Grouping

Sometimes polynomials with more than three terms can be **factored by grouping.**

Example 8 Factoring by Grouping

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor groups.} \\ &= (x - 2)(x^2 - 3) && \text{Distributive Property} \end{aligned}$$

STUDY TIP

In some cases, several different groupings will work. In Example 8, a different grouping could have been used.

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 & \\ = (x^3 - 3x) - (2x^2 - 6) & \\ = x(x^2 - 3) - 2(x^2 - 3) & \\ = (x^2 - 3)(x - 2) & \end{aligned}$$

Checkpoint 8

Factor the polynomial $x^3 + x^2 + 5x + 5$. ■

Factoring by grouping can save you some of the trial and error involved in factoring a trinomial. To factor a trinomial of the form $ax^2 + bx + c$ by grouping, rewrite the middle term using the sum of two factors of the product ac that add up to b. This technique is illustrated in Example 9.

Example 9 Factoring a Trinomial by Grouping

Use factoring by grouping to factor $2x^2 + 5x - 3$.

SOLUTION In the trinomial $2x^2 + 5x - 3$, $a = 2$ and $c = -3$, so the product ac is -6. Notice that -6 factors as $(6)(-1)$, and $6 - 1 = 5 = b$. So, you can rewrite the middle term as $5x = 6x - x$.

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.} \\ &= (2x^2 + 6x) - (x + 3) && \text{Group terms.} \\ &= 2x(x + 3) - (x + 3) && \text{Factor groups.} \\ &= (x + 3)(2x - 1) && \text{Distributive Property} \end{aligned}$$

The trinomial factors as $2x^2 + 5x - 3 = (x + 3)(2x - 1)$.

Checkpoint 9

Use factoring by grouping to factor $2x^2 + 5x - 12$. ■



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SUMMARIZE (Section 0.6)

1. Describe what it means to completely factor a polynomial *(page 48)*. For an example of factoring out common factors, see Example 1.
2. Make a list of the special polynomial forms of factoring *(page 49)*. For examples of factoring these special forms, see Examples 3, 4, and 5.
3. Describe how to factor a trinomial of the form $ax^2 + bx + c$ *(page 51)*. For examples of factoring trinomials of this form, see Examples 6 and 7.
4. Explain how to factor a polynomial by grouping *(page 52)*. For examples of factoring by grouping, see Examples 8 and 9.

SKILLS WARM UP 0.6 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.5.

In Exercises 1–10, find the product.

1. $3x(5x - 2)$
2. $-2y(y + 1)$
3. $(2x + 3)^2$
4. $(3x - 8)^2$
5. $(2x - 3)(x + 8)$
6. $(4 - 5z)(1 + z)$
7. $(2y + 1)(2y - 1)$
8. $(x + a)(x - a)$
9. $(x + 4)^3$
10. $(2x - 3)^3$

Exercises 0.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Factoring Out Common Factors In Exercises 1–6, factor out the common factor. *See Example 1.*

1. $3x + 6$
2. $6y - 30$
3. $3x^3 - 6x$
4. $4x^3 - 6x^2 + 12x$
5. $3x(x - 5) + 8(x - 5)$
6. $4(5x - 4)^2 + x(5x - 4)$

Factoring the Difference of Two Squares In Exercises 7–14, factor the difference of two squares. *See Example 3.*

7. $x^2 - 36$
8. $x^2 - \frac{1}{9}$
9. $16x^2 - 9y^2$
10. $x^2 - 49y^2$
11. $(x - 1)^2 - 4$
12. $25 - (z + 5)^2$
13. $81x^4 - 1$
14. $x^4 - 256$

Factoring Out a Common Factor First In Exercises 15–20, factor the expression. *See Example 2.*

15. $3x^2 - 3$
16. $8x^2 - 18$
17. $8x^2 - 50y^2$
18. $48 - 27x^2$
19. $25(x - 1) - x^2(x - 1)$
20. $4x^2(x + 2) - 36(x + 2)$

Factoring a Perfect Square Trinomial In Exercises 21–26, factor the perfect square trinomial. *See Example 4.*

21. $x^2 - 4x + 4$
22. $x^2 + 10x + 25$
23. $4y^2 + 12y + 9$
24. $9x^2 - 12x + 4$
25. $y^2 - \frac{2}{3}y + \frac{1}{9}$
26. $z^2 + z + \frac{1}{4}$

Factoring the Sum or Difference of Cubes In Exercises 27–32, factor the sum or difference of cubes. *See Example 5.*

27. $x^3 - 8$
28. $x^3 - 27$
29. $y^3 + 125$
30. $y^3 + 1000$
31. $x^3 - \frac{8}{27}$
32. $x^3 + \frac{8}{125}$

Factoring a Trinomial In Exercises 33–46, factor the trinomial. *See Examples 6 and 7.*

33. $x^2 + x - 2$
34. $x^2 + 6x + 8$
35. $w^2 - 5w + 6$
36. $z^2 - z - 6$
37. $y^2 + y - 20$
38. $z^2 - 4z - 21$
39. $x^2 - 30x + 200$
40. $x^2 - 5x - 150$
41. $3x^2 - 5x + 2$
42. $2x^2 - x - 1$
43. $9x^2 - 3x - 2$
44. $12y^2 + 7y + 1$
45. $6x^2 + 37x + 6$
46. $5u^2 + 13u - 6$

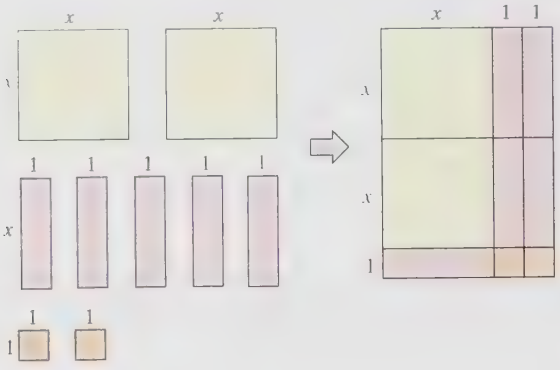
Factoring by Grouping In Exercises 47–56, factor by grouping. *See Examples 8 and 9.*

47. $x^3 - x^2 + 2x - 2$
48. $x^3 + 5x^2 - 5x - 25$
49. $2x^3 - x^2 - 6x + 3$
50. $5x^3 - 10x^2 + 3x - 6$
51. $6 + 2y - 3y^3 - y^4$
52. $z^5 + 2z^3 + z^2 + 2$
53. $2x^2 + 3x - 2$
54. $3x^2 + 5x - 2$
55. $6x^2 - 7x - 10$
56. $8x^2 + 11x + 3$

Factoring Completely In Exercises 57–80, completely factor the expression. *See Examples 1–9.*

57. $4x^2 - 8x$
58. $12x^3 - 48x$
59. $y^3 - \frac{1}{9}y$
60. $x^3 - \frac{1}{4}x$
61. $3x^2 - 48$
62. $7y^2 - 63$
63. $x^2 - 2x + 1$
64. $9x^2 - 6x + 1$
65. $1 - 4x + 4x^2$
66. $16 + 6x - x^2$
67. $2y^3 - 7y^2 - 15y$
68. $3x^4 + x^3 - 10x^2$
69. $5x^2 - 4x + 3$
70. $7x^2 + 5x - 4$
71. $3x^3 + x^2 + 15x + 5$
72. $5 - x + 5x^2 - x^3$
73. $x^4 - 4x^3 + x^2 - 4x$
74. $3u - 2u^2 + 6 - u^3$
75. $25 - (x + 5)^2$
76. $(t - 1)^2 - 49$
77. $(x^2 + 1)^2 - 4x^2$
78. $(x^2 + 8)^2 - 36x^2$
79. $2t^3 - 16$
80. $3x^3 + 81$

Geometric Modeling In Exercises 81–84, make a "geometric factoring model" to represent the given factorization. For instance, a factoring model for $2x^2 + 5x + 2 = (2x + 1)(x + 2)$ is shown below.



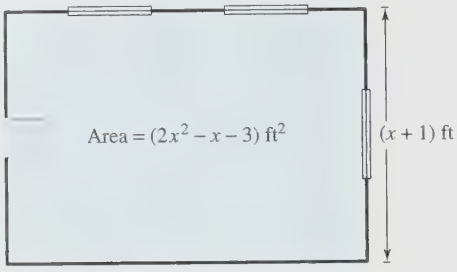
81. $x^2 + 3x + 2 = (x + 2)(x + 1)$

82. $x^2 + 4x + 3 = (x + 3)(x + 1)$

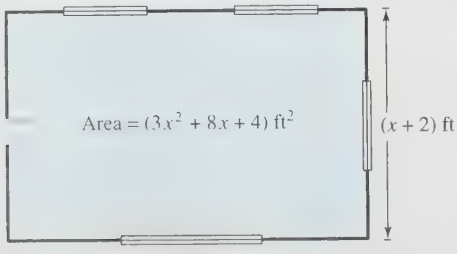
83. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

84. $3x^2 + 7x + 2 = (3x + 1)(x + 2)$

85. Geometry The room shown in the figure has a floor space of $(2x^2 - x - 3)$ square feet. The width of the room is $(x + 1)$ feet. What is the length?



86. Geometry The room shown in the figure has a floor space of $(3x^2 + 8x + 4)$ square feet. The width of the room is $(x + 2)$ feet. What is the length?



87. Choosing a Method Factor each trinomial. State whether you used factoring by grouping or factoring by trial and error.

(a) $x^2 + 11x + 24$

(b) $3x^2 + 7x - 20$

88. Think About It Find all integers b for which $x^2 + bx + 24$ can be factored. Describe how you found these values of b.

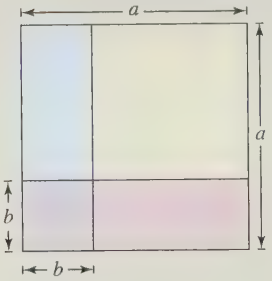
89. Think About It Find all integers $c > 0$ for which $x^2 + 8x + c$ can be factored. Describe how you found these values of c.

90. Error Analysis Describe and correct the error.

$x^3 - 8 = (x - 2)(x^2 + 4)$.

91. Think About It Describe two different ways to factor $2x^2 - 7x - 15$.

92. HOW DO YOU SEE IT? The figure shows a large square with an area of a^2 that contains a smaller square with an area of b^2.



(a) Describe the regions that represent $a^2 - b^2$. How can you rearrange these regions to show that $a^2 - b^2 = (a - b)(a + b)$?

(b) How can you use the figure to show that $(a - b)^2 = a^2 - 2ab + b^2$?

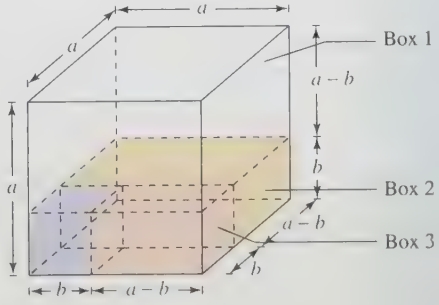
(c) Draw another figure to show that $(a + b)^2 = a^2 + 2ab + b^2$. Explain how the figure shows this.

93. Geometric Modeling The figure shows a large cube with a volume of a^3 that contains a smaller cube with a volume of b^3. When the smaller cube is removed, the remaining solid has a volume of

$a^3 - b^3$

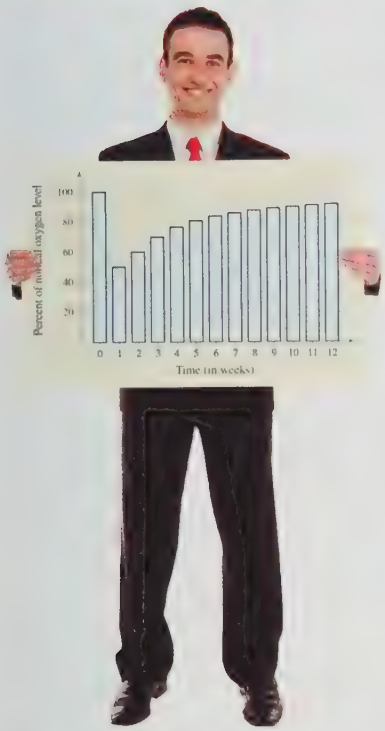
and consists of the three rectangular boxes labeled Box 1, Box 2, and Box 3. Explain how you can use the figure to obtain the factoring formula

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.



0.7 Fractional Expressions

- Find the domain of an algebraic expression.
- Simplify a rational expression and perform operations with rational expressions.
- Simplify a complex fraction.



In Exercise 72 on page 61, you will draw conclusions about the oxygen level of a pond based on a bar graph obtained using a rational model.

Domain of an Expression

The set of all real numbers for which an algebraic expression is defined is called the **domain** of the expression. For instance, the domain of

$$\frac{1}{x}$$

is all real numbers other than $x = 0$. Two algebraic expressions are **equivalent** if they have the same domain and yield the same values for all numbers in their domain. For instance, the expressions below are equivalent.

$$[(x + 1) + (x + 2)] \qquad 2x + 3$$

Example 1 Finding the Domain of an Algebraic Expression

a. The domain of the polynomial

$$2x^3 + 3x + 4$$

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers (unless the domain is specifically restricted).

b. The domain of the polynomial

$$x^2 + 5x + 2, \quad x > 0$$

is the set of positive real numbers, because the polynomial is specifically restricted to that set.

c. The domain of the radical expression

$$\sqrt{x}$$

is the set of nonnegative real numbers, because the square root of a negative number is not a real number.

d. The domain of the expression

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, because the value $x = 3$ results in division by zero, which is undefined.

STUDY TIP

The domain of an algebraic expression does not include any value that creates *division by zero or the square root of a negative number.*

Checkpoint 1

Find the domain of each expression.

a. $\frac{1}{x - 5}$

b. $1 - 2x - x^2$

c. $\sqrt{4x}$

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Simplifying Rational Expressions

The quotient of two algebraic expressions is a **fractional expression.** Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \quad \frac{2x-1}{x+1}, \quad \text{or} \quad \frac{x^2-1}{x^2+1}$$

is a **rational expression.** Recall that a fraction is in simplest form if its numerator and denominator have no factors in common aside from ± 1. To write a fraction in simplest form, divide out common factors.

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}, \quad b \neq 0, \quad c \neq 0$$

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials.

Example 2 Simplifying a Rational Expression

Write the expression $\dfrac{x^2+4x-12}{3x-6}$ in simplest form.

SOLUTION

$$\frac{x^2+4x-12}{3x-6} = \frac{(x+6)(x-2)}{3(x-2)}$$ Factor completely.

$$= \frac{(x+6)(x-2)}{3(x-2)}$$ Divide out common factors.

$$= \frac{x+6}{3}, \quad x \neq 2$$ Simplify.

Note that the original expression is undefined when $x = 2$ (because division by zero is undefined). Because this is not obvious in the simplified expression, you must add the domain restriction $x \neq 2$ to the simplified expression to make it *equivalent* to the original expression.

✓ Checkpoint 2

Write the expression $\dfrac{2x^2-2}{3x-3}$ in simplest form. ■

In Example 2, do not make the mistake of trying to simplify further by dividing out *terms*.

$$\frac{x+6}{3} = \frac{x+6}{3} = x+2$$

Remember that to simplify fractions, you divide out *factors*, not terms.

When simplifying rational expressions, be sure to factor each polynomial completely before concluding that the numerator and denominator have no factors in common. Moreover, changing the sign of a factor may allow further simplification, as demonstrated in part (b) of the next example.

STUDY TIP

In Example 3(b), note that when factoring completely, $4 - x = -(x - 4)$.

Example 3 **Simplifying Rational Expressions**

a. $\dfrac{x^3 - 4x}{x^2 + x - 2} = \dfrac{x(x+2)(x-2)}{(x+2)(x-1)} = \dfrac{x(x-2)}{x-1}, \quad x \neq -2$

Factor completely and divide out common factors.

b. $\dfrac{12 + x - x^2}{2x^2 - 9x + 4} = \dfrac{(4-x)(3+x)}{(2x-1)(x-4)} = \dfrac{-(x-4)(3+x)}{(2x-1)(x-4)}$

Factor completely and divide out common factors.

$= -\dfrac{3+x}{2x-1}, \quad x \neq 4$

Simplify.

Checkpoint 3

Write the expression $\dfrac{3 - 2x - x^2}{2x^2 - 2}$ in simplest form.

To multiply or divide rational expressions, use the properties of fractions (see Section 0.2). Recall that to divide fractions you invert the divisor and multiply.

Example 4 **Multiplying Rational Expressions**

$\dfrac{6x^2 - 6x}{x^2 + 2x - 3} \cdot \dfrac{x^2 + x - 6}{2x}$

Original product

$= \dfrac{6x(x-1)(x+3)(x-2)}{(x-1)(x+3)(2x)}$

Factor and multiply.

$= \dfrac{3(2x)(x-1)(x+3)(x-2)}{(x-1)(x+3)(2x)}$

Divide out common factors.

$= 3(x-2), \quad x \neq -3, x \neq 0, x \neq 1$

Simplify.

Checkpoint 4

Multiply and simplify: $\dfrac{3}{x-2} \cdot \dfrac{x-2}{3x+3}$.

Example 5 **Dividing Rational Expressions**

$\dfrac{2x}{3x - 12} \div \dfrac{x^2 - 2x}{x^2 - 6x + 8} = \dfrac{2x}{3x - 12} \cdot \dfrac{x^2 - 6x + 8}{x^2 - 2x}$

Invert and multiply.

$= \dfrac{(2x)(x-2)(x-4)}{(3)(x-4)(x)(x-2)}$

Factor and multiply.

$= \dfrac{(2x)(x-2)(x-4)}{(3)(x-4)(x)(x-2)}$

Divide out common factors.

$= \dfrac{2}{3}, \quad x \neq 0, x \neq 2, x \neq 4$

Simplify.

Checkpoint 5

Divide and simplify: $\dfrac{4x + 4y}{5} \div \dfrac{x + y}{2}$.

To add or subtract rational expressions, use the least common denominator (LCD) method or the following basic property of fractions (see Section 0.2).

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad b \neq 0, d \neq 0$$

This property is efficient for adding or subtracting *two* fractions that have no common factors in their denominators.

Example 6 **Adding Rational Expressions**

$$\frac{x}{x-3} + \frac{2}{3x+4} = \frac{x(3x+4) + 2(x-3)}{(x-3)(3x+4)} \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$= \frac{3x^2 + 4x + 2x - 6}{(x-3)(3x+4)} \qquad \text{Distributive Property}$$

$$= \frac{3x^2 + 6x - 6}{(x-3)(3x+4)} \qquad \text{Combine like terms.}$$

$$= \frac{3(x^2 + 2x - 2)}{(x-3)(3x+4)} \qquad \text{Factor.}$$

Checkpoint 6

Subtract: $\frac{4}{x} - \frac{2x}{3}$.

For fractions with repeated factors in their denominators, the LCD method works well.

STUDY TIP

Recall that the least common denominator of two or more fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator.

Example 7 **Combining Rational Expressions: The LCD Method**

Perform the indicated operations and simplify: $\frac{3}{x-1} - \frac{2}{x} + \frac{x+3}{x^2-1}$.

SOLUTION Using the factored denominators $(x-1)$, x, and $(x+1)(x-1)$, you can see that the least common denominator is $x(x+1)(x-1)$.

$$\frac{3}{x-1} - \frac{2}{x} + \frac{x+3}{x^2-1}$$

$$= \frac{3(x)(x+1)}{x(x+1)(x-1)} - \frac{2(x+1)(x-1)}{x(x+1)(x-1)} + \frac{(x+3)(x)}{x(x+1)(x-1)}$$

$$= \frac{3(x)(x+1) - 2(x+1)(x-1) + (x+3)(x)}{x(x+1)(x-1)}$$

$$= \frac{3x^2 + 3x - 2x^2 + 2 + x^2 + 3x}{x(x+1)(x-1)}$$

$$= \frac{2x^2 + 6x + 2}{x(x+1)(x-1)} = \frac{2(x^2 + 3x + 1)}{x(x+1)(x-1)}$$

Checkpoint 7

Perform the indicated operations and simplify:

$$\frac{5}{x} + \frac{4}{x-1} - \frac{4}{x(x-1)}.$$

Complex Fractions

Fractional expressions with separate fractions in the numerator and/or the denominator are called **complex fractions.** Here are two examples.

$$\frac{\left(\frac{1}{x}\right)}{x^2+1} \quad \text{and} \quad \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2+1}\right)}$$

A complex fraction can be simplified by combining the fractions in its numerator into a single fraction and then combining the fractions in its denominator into a single fraction. Then invert the denominator and multiply.

Example 8 **Simplifying a Complex Fraction**

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left[\frac{2-3(x)}{x}\right]}{\left[\frac{1(x-1)-1}{x-1}\right]}$$ Combine fractions.

$$= \frac{\left(\frac{2-3x}{x}\right)}{\left(\frac{x-2}{x-1}\right)}$$ Simplify.

$$= \frac{2-3x}{x} \cdot \frac{x-1}{x-2}$$ Invert and multiply.

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

✓Checkpoint 8

Simplify the complex fraction $\dfrac{\left(\frac{x}{3}-1\right)}{x-3}$.

Another way to simplify the complex fraction in Example 8 is to multiply the numerator and denominator by the LCD of all fractions in its numerator and denominator, as shown below. After simplifying, the same result is obtained.

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} \cdot \frac{x(x-1)}{x(x-1)}$$ LCD is $x(x-1)$.



SUMMARIZE (Section 0.7)

1. State the definition of the domain of an algebraic expression *(page 55)*. For an example of finding the domain of an algebraic expression, see Example 1.
2. State the definition of a rational expression and describe how to simplify a rational expression *(page 56)*. For examples of simplifying rational expressions, see Examples 2, 3, 4, 5, 6, and 7.
3. State the definition of a complex fraction *(page 59)*. For an example of simplifying a complex fraction, see Example 8.

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SKILLS WARM UP 0.7 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.6.

In Exercises 1–10, completely factor the polynomial.

1. $5x^2 - 15x^3$
2. $16x^2 - 9$
3. $9x^2 - 6x + 1$
4. $9 + 12y + 4y^2$
5. $z^2 + 4z + 3$
6. $2x^2 - 30x + 100$
7. $3 + 8x - 3x^2$
8. $3x^2 - 46x + 15$
9. $s^3 + s^2 - 4s - 4$
10. $2x^3 - 128$

Exercises 0.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Checking Domain Values In Exercises 1–4, determine whether each value of x is in the domain of the expression.

1. $\dfrac{x+2}{4x+2}$ (a) $x = -\dfrac{1}{2}$ (b) $x = 2$
2. $\dfrac{2x+3}{x-4}$ (a) $x = -\dfrac{3}{2}$ (b) $x = 4$
3. $\sqrt{2x+4}$ (a) $x = -2$ (b) $x = 2$
4. $\sqrt{3x-9}$ (a) $x = -3$ (b) $x = 3$

Finding the Domain of an Algebraic Expression In Exercises 5–12, find the domain of the expression. *See Example 1.*

5. $3x^2 - 4x + 7$
6. $6x^2 - 9,\ x > 0$
7. $\dfrac{1}{3-x}$
8. $\dfrac{x+1}{2x+1}$
9. $\dfrac{x-1}{x^2-4x}$
10. $\dfrac{4x+3}{x^2-36}$
11. $\sqrt{x+1}$
12. $\sqrt{x-1}$

Equivalent Rational Expressions In Exercises 13–18, find the missing factor and state any domain restrictions necessary to make the two fractions equivalent.

13. $\dfrac{5}{2x} = \dfrac{5(\quad)}{6x^2}$
14. $\dfrac{3}{4} = \dfrac{3(\quad)}{4(x+1)}$
15. $\dfrac{x+1}{x} = \dfrac{(x+1)(\quad)}{x(x-2)}$
16. $\dfrac{3y-4}{y+1} = \dfrac{(3y-4)(\quad)}{y^2-1}$
17. $\dfrac{3x}{x-3} = \dfrac{3x(\quad)}{x^2-x-6}$
18. $\dfrac{1-z}{z^2} = \dfrac{(1-z)(\quad)}{z^3+z^2}$

Simplifying a Rational Expression In Exercises 19–34, write the rational expression in simplest form. *See Examples 2 and 3.*

19. $\dfrac{15x^2}{10x}$
20. $\dfrac{24y^3}{56y^7}$
21. $\dfrac{2x}{4x+4}$
22. $\dfrac{9x^2+9x}{2x+2}$
23. $\dfrac{x-5}{10-2x}$
24. $\dfrac{3-x}{8x-24}$
25. $\dfrac{x^2-25}{5-x}$
26. $\dfrac{x^2-16}{4-x}$
27. $\dfrac{x^3+5x^2+6x}{x^2-4}$
28. $\dfrac{x^3-2x^2-3x}{x^2-9}$
29. $\dfrac{y^2-7y+12}{y^2+3y-18}$
30. $\dfrac{x^2+8x-20}{x^2+11x+10}$
31. $\dfrac{2-x+2x^2-x^3}{x-2}$
32. $\dfrac{x^2-9}{x^3+x^2-9x-9}$
33. $\dfrac{z^3-27}{z^2+3z+9}$
34. $\dfrac{y^3-2y^2-8y}{y^3+8}$

Multiplying or Dividing Rational Expressions In Exercises 35–48, perform the indicated operations and simplify. *See Examples 4 and 5.*

35. $\dfrac{5}{x-1} \cdot \dfrac{x-1}{25(x-2)}$
36. $\dfrac{x+13}{x^3(3-x)} \cdot \dfrac{x(x-3)}{5}$
37. $\dfrac{(x-9)(x+7)}{x+1} \cdot \dfrac{x}{9-x}$
38. $\dfrac{(x+5)(x-3)}{x+2} \cdot \dfrac{1}{(x+5)(x+2)}$
39. $\dfrac{r}{r-1} \cdot \dfrac{r^2-1}{r^2}$
40. $\dfrac{4y-16}{5y+15} \cdot \dfrac{2y+6}{4-y}$
41. $\dfrac{t^2-t-6}{t^2+6t+9} \cdot \dfrac{t+3}{t^2-4}$
42. $\dfrac{y^3-8}{2y^3} \cdot \dfrac{4y}{y^2-5y+6}$
43. $\dfrac{x^2+x-2}{x^3+x^2} \cdot \dfrac{x}{x^2+3x+2}$
44. $\dfrac{x^3-8}{x+1} \cdot \dfrac{x^2-1}{x^3-3x^2+2x}$
45. $\dfrac{3(x+y)}{4} \div \dfrac{x+y}{2}$
46. $\dfrac{x+2}{5(x-3)} \div \dfrac{x-2}{5(x-3)}$

47. $\dfrac{x^2}{(x+1)^2} \div \dfrac{x}{(x+1)^3}$ **48.** $\dfrac{x^2-1}{x} \div \dfrac{(x-1)^2}{x}$

Finding the Least Common Denominator In Exercises 49–52, find the least common denominator of the expressions.

49. $\dfrac{10}{x+4}, \dfrac{x+1}{x^2-2x-24}$ **50.** $\dfrac{x}{x+2}, \dfrac{3}{x^2+5x+6}$

51. $\dfrac{1}{x^2}, \dfrac{1}{x-1}, \dfrac{1}{x^2-x}$ **52.** $\dfrac{1}{x}, \dfrac{1}{x^2+3x}, \dfrac{1}{x+3}$

Adding or Subtracting Rational Expressions In Exercises 53–62, perform the indicated operations and simplify. *See Examples 6 and 7.*

53. $\dfrac{4x}{x-2} + \dfrac{x}{x-2}$ **54.** $\dfrac{3x-2}{x+1} + \dfrac{2-x}{x+1}$

55. $\dfrac{3x}{x-4} - \dfrac{x}{4-x}$ **56.** $\dfrac{4x}{3-x} - \dfrac{5x}{x-3}$

57. $4 + \dfrac{3}{x-5}$ **58.** $\dfrac{4}{x+2} - 6$

59. $\dfrac{2}{x^2-4} - \dfrac{1}{x^2-3x+2}$ **60.** $\dfrac{x}{x^2+x-2} - \dfrac{1}{x+2}$

61. $-\dfrac{1}{x} + \dfrac{2}{x^2+1} + \dfrac{1}{x^3+x}$ **62.** $\dfrac{2}{x+1} + \dfrac{2}{x-1} + \dfrac{1}{x^2-1}$

Simplifying a Complex Fraction In Exercises 63–68, simplify the complex fraction. *See Example 8.*

63. $\dfrac{\left(\dfrac{x}{2}-1\right)}{(x-2)}$ **64.** $\dfrac{(x-3)}{\left(\dfrac{x}{4}-\dfrac{4}{x}\right)}$

65. $\dfrac{\left(\dfrac{1}{x}-\dfrac{1}{x+1}\right)}{\left(\dfrac{1}{x+1}\right)}$ **66.** $\dfrac{\left(\dfrac{5}{y}-\dfrac{6}{2y+1}\right)}{\left(\dfrac{5}{y}+4\right)}$

67. $\dfrac{\left(\sqrt{x}-\dfrac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$ **68.** $\dfrac{\left(\dfrac{1}{\sqrt{2y}}+\sqrt{2y}\right)}{\sqrt{2y}}$

Monthly Payment In Exercises 69 and 70, use the formula for the approximate annual interest rate r of a monthly installment loan

$$r = \frac{\left[\dfrac{24(NM-P)}{N}\right]}{\left(P + \dfrac{NM}{12}\right)}$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

69. (a) Approximate the annual interest rate r for a four-year car loan of \$18,000 with monthly payments of \$415.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

70. (a) Approximate the annual interest rate r for a five-year car loan of \$20,000 with monthly payments of \$415.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

71. Refrigeration Food at a temperature of 75°F is placed in a 40°F refrigerator. A model for the temperature T (in degrees Fahrenheit) of the food after t hours is

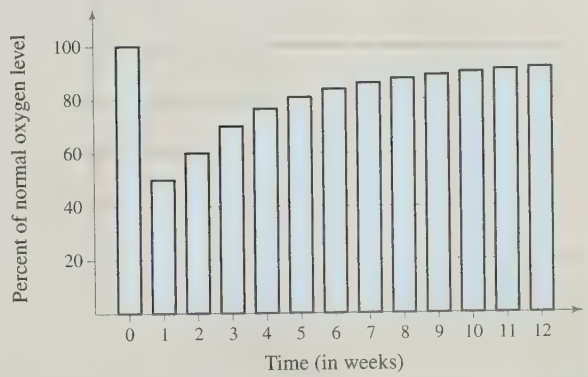
$$T = 10\left(\frac{4t^2+16t+75}{t^2+4t+10}\right), \quad t \geq 0.$$

Sketch a bar graph showing the temperature of the food at $t = 0, 1, 2, 3, 4$, and 5 hours. According to the bar graph, does it seem likely that the temperature of the food will drop to 40°F within 7 hours of being placed in the refrigerator? Explain.

72. HOW DO YOU SEE IT? The mathematical model

$$P = 100\left(\frac{t^2-t+1}{t^2+1}\right), \quad t \geq 0$$

gives the percent P of the normal level of oxygen in a pond, where t is the time (in weeks) after organic waste is dumped into the pond. The bar graph shows the situation. What conclusions can you draw from the bar graph?



True or False? In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.

73. $\dfrac{x^{2n}-1^{2n}}{x^n-1^n} = x^n + 1^n$ **74.** $\dfrac{x^2-3x+2}{x-1} = x-2$

75. Writing In simplifying a rational expression, when is it necessary to indicate domain restrictions?

76. Think About It Is it true that

$$\frac{ax-b}{b-ax} = -1$$

for all nonzero numbers a and b? Explain.

ALGEBRA TUTOR

Order of Operations

Much of the algebra in this chapter involves evaluation of algebraic expressions. When you evaluate an algebraic expression, you need to know the priorities assigned to different operations. These priorities are called the *order of operations.*

1. Perform operations inside *symbols of grouping* or *absolute value symbols*, starting with the innermost symbol.
2. Evaluate all *exponential* expressions.
3. Perform all *multiplications* and *divisions* from left to right.
4. Perform all *additions* and *subtractions* from left to right.

Example 1 Using Order of Operations

Evaluate each expression.

a. $19 - 3 \cdot 6$ **b.** $20 - 2 \cdot 3^2 + 11$

c. $3 + 8 \div 2 \cdot 2$ **d.** $(7 - 2) - (4 - 5)$

e. $7 - [(5 \cdot 3) + 2^3]$ **f.** $16 + (12 - |4 - 8|)$

SOLUTION

a. $19 - 3 \cdot 6 = 19 - 18$ Multiply.

$= 1$ Subtract.

b. $20 - 2 \cdot 3^2 + 11 = 20 - 2 \cdot 9 + 11$ Evaluate exponential expression.

$= 20 - 18 + 11$ Multiply.

$= 2 + 11$ Subtract.

$= 13$ Add.

c. $3 + 8 \div 2 \cdot 2 = 3 + 4 \cdot 2$ Divide.

$= 3 + 8$ Multiply.

$= 11$ Add.

d. $(7 - 2) - (4 - 5) = 5 - (-1)$ Subtract inside parentheses.

$= 6$ Subtract.

e. $7 - [(5 \cdot 3) + 2^3] = 7 - [15 + 2^3]$ Multiply inside parentheses.

$= 7 - [15 + 8]$ Evaluate exponential expression.

$= 7 - 23$ Add inside brackets.

$= -16$ Subtract.

f. $16 + (12 - |4 - 8|) = 16 + (12 - |-4|)$ Subtract inside absolute value symbols.

$= 16 + (12 - 4)$ Evaluate absolute value.

$= 16 + 8$ Subtract inside parentheses.

$= 24$ Add.

TECH TUTOR



Most scientific and graphing calculators use the same order of operations listed above. Try entering the expressions in Example 1 into your calculator. Do you get the same results?



TECH TUTOR

Symbolic algebra systems can simplify algebraic expressions. If you have access to such a system, try using it to simplify the expressions in this Algebra Tutor.

Simplifying Expressions

A second algebraic skill studied in this chapter is simplifying expressions by combining like terms.

Like terms have the same variables raised to the same powers. For instance, $5x^2$ and $7x^2$ are like terms. Constant terms, such as 8 and -3, are also like terms. An expression is said to be *simplified* when it contains no symbols of grouping and all like terms are combined. The Distributive Property allows you to combine like terms by adding coefficients. For instance,

$$5x^2 + 7x^2 = (5 + 7)x^2 = 12x^2.$$

Example 2 **Simplifying Expressions by Combining Like Terms**

Simplify each expression.

a. $-8x^2 + x^2 - 9x^2$

b. $7 - x + 20 + 4x - 1$

c. $4x^3 + 5x^2 - 9x^3 - 2x^2$

d. $x^2 - 8x + x + 6x^2 - 3x$

e. $12x - 10y - 5x + y$

f. $4(w + 9) - 5(w - 3)$

g. $3z + 1 - 6z^2 - z - 4 + 11z^2$

SOLUTION

a. $-8x^2 + x^2 - 9x^2 = (-8 + 1 - 9)x^2$ Distributive Property

$= -16x^2$ Add coefficients.

b. $7 - x + 20 + 4x - 1 = (-x + 4x) + (7 + 20 - 1)$ Group like terms.

$= 3x + 26$ Combine like terms.

c. $4x^3 + 5x^2 - 9x^3 - 2x^2 = (4x^3 - 9x^3) + (5x^2 - 2x^2)$ Group like terms.

$= -5x^3 + 3x^2$ Combine like terms.

d. $x^2 - 8x + x + 6x^2 - 3x = (x^2 + 6x^2) + (-8x + x - 3x)$ Group like terms.

$= 7x^2 - 10x$ Combine like terms.

e. $12x - 10y - 5x + y = (12x - 5x) + (-10y + y)$ Group like terms.

$= 7x - 9y$ Combine like terms.

f. $4(w + 9) - 5(w - 3) = 4w + 36 - 5w + 15$ Distributive Property

$= (4w - 5w) + (36 + 15)$ Group like terms.

$= -w + 51$ Combine like terms.

g. $3z + 1 - 6z^2 - z - 4 + 11z^2$

$= (-6z^2 + 11z^2) + (3z - z) + (1 - 4)$ Group like terms.

$= 5z^2 + 2z - 3$ Combine like terms.

SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 66. Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 0.1

	Review Exercises
■ Classify real numbers as natural numbers, integers, rational numbers, or irrational numbers.	*1, 2*
■ Order real numbers.	*3, 4*
■ Use and interpret inequality notation.	*5–10*
■ Find the absolute value of a number and use absolute value to find the distance between two numbers.	*11–18*

Section 0.2

■ Identify the terms of an algebraic expression.	*19, 20*
■ Evaluate an algebraic expression.	*21, 22*
■ Identify basic rules of algebra.	*23–26*
■ Perform operations on real numbers.	*27–30*
■ Use a calculator to evaluate an expression and round decimal numbers.	*31, 32*

Section 0.3

■ Use properties of exponents to simplify expressions with exponents.

$a^m a^n = a^{m+n}$ $\quad \dfrac{a^m}{a^n} = a^{m-n}$ $\quad (ab)^m = a^m b^m$

$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ $\quad (a^m)^n = a^{mn}$ $\quad a^{-n} = \dfrac{1}{a^n}$

$a^0 = 1$ $\quad \left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$ $\quad |a^2| = |a|^2 = a^2$ *33–36*

■ Use scientific notation.	*37–40*
■ Use a calculator to evaluate exponential expressions.	*41, 42*

■ Use interest formulas to solve application problems.

Simple interest: $A = P(1 + rt)$

Compound interest: $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$ *43, 44*

* A wide range of valuable study aids are available to help you master the material in this chapter. The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *www.cengagebrain.com* offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

Section 0.4

Review Exercises

- Simplify and evaluate expressions involving radicals.

$\sqrt[n]{a^m} = (\sqrt[n]{a})^m \qquad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} \qquad (\sqrt[n]{a})^n = a$ For n even, $\sqrt[n]{a^n} = |a|$. For n odd, $\sqrt[n]{a^n} = a$. *45–52*

- Rationalize a denominator by using its conjugate. *53, 54*
- Combine like radicals. *55, 56*
- Use properties of rational exponents. *57–60*
- Use a calculator to evaluate a radical. *61, 62*

Section 0.5

- Write a polynomial in standard form. *63–72*
- Add and subtract polynomials by combining like terms. *63–66*
- Multiply polynomials using the FOIL Method or a vertical format. *67, 69, 70*
- Use special products to multiply polynomials.

$(u + v)(u - v) = u^2 - v^2 \qquad (u \pm v)^2 = u^2 \pm 2uv + v^2$

$(u \pm v)^3 = u^3 \pm 3u^2v + 3uv^2 \pm v^3$ *68, 71, 72*

- Use polynomials to solve application problems. *73, 74*

Section 0.6

- Factor a polynomial by factoring out common factors. *75, 76, 78, 80*
- Factor a polynomial in a special form.

$u^2 - v^2 = (u + v)(u - v) \qquad u^2 \pm 2uv + v^2 = (u \pm v)^2$

$u^3 \pm v^3 = (u \pm v)(u^2 \mp uv + v^2)$ *75–78*

- Factor a trinomial as the product of two binomials. *79, 80*
- Factor a polynomial by grouping. *81, 82*

Section 0.7

- Find the domain of an algebraic expression by finding values of the variable that make a denominator zero or a radicand negative. *83–88*
- Simplify a rational expression by dividing out common factors from the numerator and denominator. *89–92*
- Perform operations with rational expressions by using properties of fractions. *93–98*
- Simplify a complex fraction. *99, 100*

Study Strategies

- **Use the Skills Warm-Up Exercises** Each exercise set in this text (except the set for Section 0.1) begins with a set of skills warm-up exercises. You should begin each homework session by quickly working through all of these exercises (all are answered in the back of the text). The "old" skills covered in these exercises are needed to master the "new" skills in the section exercise set. The skills warm-up exercises remind you that mathematics is cumulative—to be successful in this course, you must retain "old" skills.
- **Use the Additional Study Aids** The additional study aids were prepared specifically to help you master the concepts discussed in the text. They are the *Student Solutions Manual*, the *Graphing Technology Guide*, and the *Instructional DVD*.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises

Classifying Real Numbers In Exercises 1 and 2, determine which numbers are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers.

1. $\{9, -7, -\frac{8}{9}, \frac{5}{2}, \sqrt{6}, 0.4\}$ **2.** $\{\sqrt{8}, -8, -\frac{10}{3}, 0, 5.2, \frac{3}{7}\}$

Comparing Numbers on the Real Number Line In Exercises 3 and 4, plot the two real numbers on the real number line and place the appropriate inequality sign (< or >) between them.

3. $-4, -3$ **4.** $\frac{1}{5}, \frac{1}{6}$

Interpreting an Inequality In Exercises 5 and 6, give a verbal description of the subset of real numbers that is represented by the inequality, and sketch the subset on the real number line.

5. $x \leq -6$ **6.** $x > 5$

Inequalities and Subsets of Real Numbers In Exercises 7–10, use inequality notation to describe the subset of real numbers.

7. x is nonnegative. **8.** x is at most 7.

9. x is greater than 2 and less than or equal to 5.

10. x is less than or equal to -2 or x is greater than 2.

Finding Absolute Value In Exercises 11 and 12, evaluate the expression.

11. $-|-14|$ **12.** $|4 - 2|$

The Distance Between Two Numbers In Exercises 13–16, find the distance between a and b.

13. $a = -14, \quad b = -18$ **14.** $a = -1, \quad b = -5$

15. $a = 2, \quad b = -8$ **16.** $a = 10, \quad b = -7$

Using Absolute Value Notation In Exercises 17 and 18, use absolute value notation to describe the sentence.

17. The distance between x and 7 is at least 4.

18. The distance between x and -22 is no more than 10.

Identifying Terms In Exercises 19 and 20, identify the terms of the algebraic expression.

19. $5x^3 - 2x + 7$ **20.** $7x^5 + 4x^2 - 8$

Evaluating Algebraic Expressions In Exercises 21 and 22, evaluate the expression for each value of x.

21. $-4x^2 - 6x$ (a) $x = -1$ (b) $x = 0$

22. $12 - 5x^2$ (a) $x = -2$ (b) $x = 3$

Basic Rules of Algebra In Exercises 23–26, identify the rule of algebra illustrated by the statement.

23. $5(x^2 + x) = 5x^2 + 5x$

24. $(x + 6) + 3 = x + (6 + 3)$

25. $3x + 7 = 7 + 3x$ **26.** $(x^2 - 1)\left(\frac{1}{x^2 - 1}\right) = 1$

Properties of Zero and Properties of Fractions In Exercises 27–30, perform the indicated operation(s). (Write fractional answers in simplest form.)

27. $-3(0) - 2(4 - 5)$ **28.** $\frac{-3 + 3}{9} - 5(2 - 3)$

29. $\frac{1}{2} + \frac{1}{3} - \frac{1}{6}$ **30.** $\frac{5}{12} + \frac{3}{5}$

Using a Calculator In Exercises 31 and 32, use a calculator to evaluate the expression. (Round to two decimal places.)

31. $4\left(\frac{1}{6} - \frac{1}{7}\right)$ **32.** $-2 + 3\left(\frac{1}{2} - \frac{1}{3}\right)$

Using Properties of Exponents In Exercises 33–36, simplify the expression.

33. $\frac{(4x)^2}{2x}$ **34.** $(-x)^2(-3x)^3$

35. $\frac{10x^2}{2x^6}$ **36.** $2x(5x^2)^3$

Converting to Scientific Notation In Exercises 37 and 38, write the number in scientific notation.

37. 308,700,000 **38.** 0.0002036

Converting to Decimal Notation In Exercises 39 and 40, write the number in decimal notation.

39. 7.28×10^6 **40.** 8.4×10^{-5}

Using a Calculator In Exercises 41 and 42, use a calculator to evaluate the expression. (Round to three decimal places.)

41. $1800(1 + 0.08)^{24}$ **42.** $50{,}000\left(1 + \frac{0.075}{12}\right)^{48}$

Finding the Balance in an Account In Exercises 43 and 44, you deposit \$8000 in an account earning the interest described. Complete the table showing the balance over time.

Year	5	10	15	20	25
Balance					

43. Annual interest rate of 4.5% compounded monthly

44. Annual interest rate of 4% compounded quarterly

Converting Between Forms In Exercises 45 and 46, fill in the missing form.

	Radical Form	*Rational Exponent Form*
45.	$\sqrt{16} = 4$	
46.		$16^{1/4} = 2$

Evaluating an Expression Involving Radicals In Exercises 47 and 48, evaluate the expression.

47. $\sqrt{169}$ **48.** $\sqrt[3]{125}$

Simplifying a Radical Expression In Exercises 49–52, simplify the expression.

49. $\sqrt{4x^4}$ **50.** $\sqrt[3]{54x^3}$

51. $\sqrt{6}\sqrt{2}$ **52.** $\sqrt{6}\sqrt{3}$

Rationalizing a Denominator In Exercises 53 and 54, rewrite the expression by rationalizing the denominator. Simplify your answer.

53. $\dfrac{1}{2 - \sqrt{3}}$ **54.** $\dfrac{2}{3 + \sqrt{5}}$

Simplifying Expressions In Exercises 55–58, simplify the expression.

55. $2\sqrt{x} - 5\sqrt{x}$ **56.** $\sqrt{72} + \sqrt{128}$

57. $(64)^{-2/3}$ **58.** $4^{1/3} \cdot 4^{5/3}$

Reducing the Index In Exercises 59 and 60, use rational exponents to reduce the index of the radical.

59. $\sqrt[4]{5^2}$ **60.** $\sqrt[8]{x^4}$

Evaluating Radicals with a Calculator In Exercises 61 and 62, use a calculator to approximate the number. (Round your answer to three decimal places.)

61. $\sqrt{127}$ **62.** $\sqrt[3]{52}$

Operations with Polynomials In Exercises 63–72, perform the indicated operation(s) and write the resulting polynomial in standard form.

63. $(x - 3) - (2x - 8)$ **64.** $(x^2 + 2) + (2 - 4x)$

65. $x(x - 2) - 2(3x + 7)$ **66.** $2x(x + 1) + 3(x^2 - x)$

67. $(x + 1)(x - 2)$ **68.** $(2x - 5)(2x + 5)$

69. $(x + 4)(x^2 - 4x + 16)$ **70.** $(x - 2)(x^2 + 6x + 9)$

71. $(x + 4)^2$ **72.** $(2x + 1)^3$

73. Cell Sites The numbers of mobile telephone cell sites in the United States from 2005 through 2010 can be approximated by the polynomial $14{,}945.3x + 105{,}238$, where x represents the year, with $x = 5$ corresponding to 2005. Evaluate the polynomial at $x = 10$. Then interpret your result in the context of the situation. *(Source: CTIA-The Wireless Association)*

74. Mobile Home Prices The average sale price (in dollars) of a newly manufactured residential mobile home in the United States each year from 2004 through 2009 can be approximated by the polynomial

$$-707.14x^2 + 10{,}104.3x + 29{,}311$$

where x represents the year, with $x = 4$ corresponding to 2004. Evaluate the polynomial at $x = 4$, $x = 8$, and $x = 9$. Interpret your results in the context of the situation. *(Source: U.S. Census Bureau)*

Factoring Completely Exercises 75–82, completely factor the expression.

75. $4x^2 - 36$ **76.** $x^3 - 16x$

77. $8x^3 - 125$ **78.** $2x^6 - 16x^3$

79. $x^2 - 4x - 5$ **80.** $-3x^2 - 6x + 3x^3$

81. $x^3 - 4x^2 - 2x + 8$ **82.** $x^3 - 2x^2 - 9x + 18$

Finding the Domain of an Algebraic Expression In Exercises 83–86, find the domain of the expression.

83. $\dfrac{2x + 1}{x - 4}$ **84.** $\dfrac{x - 3}{7 - x}$

85. $2\sqrt{x + 1}$ **86.** $\sqrt{x - 5}$

Equivalent Rational Expressions In Exercises 87 and 88, find the missing factor and state any domain restrictions necessary to make the two fractions equivalent.

87. $\dfrac{4}{3x} = \dfrac{4(\quad)}{9x^2}$ **88.** $\dfrac{5}{7} = \dfrac{5(\quad)}{7(x + 2)}$

Simplifying a Rational Expression In Exercises 89–92, write the rational expression in simplest form.

89. $\dfrac{x^2 - 4}{2x + 4}$ **90.** $\dfrac{2x^2 + 4x}{2x}$

91. $\dfrac{x^3 - 9x}{x^3 - 4x^2 + 3x}$ **92.** $\dfrac{x^3 + 64}{x^2 - x - 20}$

Operations with Rational Expressions In Exercises 93–98, perform the indicated operation(s) and simplify.

93. $\dfrac{2x - 1}{x + 1} \cdot \dfrac{x^2 - 1}{2x^2 - 7x + 3}$ **94.** $\dfrac{x + 2}{x - 4} \div \dfrac{2x + 4}{8x}$

95. $\dfrac{x}{x - 1} + \dfrac{2x}{x - 2}$ **96.** $\dfrac{2}{x + 2} - \dfrac{3}{x - 2}$

97. $\dfrac{2}{x - 1} + \dfrac{4}{x + 1} + \dfrac{8}{x^2 - 1}$ **98.** $\dfrac{1}{x - 1} + \dfrac{2}{x} - \dfrac{1}{x^2 - x}$

Simplifying a Complex Fraction In Exercises 99 and 100, simplify the complex fraction.

99. $\dfrac{\left(\dfrac{x^2 - 1}{x}\right)}{\left[\dfrac{(x - 1)^2}{x}\right]}$ **100.** $\dfrac{(x - 4)}{\left(\quad\right)}$

TEST YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

1. Evaluate the expression $-3x^2 - 5x$ at $x = -3$.
2. Complete the table at the left given that \$4000 is deposited in an account with an annual interest rate of 3.5%, compounded monthly. What can you conclude from the table?

Year	Balance
5	
10	
15	
20	
25	

Table for 2

In Exercises 3–8, simplify the expression.

3. $8(-2x^2)^3$
4. $3\sqrt{x} - 7\sqrt{x}$
5. $5^{1/4} \cdot 5^{7/4}$
6. $\sqrt{48} - \sqrt{80}$
7. $\sqrt{12x^3}$
8. $\dfrac{2}{5 - \sqrt{7}}$

In Exercises 9 and 10, write the polynomial in standard form.

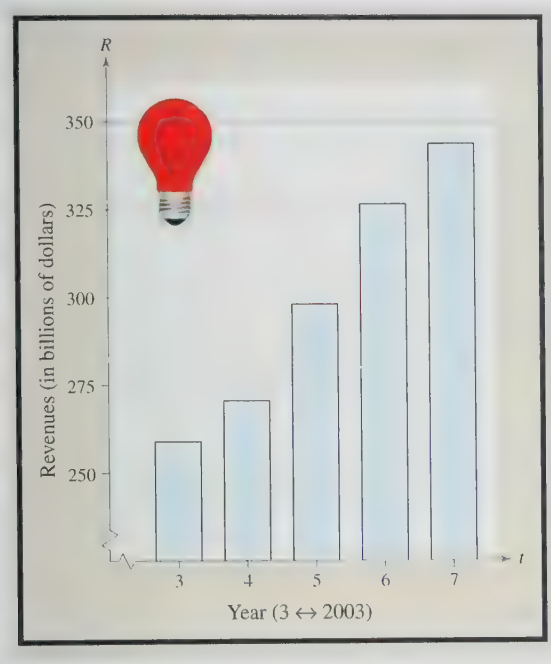
9. $(3x + 7)^2$
10. $3x(x + 5) - 2x(4x - 7)$

In Exercises 11–14, completely factor the expression.

11. $5x^2 - 80$
12. $4x^2 + 12x + 9$
13. $x^3 - 6x^2 - 3x + 18$
14. $x^3 + 2x^2 - 4x - 8$
15. Simplify: $\dfrac{x^2 - 16}{3x + 12}$.
16. Multiply and simplify: $\dfrac{3x - 5}{x + 3} \cdot \dfrac{x^2 + 7x + 12}{9x^2 - 25}$.
17. Add and simplify: $\dfrac{x}{x - 3} + \dfrac{3x}{x - 4}$.
18. Subtract and simplify: $\dfrac{3}{x + 5} - \dfrac{4}{x - 2}$.

In Exercises 19 and 20, find the domain of the expression.

19. $4\sqrt{x - 10}$
20. $\dfrac{3}{x + 9}$
21. Simplify the complex fraction $\dfrac{\left(\dfrac{2x - 9}{x - 1}\right)}{\left(\dfrac{3}{x - 1} + \dfrac{1 - x}{x + 2}\right)}$.
22. **Movie Price** The average price of a movie ticket in the United States each year from 2000 through 2009 can be approximated by the polynomial $0.223x + 5.35$, where x represents the year, with $x = 0$ corresponding to 2000. Evaluate the polynomial at $x = 0$ and $x = 9$. Then explain the meaning of your results in the context of the situation. *(Source: National Association of Theatre Owners)*



Example 9 on page 99 shows how a quadratic equation can be used to approximate revenues for the electric power industry in the United States for given years.

1 Equations and Inequalities

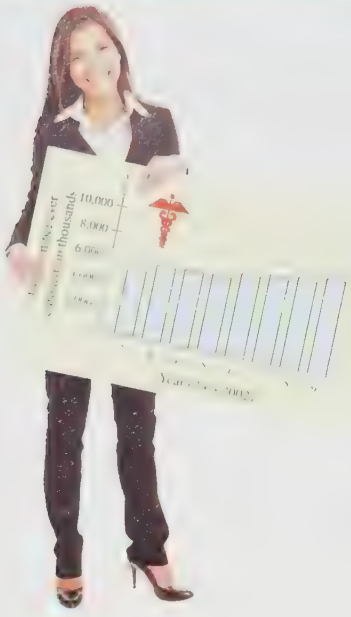
1.1 Linear Equations
1.2 Mathematical Modeling
1.3 Quadratic Equations
1.4 The Quadratic Formula
1.5 Other Types of Equations
1.6 Linear Inequalities
1.7 Other Types of Inequalities



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1.1 Linear Equations

- Classify an equation as an identity or a conditional equation.
- Solve a linear equation in one variable.
- Solve an equation involving fractional expressions.
- Use a linear model to solve an application problem.



In Exercise 79 on page 79, you will use a linear model to determine the number of enrollees in a health insurance program.

Equations and Solutions

An **equation** is a statement that two algebraic expressions are equal. Some examples of equations in x are

$$3x - 5 = 7, \quad x^2 - x - 6 = 0, \quad \text{and} \quad \sqrt{2x} = 4.$$

To **solve** an equation in x means to find all values of x for which the equation is **true.** Such values are called **solutions.** For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$, because $3(4) - 5 = 7$ is a true statement.

An equation that is true for *every* real number in the domain of the variable is called an **identity.** Two examples of identities are

$$x^2 - 9 = (x + 3)(x - 3) \quad \text{and} \quad \frac{x}{3x^2} = \frac{1}{3x}, \quad x \neq 0.$$

The first equation is an identity because it is a true statement for all real values of x. The second equation is an identity because it is true for all nonzero real values of x.

An equation that is true for just *some* (or even none) of the real numbers in the domain of the variable is called a **conditional equation.** For example, the equation $x^2 - 9 = 0$ is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation.

Example 1 Classifying Equations

Determine whether each equation is an identity or a conditional equation.

a. $2(x + 3) = 2x + 6$

b. $2(x + 3) = x + 6$

c. $2(x + 3) = 2x + 3$

SOLUTION

a. This equation is an identity because it is true for every real value of x.

b. This equation is a conditional equation because $x = 0$ is the only value in the domain for which the equation is true.

c. This equation is a conditional equation because there are no real number values of x for which the equation is true.

✓ Checkpoint 1

Determine whether the equation $4(x + 1) = 4x + 4$ is an identity or a conditional equation.

Equations are used in algebra for two distinct purposes: (1) *identities* are usually used to state mathematical properties and (2) *conditional equations* are usually used to model and solve problems that occur in real life.

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Linear Equations in One Variable

The most common type of conditional equation is a **linear equation.**

Definition of a Linear Equation

A **linear equation** in one variable x is an equation that can be written in the standard form

$$ax + b = 0$$

where a and b are real numbers with $a \neq 0$.

A linear equation in x has exactly one solution. To see this, consider the following steps. (Remember that $a \neq 0$.)

$ax + b = 0$ Original equation

$ax = -b$ Subtract b from each side.

$x = -\frac{b}{a}$ Divide each side by a.

So, the equation $ax + b = 0$ has exactly one solution,

$$x = -\frac{b}{a}.$$

To solve a linear equation in x, isolate x by forming a sequence of **equivalent** (and usually simpler) equations, each having the same solution as the original equation. The operations that yield equivalent equations come from the basic rules of algebra reviewed in Section 0.2.



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Forming Equivalent Equations

A given equation can be transformed into an equivalent equation by one or more of the following steps.

	Given Equation	*Equivalent Equation*
1. Remove symbols of grouping, combine like terms, or simplify one or both sides of the equation.	$2x - x = 4$ $3(x - 2) = 5$	$x = 4$ $3x - 6 = 5$
2. Add (or subtract) the same quantity to (from) *each* side of the equation.	$x + 1 = 6$ $x - 4 = 3$	$x = 5$ $x = 7$
3. Multiply (or divide) *each* side of the equation by the same *nonzero* quantity.	$2x = 6$ $\frac{x}{8} = 4$	$x = 3$ $x = 32$
4. Interchange the two sides of the equation.	$2 = x$	$x = 2$

The steps for solving a linear equation in x written in standard form are shown in Example 2.

ALGEBRA TUTOR

For help in solving equations similar to the one in Example 2, see the review of solving equations on page 148.

Example 2 Solving a Linear Equation

Solve $3x - 6 = 0$.

SOLUTION

$3x - 6 = 0$	Write original equation.
$3x = 6$	Add 6 to each side.
$x = 2$	Divide each side by 3.

✓ Checkpoint 2

Solve each equation.

a. $5 + 5x = 15$ **b.** $10 - x = 11$

After solving an equation, you should **check each solution** in the *original* equation. For instance, in Example 2, you can check that 2 is the solution by substituting 2 for x in the original equation

$$3x - 6 = 0.$$

CHECK

$3x - 6 = 0$	Write original equation.
$3(2) - 6 \stackrel{?}{=} 0$	Substitute 2 for x.
$6 - 6 = 0$	Solution checks. ✓

Example 3 Solving a Linear Equation

Solve $6(x - 1) + 4 = 3(7x + 1)$.

STUDY TIP

You may think a solution to a problem looks easy when it is worked out in class, but you may not know where to begin when solving the problem on your own. Keep in mind that many problems involve some trial and error before a solution is found.

SOLUTION

$6(x - 1) + 4 = 3(7x + 1)$	Write original equation.
$6x - 6 + 4 = 21x + 3$	Distributive Property
$6x - 2 = 21x + 3$	Simplify.
$-15x = 5$	Add 2 to, and subtract $21x$ from, each side.
$x = -\frac{1}{3}$	Divide each side by -15.

The solution is $x = -\frac{1}{3}$.

CHECK

$6(x - 1) + 4 = 3(7x + 1)$	Write original equation.
$6(-\frac{1}{3} - 1) + 4 \stackrel{?}{=} 3[7(-\frac{1}{3}) + 1]$	Substitute $-\frac{1}{3}$ for x.
$6(-\frac{4}{3}) + 4 \stackrel{?}{=} 3(-\frac{7}{3} + 1)$	Simplify.
$-8 + 4 \stackrel{?}{=} -7 + 3$	Simplify.
$-4 = -4$	Solution checks. ✓

✓ Checkpoint 3

Solve each equation.

a. $3(x + 5) = x - 7$ **b.** $2(x + 2) + 6 = 4(2x - 3)$

Some equations in one variable have *infinitely many solutions*. To recognize an equation of this type, perform the regular steps for solving the equation. If, when writing equivalent equations, you reach a statement that is true for all values in the domain of the variable, then the equation is an identity and has infinitely many solutions.

Example 4 An Equation with Infinitely Many Solutions

Solve $x + 4(x - 2) = 3x + 2(x - 4)$.

SOLUTION

$x + 4(x - 2) = 3x + 2(x - 4)$	Write original equation.
$x + 4x - 8 = 3x + 2x - 8$	Distributive Property
$5x - 8 = 5x - 8$	Simplify.
$-8 = -8$	Subtract $5x$ from each side.

Because the last equation is true for every real value of x, the original equation is an identity and you can conclude that it has infinitely many solutions.

✓Checkpoint 4

Solve $x + 5 + 3(2x + 1) = 7x + 8$.

It is also possible for an equation in one variable to have *no solution*. When solving an equation of this type, you will reach a statement that is not true for any value of the variable.

Example 5 An Equation with No Solution

Solve $4x - 9 + 2(x + 8) = 1 + 6(x + 4)$.

SOLUTION

$4x - 9 + 2(x + 8) = 1 + 6(x + 4)$	Write original equation.
$4x - 9 + 2x + 16 = 1 + 6x + 24$	Distributive Property
$6x + 7 = 6x + 25$	Simplify.
$7 \neq 25$	Subtract $6x$ from each side.

Because the statement

$7 = 25$

is not true, you can conclude that the original equation has no solution.

✓Checkpoint 5

Solve $1 + 4(x + 1) = 4(2 + x)$.

Equations in one variable with infinitely many solutions or no solution are not linear because they cannot be written in the standard form

$ax + b = 0$.

Note that a linear equation in x has exactly one solution.

Equations Involving Fractional Expressions

To solve an equation involving fractional expressions, you can multiply every term in the equation by the least common denominator (LCD) of the terms.

Example 6 **An Equation Involving Fractional Expressions**

$$\frac{x}{3} + \frac{3x}{4} = 2$$ Original equation

$$(12)\frac{x}{3} + (12)\frac{3x}{4} = (12)2$$ Multiply each term by the least common denominator.

$$4x + 9x = 24$$ Simplify.

$$13x = 24$$ Combine like terms.

$$x = \frac{24}{13}$$ Divide each side by 13.

✓Checkpoint 6

Solve $\frac{4x}{5} - \frac{x}{2} = 9$.

When multiplying or dividing an equation by a *variable expression*, it is possible to introduce an **extraneous** solution—one that does not satisfy the original equation. In such cases, a check is especially important.

Example 7 **An Equation with an Extraneous Solution**

Solve $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$.

SOLUTION The least common denominator is $x^2 - 4 = (x + 2)(x - 2)$. Multiply each term by this LCD and simplify.

$$\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$$ Write original equation.

$$\frac{1}{x-2}(x+2)(x-2) = \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2)$$

$$x + 2 = 3(x - 2) - 6x, \quad x \neq \pm 2$$ Simplify.

$$x + 2 = 3x - 6 - 6x$$ Distributive Property

$$4x = -8$$ Simplify.

$$x = -2$$ Extraneous solution

By checking $x = -2$, you can see that it yields a denominator of zero for the fraction

$$\frac{3}{x+2}.$$

So, $x = -2$ is extraneous, and the equation has *no solution*.

✓Checkpoint 7

Solve $\frac{1}{x-4} + \frac{1}{x} = \frac{4}{x(x-4)}$.

TECH TUTOR



Use the *table* feature of your graphing utility to check the solution in Example 3. In the equation editor, enter the expression to the left of the equal sign in y_1 and enter the expression to the right of the equal sign in y_2, as follows.

$$y_1 = 6(x - 1) + 4$$

$$y_2 = 3(7x + 1)$$

Set the *table* feature to ASK mode. When you enter the solution $-\frac{1}{3}$ for x, both y_1 and y_2 are -4, as shown.

X	Y1	Y2
-.3333	-4	-4

X=-.33333333333

Similarly, a graphing utility can help you determine if a solution is extraneous. For instance, enter the equation from Example 7 into the graphing utility's equation editor. Then, use the *table* feature in ASK mode to enter -2 for x. You will see that the graphing utility displays ERROR in the y_2 column. So, the solution $x = -2$ is extraneous.

An equation with a *single fraction* on each side can be cleared of denominators by **cross-multiplying,** which is equivalent to multiplying each side of the equation by the least common denominator and then simplifying.

Example 8 Cross-Multiplying to Solve an Equation

Solve $\frac{3y-2}{2y+1} = \frac{6y-9}{4y+3}$.

SOLUTION

$$\frac{3y-2}{2y+1} = \frac{6y-9}{4y+3}$$ Write original equation.

$$(3y-2)(4y+3) = (6y-9)(2y+1)$$ Cross-multiply.

$$12y^2 + y - 6 = 12y^2 - 12y - 9$$ Multiply.

$$13y = -3$$ Isolate y-term on left.

$$y = -\frac{3}{13}$$ Divide each side by 13.

The solution is $y = -\frac{3}{13}$. Check this in the original equation.

TECH TUTOR

When using the *equation editor* of a graphing utility, you must enter equations in terms of x. So, to enter an equation like the one shown in Example 8, replace y with x, as shown.

$y_1 = (3x - 2)/(2x + 1)$

$y_2 = (6x - 9)/(4x + 3)$

Checkpoint 8

Solve $\frac{3x-6}{x+10} = \frac{3}{4}$.

Example 9 Using a Calculator to Solve an Equation

Solve $\frac{1}{9.38} - \frac{3}{x} = \frac{5}{0.3714}$.

SOLUTION To minimize roundoff error, solve for x in the equation before performing any calculations. The least common denominator is $(9.38)(0.3714)(x)$.

$$\frac{1}{9.38} - \frac{3}{x} = \frac{5}{0.3714}$$

$$(9.38)(0.3714)(x)\left(\frac{1}{9.38} - \frac{3}{x}\right) = (9.38)(0.3714)(x)\left(\frac{5}{0.3714}\right)$$

$$0.3714x - 3(9.38)(0.3714) = 5(9.38)(x), \quad x \neq 0$$

$$[0.3714 - 5(9.38)]x = 3(9.38)(0.3714)$$

$$x = \frac{3(9.38)(0.3714)}{0.3714 - 5(9.38)}$$

$$x \approx -0.225$$ Round to three decimal places.

The solution is $x \approx -0.225$. Check this in the original equation.

STUDY TIP

Because of roundoff error, a check of a decimal solution may not yield exactly the same values for each side of the original equation. The difference, however, should be quite small.

Checkpoint 9

Solve $\frac{5}{x} + \frac{1}{2.7} = \frac{4}{0.6}$.

Application

Example 10 **Hourly Earnings**

The mean hourly earnings y (in dollars) of employees in the manufacturing industry in the United States from 2003 to 2008 can be modeled by the linear equation

$$y = 0.388t + 14.58, \quad 3 \le t \le 8$$

where t represents the year, with $t = 3$ corresponding to 2003. Use the model to estimate the year in which the mean hourly earnings were \$17.30. *(Source: U.S. Bureau of Labor Statistics)*

SOLUTION To determine when the mean hourly earnings were \$17.30, solve the model for t when $y = 17.3$.

$$y = 0.388t + 14.58 \qquad \text{Write original model.}$$

$$17.3 = 0.388t + 14.58 \qquad \text{Substitute 17.3 for } y.$$

$$2.72 = 0.388t \qquad \text{Subtract 14.58 from each side.}$$

$$t = \frac{2.72}{0.388} \qquad \text{Divide each side by 0.388.}$$

$$t \approx 7 \qquad \text{Use a calculator.}$$

Because $t = 3$ corresponds to 2003, it follows that $t = 7$ corresponds to 2007. See Figure 1.1. So, the mean hourly earnings were \$17.30 in 2007.

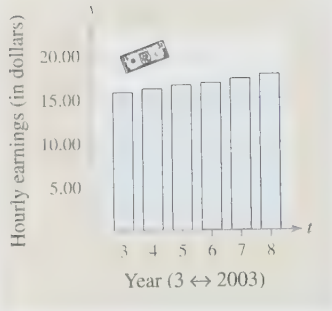


FIGURE 1.1

Checkpoint 10

The mean hourly earnings y (in dollars) of employees in the trade, transportation, and utilities industry in the United States from 2000 to 2008 can be modeled by the linear equation

$$y = 0.350t + 13.29, \quad 0 \le t \le 8$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to estimate the year in which the mean hourly earnings were \$14.35. *(Source: U.S. Bureau of Labor Statistics)*



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SUMMARIZE (Section 1.1)

1. State the definitions of an identity and a conditional equation *(page 70)*. For an example of identifying identities and conditional equations, see Example 1.
2. State the definition of a linear equation and list the four steps that can be used to form an equivalent equation *(page 71)*. For examples of solving linear equations, see Examples 2 and 3.
3. Describe two methods for solving equations involving fractional expressions *(pages 74 and 75)*. For examples of solving equations involving fractional expressions, see Examples 6, 7, 8, and 9.
4. Describe a real-life example of how a linear equation can be used to analyze the mean hourly earnings of employees in a particular industry *(page 76, Example 10)*.

SKILLS WARM UP 1.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.5 and 0.7.

In Exercises 1–10, perform the indicated operations and simplify your answer.

1. $(2x - 4) - (5x + 6)$
2. $(3x - 5) + (2x - 7)$
3. $2(x + 1) - (x + 2)$
4. $-3(2x - 4) + 7(x + 2)$
5. $\frac{x}{2} + \frac{x}{7}$
6. $x - \frac{x}{4}$
7. $\frac{1}{x + 1} - \frac{1}{x}$
8. $\frac{5}{x} + \frac{3}{x}$
9. $\frac{4}{x} + \frac{3}{x - 2}$
10. $\frac{1}{x + 1} - \frac{1}{x - 1}$

Exercises 1.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Classifying Equations In Exercises 1–6, determine whether the equation is an identity or a conditional equation. *See Example 1.*

1. $2(x - 1) = 2x - 2$
2. $3(x + 2) = 3x + 6$
3. $2(x - 1) = 3x + 4$
4. $3(x + 2) = 2x + 4$
5. $2(x + 1) = 2x + 1$
6. $3(x + 4) = 3x + 4$

Checking Solutions In Exercises 7–16, determine whether each value of x is a solution of the equation.

	Equation	*Values*	
7.	$5x - 3 = 3x + 5$	(a) $x = 0$	(b) $x = -5$
		(c) $x = 4$	(d) $x = 10$
8.	$7 - 3x = 5x - 17$	(a) $x = -3$	(b) $x = 0$
		(c) $x = 8$	(d) $x = 3$
9.	$3x^2 + 2x - 5 = 2x^2 - 2$	(a) $x = -3$	(b) $x = 1$
		(c) $x = 4$	(d) $x = -5$
10.	$5x^3 + 2x - 3 = 4x^3 + 2x - 11$	(a) $x = 2$	(b) $x = -2$
		(c) $x = 0$	(d) $x = 10$
11.	$\frac{5}{2x} - \frac{4}{x} = 3$	(a) $x = -\frac{1}{2}$	(b) $x = 4$
		(c) $x = 0$	(d) $x = \frac{1}{4}$
12.	$3 + \frac{1}{x + 2} = 4$	(a) $x = -1$	(b) $x = -2$
		(c) $x = 0$	(d) $x = 5$
	Equation	*Values*	
13.	$(x + 5)(x - 3) = 20$	(a) $x = 3$	(b) $x = -2$
		(c) $x = 0$	(d) $x = -7$
14.	$(3x + 5)(2x - 7) = 0$	(a) $x = -\frac{5}{3}$	(b) $x = -\frac{2}{7}$
		(c) $x = \frac{2}{3}$	(d) $x = \frac{3}{2}$
15.	$\sqrt{2x - 3} = 3$	(a) $x = 6$	(b) $x = -3$
		(c) $x = -\frac{1}{3}$	(d) $x = -2$
16.	$\sqrt[3]{x - 8} = 3$	(a) $x = 2$	(b) $x = -5$
		(c) $x = 35$	(d) $x = 8$

Solving an Equation In Exercises 17–32, solve the equation and check your solution. (Some equations have no solution.) *See Examples 2, 3, 4, and 5.*

17. $x + 10 = 15$
18. $9 - x = 13$
19. $7 - 2x = 15$
20. $7x + 2 = 16$
21. $8x - 5 = 3x + 10$
22. $7x + 3 = 3x - 13$
23. $5x - x = 4x - 4$
24. $3 - x = 4x - 5x$
25. $4x - 8 = -4(2 - x)$
26. $3(3 - x) = 9 - 3x$
27. $2(x + 5) - 7 = 3(x - 2)$
28. $2(13t - 15) + 3(t - 19) = 0$
29. $6[x - (2x + 3)] = 8 - 5x$
30. $3[2x - (x + 7)] = 5(x - 3)$
31. $x - 3(2x + 3) = 8 - 5x$
32. $9x - 10 = 5x + 2(2x - 5)$

An Equation Involving Fractional Expressions In Exercises 33–48, solve the equation and check your solution. (Some equations have no solution.) *See Examples 6, 7, and 8.*

33. $\frac{5x}{4} + \frac{1}{2} = \frac{1}{2}$

34. $\frac{x}{5} - \frac{x}{2} = 3$

35. $\frac{3}{2}(z + 5) = \frac{1}{4}(z + 24)$

36. $\frac{3x}{2} + \frac{1}{4}(x - 2) = 10$

37. $\frac{100 - 4u}{3} = \frac{5u + 6}{4} + 6$

38. $\frac{17 + y}{y} + \frac{32 + y}{y} = 100$

39. $\frac{5x - 4}{5x + 4} = \frac{2}{3}$

40. $\frac{10x + 3}{5x + 6} = \frac{1}{2}$

41. $10 - \frac{13}{x} = 4 + \frac{5}{x}$

42. $\frac{15}{x} - 4 = \frac{6}{x} + 3$

43. $\frac{1}{x - 3} + \frac{1}{x + 3} = \frac{10}{x^2 - 9}$

44. $\frac{1}{x - 2} + \frac{3}{x + 3} = \frac{4}{x^2 + x - 6}$

45. $\frac{6}{(x - 3)(x - 1)} = \frac{3}{x - 3} + \frac{4}{x - 1}$

46. $\frac{2}{(x - 4)(x - 2)} = \frac{1}{x - 4} + \frac{2}{x - 2}$

47. $\frac{x - 3}{2x + 1} = \frac{x - 5}{2x + 4}$

48. $\frac{2x - 3}{6x - 5} = \frac{x + 2}{3x + 4}$

Solving an Equation In Exercises 49–56, solve the equation and check your solution. (Some equations have no solution.)

49. $0.25x + 0.75(10 - x) = 3$

50. $0.60x + 0.40(100 - x) = 50$

51. $(x + 2)^2 + 5 = (x + 3)^2$

52. $(x + 1)^2 + 2(x - 2) = (x + 1)(x - 2)$

53. $(x + 2)^2 - x^2 = 4(x + 1)$

54. $4(x + 1) - 3x = x + 5$

55. $(2x + 1)^2 = 4(x^2 + x + 1)$

56. $(2x - 1)^2 = 4(x^2 - x + 6)$

The symbol  indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.

57. Error Analysis A student states that the solution of the equation

$$\frac{2}{x(x - 2)} + \frac{5}{x} = \frac{1}{x - 2}$$

is $x = 2$. Describe and correct the student's error.

58. Error Analysis A student states that the equation

$$3(x + 2) = -3x + 6$$

is an identity. Describe and correct the student's error.

59. Writing Explain why a solution of an equation involving fractional expressions may be extraneous.

60. Writing Describe two methods you can use to check a solution of an equation involving fractional expressions.

61. Writing What is meant by "equivalent equations?" Give an example of two equivalent equations.

62. Think About It For what value(s) of b does the equation $7x + 3 = 7x + b$ have infinitely many solutions? no solution?

Using a Calculator to Solve an Equation In Exercises 63–68, use a calculator to solve the equation. (Round your solution to three decimal places.) *See Example 9.*

63. $0.275x + 0.725(500 - x) = 300$

64. $2.763 - 4.5(2.1x - 5.1432) = 6.32x + 5$

65. $\frac{x}{0.6321} + \frac{x}{0.0692} = 1000$

66. $(x + 5.62)^2 + 10.83 = (x + 7)^2$

67. $\frac{2}{7.398} - \frac{4.405}{x} = \frac{1}{x}$

68. $\frac{x}{2.625} + \frac{x}{4.875} = 1$

69. Think About It What method or methods would you recommend for checking the solutions to Exercises 63–68 using your graphing utility?



70. Think About It In Exercises 63–68, your answers are rounded to three decimal places. What effect does rounding have as you check a solution?

Roundoff Error In Exercises 71–74, evaluate the expression in two ways. (a) Calculate entirely on your calculator using appropriate parentheses, and then round the answer to two decimal places. (b) Round both the numerator and the denominator to two decimal places before dividing, and then round the final answer to two decimal places. (c) Then describe why roundoff error is introduced by the second method.

71. $\frac{1 + 0.73205}{1 - 0.73205}$

72. $\frac{1 + 0.86603}{1 - 0.86603}$

73. $\dfrac{333 + \dfrac{1.98}{0.74}}{4 + \dfrac{6.25}{3.15}}$

74. $\dfrac{1.73205 - 1.19195}{3 - (1.73205)(1.19195)}$

75. Personal Income The per capita personal income y (in dollars) in West Virginia from 2000 through 2009 can be approximated by the linear equation

$$y = 1114.4t + 21{,}952, \quad 0 \le t \le 9$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to estimate the year in which the per capita personal income was about \$32,000. *(Source: Bureau of Economic Analysis)*

76. Annual Sales The annual sales S (in billions of dollars) of Oracle Corporation from 2004 through 2009 can be approximated by the linear equation

$$S = 2.903t - 1.98, \quad 4 \le t \le 9$$

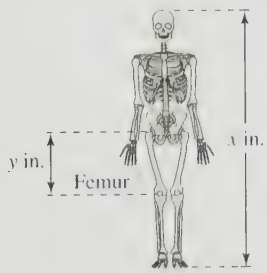
where t represents the year, with $t = 4$ corresponding to 2004. Use the model to estimate the year in which Oracle's annual sales were about \$18 billion. *(Source: Oracle Corporation)*

Forensics In Exercises 77 and 78, use the following information. The relationship between the length of an adult's femur (thigh bone) and the height of the adult can be approximated by the linear equations

$y = 0.432x - 10.44$ Female

$y = 0.449x - 12.15$ Male

where y is the length of the femur in inches and x is the height of the adult in inches (see figure).



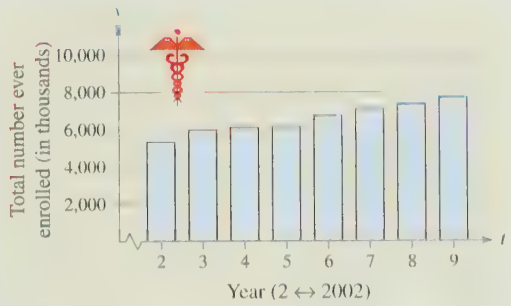
77. A crime scene investigator discovers a femur belonging to an adult human female. The bone is 18 inches long. Estimate the height of the female.

78. Officials search a forest for a missing man who is 6 feet 2 inches tall. They find an adult male femur that is 21 inches long. Is it possible that the femur belongs to the missing man?

79. Health Insurance The Children's Health Insurance Program (CHIP) tracks the total number of children ever enrolled in the program. For the years 2002 through 2009, the total numbers ever enrolled y (in thousands) are shown in the bar graph and can be approximated by

$$y = 322.5t + 4788, \quad 2 \le t \le 9$$

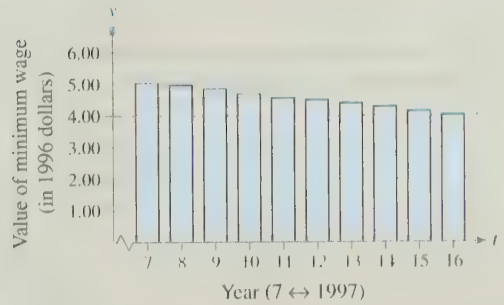
where t is the year, with $t = 2$ corresponding to 2002. *(Source: Centers for Medicare and Medicaid Services)*



(a) In what year did the total number of CHIP enrollees exceed 7 million?

(b) Use the model to predict the year in which the total number of CHIP enrollees will reach 8.3 million.

80. HOW DO YOU SEE IT? From 1997 through 2006, the federal minimum wage was \$5.15 per hour. Adjusting for inflation, the federal minimum wage's value y (in 1996 dollars) during each of these years is shown in the bar graph. *(Source: U.S. Department of Labor)*

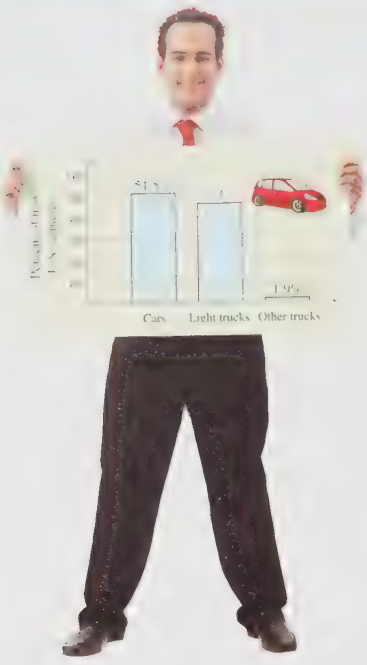


(a) In which year was the value of the minimum wage equal to \$4.85 in 1996 dollars?

(b) By 2009, the minimum wage increased to \$7.25, a value of \$5.30 in 1996 dollars. Suppose it had not increased. Use the linear relationship shown in the bar graph to predict the value (in 1996 dollars) of a 2009 minimum wage of \$5.15 per hour.

1.2 Mathematical Modeling

- Construct a mathematical model from a verbal model.
- Model and solve percent and distance problems.
- Model and solve mixture problems.
- Use common formulas to solve geometry problems.



In Exercise 37 on page 89, you will use a mathematical model to determine the number of new cars that were sold or leased in 2009.

Introduction to Problem Solving

In this section, you will use algebra to solve real-life problems. To do this, you will construct one or more equations that represent each real-life problem. This procedure is called **mathematical modeling.**

A good approach to mathematical modeling is to use two stages. First, use the verbal description of the problem to form a *verbal model.* Then, assign labels to each of the quantities in the verbal model and use the labels to form a *mathematical model* or an *algebraic equation.*

Verbal description ⇒ Verbal model ⇒ Algebraic equation

When you are trying to construct a verbal model, it is helpful to look for a *hidden equality*—a statement that two algebraic expressions are equal. These two expressions might be explicitly stated as being equal, or they might be known to be equal (based on prior knowledge or experience).

Example 1 Using a Verbal Model

You accept a job with an annual income of $47,800. This includes your salary and a $1000 year-end bonus. You are paid twice a month. What is your salary per pay period?

SOLUTION Because there are 12 months in a year and you are paid twice a month, it follows that there are 24 pay periods during the year.

Verbal Model: Income for year = 24 pay periods · Salary per pay period + Bonus

Labels: Income for year = 47,800 (dollars)
Salary per pay period = x (dollars)
Bonus = 1000 (dollars)

Equation: $47{,}800 = 24x + 1000$

$46{,}800 = 24x$

$1950 = x$

Your salary per pay period is $1950.

CHECK

$47{,}800 = 24x + 1000$ Write original equation.

$47{,}800 \stackrel{?}{=} 24(1950) + 1000$ Substitute 1950 for x.

$47{,}800 = 47{,}800$ Solution checks. ✓

✓ Checkpoint 1

You accept a job with an annual income of $51,000. This includes your salary and a $1600 year-end bonus. You are paid weekly. What is your salary per pay period? ■

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Translating Key Words and Phrases

Key Words and Phrases	*Verbal Description*	*Algebraic Statement*
Consecutive		
Next, subsequent	Consecutive integers	$n, n + 1$
Addition		
Sum, plus, greater, increased by, more than, exceeds, total of	The sum of 5 and x Seven more than y	$5 + x$ $y + 7$
Subtraction		
Difference, minus, less than, decreased by, subtracted from, reduced by, the remainder	Four decreased by b Three less than z Five subtracted from w	$4 - b$ $z - 3$ $w - 5$
Multiplication		
Product, multiplied by, twice, times, percent of	Two times x	$2x$
Division		
Quotient, divided by, per	The quotient of x and 8	$\frac{x}{8}$



Example 2 Constructing Mathematical Models

a. A salary of \$50,700 is increased by 6%. Write an equation that represents the new salary.

Verbal Model: New salary = 6%(Original salary) + Original salary

Labels:
Original salary = 50,700 (dollars)
New salary = S (dollars)
Percent = 0.06 (percent in decimal form)

Equation: $S = 0.06(50{,}700) + 50{,}700$

b. An e-reader is marked down 10% to \$225. Write an equation you can use to find the original price.

Verbal Model: Original price − 10%(Original price) = Sale price

Labels:
Original price = p (dollars)
Sale price = 225 (dollars)
Percent = 0.1 (percent in decimal form)

Equation: $p - 0.1p = 225$

Checkpoint 2

A bid price of \$6800 for an auction item is increased by 5%. Write an equation that you can use to find the new bid price.

In Example 2, notice that part of the labeling process is to list the unit of measure for each labeled quantity. Developing this habit helps in checking the validity of a verbal model.

Using Mathematical Models

Example 3 Finding the Percent of a Raise

You accept a job that pays $12.50 an hour. You are told that after a two-month probationary period, your hourly wage will be increased to $13 an hour. What percent raise will you receive after the two-month period?

SOLUTION

Verbal Model: Raise = Percent · Old wage

Labels:
Old wage = 12.5 (dollars)
Raise = 13 − 12.5 = 0.5 (dollars)
Percent = r (percent in decimal form)

Equation: $0.5 = r \cdot 12.5$

By solving this equation, you can find that you will receive a raise of

$$r = \frac{0.5}{12.5} = 0.04$$

or 4%.

Checkpoint 3

You buy stock at $25 per share. You sell the stock at $30 per share. What is the percent increase of the stock's value?

Example 4 Finding the Percent of Employees

The number of people employed by the U.S. federal government for a recent year is 2,730,040. The table shows the number of people employed in the executive branch of the federal government. What percent of employees in the U.S. federal government are in the executive branch? *(Source: U.S. Office of Personnel Management)*

	Number of Employees
Executive Office of the President	1717
Executive Departments	1,740,979
Independent Agencies	923,744

SOLUTION

Verbal Model: Executive branch employees = Percent · Federal government employees

Labels:
Federal government employees = 2,730,040 (people)
Executive branch employees = 2,666,440 (people)
Percent = r (percent in decimal form)

Equation: $2{,}666{,}440 = r \cdot 2{,}730{,}040$

By solving this equation, you can find that the percent of federal government employees in the executive branch is $r = 2{,}666{,}440/2{,}730{,}040$, or about 97.7%.

Checkpoint 4

Your income last year was $42,000. By the end of the year, you paid a total of $648 for parking fees. What percent of your income was paid for parking fees?



The President presides over the executive branch of the federal government, which employs more than 2.5 million people.

Vladislav Gurfinkel/Shutterstock.Com

Example 5 Finding the Dimensions of a Room

A rectangular family room is twice as long as it is wide, and its perimeter is 84 feet. Find the dimensions of the family room.

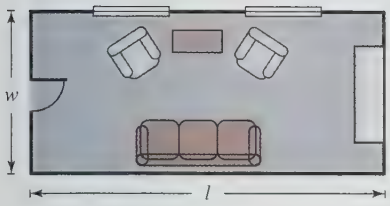


FIGURE 1.2

SOLUTION For this problem, it helps to sketch a diagram, as shown in Figure 1.2.

Verbal Model: $2 \cdot \text{Length} + 2 \cdot \text{Width} = \text{Perimeter}$

Labels:
Perimeter = 84 (feet)
Width = w (feet)
Length = $l = 2w$ (feet)

Equation:
$$2(2w) + 2w = 84$$
$$4w + 2w = 84$$
$$6w = 84$$
$$w = 14 \text{ feet}$$
$$l = 2w = 2(14) = 28 \text{ feet}$$

The dimensions of the room are 14 feet by 28 feet.

Checkpoint 5

A rectangular driveway is three times as long as it is wide, and its perimeter is 120 feet. Find the dimensions of the driveway.

Example 6 A Distance Problem

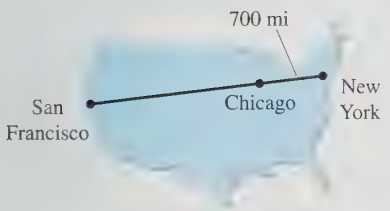


FIGURE 1.3

A plane travels nonstop from New York to San Francisco, a distance of about 2600 miles. It takes 1.5 hours to fly from New York to Chicago, a distance of about 700 miles (see Figure 1.3). Assuming the plane flies at a constant speed, how long does the entire trip take? What time (EST) should the plane leave New York to arrive in San Francisco by 5 P.M. PST (8 P.M. EST)?

SOLUTION To solve this problem, use the formula that relates distance, rate, and time. That is, (distance) = (rate)(time). Because it took the plane 1.5 hours to travel a distance of about 700 miles, you can conclude that its rate (or speed) is

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$
$$= \frac{700 \text{ miles}}{1.5 \text{ hours}} \approx 466.67 \text{ miles per hour.}$$

Because the entire trip is about 2600 miles, the time for the entire trip is

$$\text{Time} = \frac{\text{distance}}{\text{rate}}$$
$$= \frac{2600 \text{ miles}}{466.67 \text{ miles per hour}} \approx 5.57 \text{ hours.}$$

Because 0.57 hour represents about 34 minutes, you can conclude that the entire trip takes about 5 hours and 34 minutes. So, the plane must leave New York by 2:26 P.M. EST in order to arrive in San Francisco by 8 P.M. EST.

Checkpoint 6

A boat travels at full speed to an island 11 miles away. It takes 0.3 hour to travel the first 3 miles. How long does the entire trip take?

Another way to solve the distance problem in Example 6 is to use the concept of **ratio and proportion.** To do this, let x represent the time required to fly from New York to San Francisco, set up the following proportion, and solve for x.

$$\frac{\text{Time to San Francisco}}{\text{Time to Chicago}} = \frac{\text{Distance to San Francisco}}{\text{Distance to Chicago}}$$

$$\frac{x}{1.5} = \frac{2600}{700}$$

$$1.5 \cdot \frac{x}{1.5} = 1.5 \cdot \frac{2600}{700}$$

$$x \approx 5.57$$

Notice how ratio and proportion are used with a property from geometry to solve the problem in the following example.

Example 7 An Application Involving Similar Triangles

To determine the height of Petronas Tower 1 (in Kuala Lumpur, Malaysia), you measure the shadow cast by the building to be 113 meters long, as shown in Figure 1.4. Then you measure the shadow cast by a 100-centimeter post and find that its shadow is 25 centimeters long. Use this information to determine the height of Petronas Tower 1.

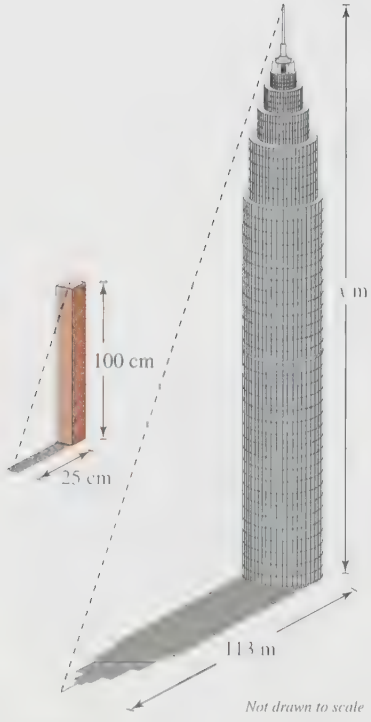


FIGURE 1.4

SOLUTION To find the height of the tower, you can use a property from geometry that states that the ratios of corresponding sides of similar triangles are equal.

Verbal Model:
$$\frac{\text{Height of tower}}{\text{Length of tower's shadow}} = \frac{\text{Height of post}}{\text{Length of post's shadow}}$$

Labels:
Height of tower = x (meters)
Length of tower's shadow = 113 (meters)
Height of post = 100 (centimeters)
Length of post's shadow = 25 (centimeters)

Equation:
$$\frac{x}{113} = \frac{100}{25}$$

$$113 \cdot \frac{x}{113} = 113 \cdot \frac{100}{25}$$

$$x = 113 \cdot 4$$

$$x = 452 \text{ meters}$$

The Petronas Tower 1 is 452 meters high.

CHECK

$$\frac{x}{113} = \frac{100}{25}$$ Write original equation.

$$\frac{452}{113} = \frac{100}{25}$$ Substitute 452 for x.

$4 = 4$ Solution checks. ✓

✓ Checkpoint 7

A tree casts a shadow that is 24 feet long. At the same time, a four-foot tall mailbox casts a shadow that is 3 feet long. How tall is the tree? ■

Mixture Problems

The next example is called a **mixture problem** because it involves two different unknown quantities that are *mixed* in a specific way. Watch for a *hidden product* in the verbal model.

TECH TUTOR



You can write a program for a programmable calculator to solve simple interest problems. Use a program with Example 8 to find how much interest was earned on just the portion of the money invested at $2\frac{1}{2}\%$.

Example 8 A Simple Interest Problem

You invested a total of \$10,000 in accounts that earned $1\frac{1}{2}\%$ and $2\frac{1}{2}\%$ simple interest. In 1 year, the two accounts earned \$207.50 in interest. How much did you invest in each account?

SOLUTION The formula for simple interest is

$$I = Prt$$

where I is the interest, P is the principal, r is the annual interest rate (in decimal form), and t is the time in years.

Verbal Model: Interest from $1\frac{1}{2}\%$ + Interest from $2\frac{1}{2}\%$ = Total interest

You can let x represent the amount invested at $1\frac{1}{2}\%$. Because the total amount invested at $1\frac{1}{2}\%$ and $2\frac{1}{2}\%$ is \$10,000, you can let $10{,}000 - x$ represent the amount invested at $2\frac{1}{2}\%$.

Labels:

Amount invested at $1\frac{1}{2}\% = x$ (dollars)

Amount invested at $2\frac{1}{2}\% = 10{,}000 - x$ (dollars)

Interest from $1\frac{1}{2}\% = Prt = (x)(0.015)(1)$ (dollars)

Interest from $2\frac{1}{2}\% = Prt = (10{,}000 - x)(0.025)(1)$ (dollars)

Total interest $= 207.50$ (dollars)

Equation:

$$\begin{aligned} 0.015x + 0.025(10{,}000 - x) &= 207.50 \\ 0.015x + 250 - 0.025x &= 207.50 \\ -0.01x &= -42.50 \\ x &= 4250 \end{aligned}$$

STUDY TIP

In Example 8, notice the hidden products in the two terms on the left side of the equation. Both hidden products come from the common formula $I = Prt$.

So, the amount invested at $1\frac{1}{2}\%$ is \$4250 and the amount invested at $2\frac{1}{2}\%$ is

$$\begin{aligned} 10{,}000 - x &= 10{,}000 - 4250 \\ &= \$5750. \end{aligned}$$

Check these results in the original statement of the problem.

CHECK

$$\underbrace{0.015(4250)}_{\text{Interest from } 1\frac{1}{2}\%} + \underbrace{0.025(10{,}000 - 4250)}_{\text{Interest from } 2\frac{1}{2}\%} \stackrel{?}{=} \underbrace{207.50}_{\text{Total interest}}$$

$$63.75 + 143.75 \stackrel{?}{=} 207.50$$

$$207.50 = 207.50$$

Solution checks. ✓

✓ Checkpoint 8

You invested a total of \$2500 in accounts that earned 1% and 2% simple interest. In 1 year, the two accounts earned a total of \$38 in interest. How much did you invest in each account?

Common Formulas

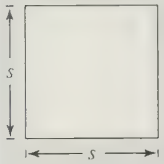
Many common types of geometric, scientific, and investment problems use ready-made equations, called **formulas.** Knowing formulas such as those in the following lists will help you translate and solve a wide variety of real-life problems involving perimeter, area, volume, temperature, interest, and distance.

Common Formulas for Area, Perimeter, and Volume

Square

$A = s^2$

$P = 4s$



Rectangle

$A = lw$

$P = 2l + 2w$



Circle

$A = \pi r^2$

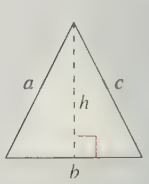
$C = 2\pi r$



Triangle

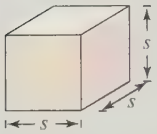
$A = \frac{1}{2}bh$

$P = a + b + c$



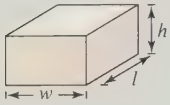
Cube

$V = s^3$



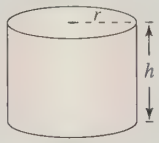
Rectangular Solid

$V = lwh$



Circular Cylinder

$V = \pi r^2 h$



Sphere

$V = \frac{4}{3}\pi r^3$



Miscellaneous Common Formulas

Temperature: $F = \frac{9}{5}C + 32$

$C = \frac{5}{9}(F - 32)$

F = degrees Fahrenheit
C = degrees Celsius

Simple Interest: $I = Prt$

I = interest
P = principal
r = interest rate
t = time

Distance: $d = rt$

d = distance traveled
r = rate
t = time

When working with applied problems, you often need to rewrite common formulas. For instance, the formula

$$P = 2l + 2w$$

for the perimeter of a rectangle can be rewritten or solved for w to produce

$$w = \tfrac{1}{2}(P - 2l).$$

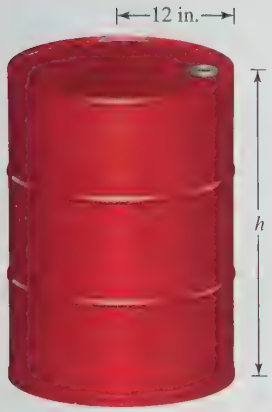


FIGURE 1.5

Example 9 Using a Formula

A cylindrical shipping container has a volume of 15,607 cubic inches and a radius of 12 inches, as shown in Figure 1.5. Find the height of the container.

SOLUTION The formula for the *volume of a cylinder* is $V = \pi r^2 h$. To find the height of the container, solve for h.

$$h = \frac{V}{\pi r^2}$$

Then, using $V = 15{,}607$ and $r = 12$, find the height.

$$h = \frac{15{,}607}{\pi(12)^2}$$ Substitute 15,607 for V and 12 for r.

$$= \frac{15{,}607}{144\pi}$$ Simplify denominator.

$$\approx 34.5$$ Use a calculator.

So, the height of the container is about 34.5 inches. You can use unit analysis to check that your answer is reasonable.

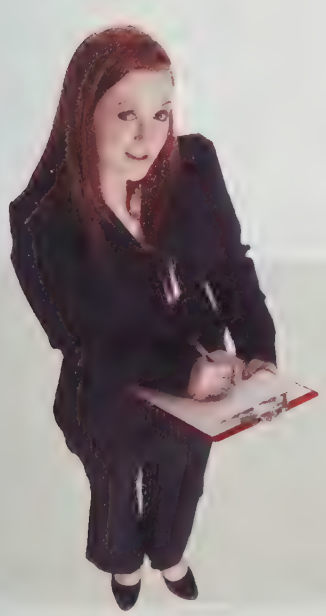
$$\frac{15{,}607 \text{ in.}^3}{144\pi \text{ in.}^2} \approx 34.5 \text{ in.}$$

Checkpoint 9

One cubic foot of water fills a cylindrical pipe with a radius of 0.5 foot. What is the height of the pipe?

Strategy for Solving Word Problems

1. *Search* for the hidden equality—two expressions said to be equal or known to be equal. A sketch may be helpful.
2. *Write* a verbal model that equates these two expressions. Identify any *hidden* products.
3. *Assign* numbers to the known quantities and letters (or algebraic expressions) to the unknown quantities.
4. *Rewrite* the verbal model as an algebraic equation using the assigned labels.
5. *Solve* the resulting algebraic equation.
6. *Check* that the answer satisfies the word problem as stated. (Remember that "solving for x" or another variable may not completely answer the question.)



SUMMARIZE (Section 1.2)

1. Describe the procedure of mathematical modeling *(page 80)*. For examples of constructing and using mathematical models, see Examples 1, 2, 3, 4, 5, 6, and 7.
2. Describe a mixture problem *(page 85)*. For an example of a mixture problem, see Example 8.
3. State the definition of a formula *(page 86)*. For an example that uses a volume formula, see Example 9.

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SKILLS WARM UP 1.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 1.1.

In Exercises 1–10, solve the equation (if possible) and check your solution.

1. $3x - 42 = 0$
2. $64 - 16x = 0$
3. $2 - 3x = 14 + x$
4. $7 + 5x = 7x - 1$
5. $5[1 + 2(x + 3)] = 6 - 3(x - 1)$
6. $2 - 5(x - 1) = 2[x + 10(x - 1)]$
7. $\frac{x}{3} + \frac{x}{2} = \frac{1}{3}$
8. $\frac{2}{x} + \frac{2}{5} = 1$
9. $1 - \frac{2}{z} = \frac{z}{z + 3}$
10. $\frac{x}{x + 1} - \frac{1}{2} = \frac{4}{3}$

Exercises 1.2

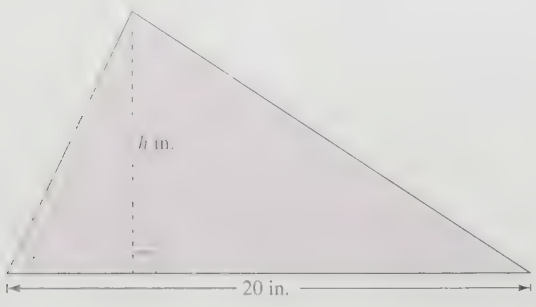
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Writing an Algebraic Expression In Exercises 1–6, (a) use the verbal description to write a verbal model, (b) assign labels to the quantities in the verbal model, and (c) use the labels to write an algebraic expression.

1. The sum of two consecutive natural numbers
2. The product of two natural numbers whose sum is 25
3. **Distance Traveled** The distance traveled in t hours by a car traveling at 50 miles per hour
4. **Travel Time** The travel time for a plane that is traveling at a rate of r miles per hour for 200 miles
5. **Geometry** The perimeter of a rectangle whose width is x and whose length is twice the width



6. **Geometry** The area of a triangle whose base is 20 inches and whose height is h inches



Writing an Algebraic Equation In Exercises 7–12, write an equation that represents the verbal description. *See Example 2.*

7. The sum of 5 and x equals 8.
8. The difference of n and 7 is 4.
9. The quotient of r and 2 is 9.
10. The product of x and 6 equals -9.
11. A salary of \$60,000 is increased by 8%.
12. A food processor is marked down 30% to \$105.

Constructing a Mathematical Model In Exercises 13–22, (a) use the verbal description to write a verbal model, (b) assign labels to the quantities in the verbal model, (c) use the labels to write a mathematical model, and (d) solve the problem. *See Examples 3 and 4.*

13. Find two consecutive numbers whose sum is 525.
14. Find three consecutive natural numbers whose sum is 804.
15. One positive number is five times another positive number. The difference of the two numbers is 148. Find the numbers.
16. One positive number is one-fifth of another number. The difference of the two numbers is 76. Find the numbers.
17. **Salary** Your weekly salary increases from \$440 to \$506. What percent is your raise?
18. **Salary** Your monthly salary increases from \$2200 to \$2376. What percent is your raise?
19. **Discount Rate** A satellite radio system for your car has been discounted by \$30. The sale price is \$119. What percent of the original list price is the discount?
20. **Discount Rate** The price of a shirt has been discounted by \$20. The sale price is \$29.95. What percent of the original list price is the discount?

21. Weekly Paycheck Your weekly paycheck is 12% *more* than your coworker's. Your two paychecks total $848. Find the amount of each paycheck.

22. Weekly Paycheck Your weekly paycheck is 12% *less* than your coworker's. Your two paychecks total $848. Find the amount of each paycheck.

Movie Series In Exercises 23–28, use the table, which shows Harry Potter movies in the order of their releases. Find the percent increase or decrease in the U.S. gross revenue of the two indicated movies. *(Source: Nash Information Services)*

Movie Title: *Harry Potter and the . . .*	U.S. gross revenue
Sorcerer's Stone (2001)	$317,557,891
Chamber of Secrets (2002)	$261,987,880
Prisoner of Azkaban (2004)	$249,538,952
Goblet of Fire (2005)	$290,013,036
Order of the Phoenix (2007)	$292,004,738
Half-Blood Prince (2009)	$301,959,197

23. *Sorcerer's Stone* (2001) to *Chamber of Secrets* (2002)

24. *Chamber of Secrets* (2002) to *Prisoner of Azkaban* (2004)

25. *Prisoner of Azkaban* (2004) to *Goblet of Fire* (2005)

26. *Goblet of Fire* (2005) to *Order of the Phoenix* (2007)

27. *Order of the Phoenix* (2007) to *Half-Blood Prince* (2009)

28. *Sorcerer's Stone* (2001) to *Half-Blood Prince* (2009)

Size Inflation In Exercises 29–32, use the following information. Restaurants tend to serve food in larger portions now than they have in the past. Several examples are shown in the table. Find the percent increase in size from the past to 2011 for the indicated food item.

Food or drink item	Past size	2011 size
Small soft drink	7 fl oz	16 fl oz
Small French fries	2.4 oz	2.5 oz
Large French fries	3.5 oz	5.4 oz
Pizza	10 in.	12 in.

29. Small soft drink

30. Small French fries

31. Large French fries

32. Pizza (Compare the areas.)

33. Comparing Calories A lunch consisting of a large deluxe cheeseburger, large fries, and a large soft drink contains 1560 calories. A lunch consisting of a small cheeseburger, small fries, and a small soft drink contains 680 calories. Find the percent change in calories from the larger to the smaller lunch.

34. Comparing Calories One slice (or one-eighth) of a 14-inch cheese pizza has 250 calories. One slice of a pepperoni pizza has 280 calories. Find the percent change in calories from a slice of the cheese pizza to a slice of the pepperoni pizza.

35. Salary You accept a new job with a starting salary of $35,000. You receive an 8% raise at the start of your second year, a 7.8% raise at the start of your third year, and a 9.4% raise at the start of your fourth year.

(a) Find your salary for the second year.

(b) Find your salary for the third year.

(c) Find your salary for the fourth year.

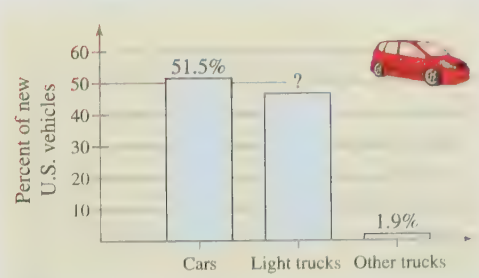
36. Salary You accept a new job with a starting salary of $48,000. You receive a 4% raise at the start of your second year, a 5.5% raise at the start of your third year, and an 11.4% raise at the start of your fourth year.

(a) Find your salary for the second year.

(b) Find your salary for the third year.

(c) Find your salary for the fourth year.

37. New Vehicle Sales In 2009, about 10,601,000 new motor vehicles (cars, light trucks, and other types of trucks) were sold or leased in the United States. The bar graph shows the percent of vehicles sold or leased by type. *(Source: U.S. Bureau of Economic Analysis)*

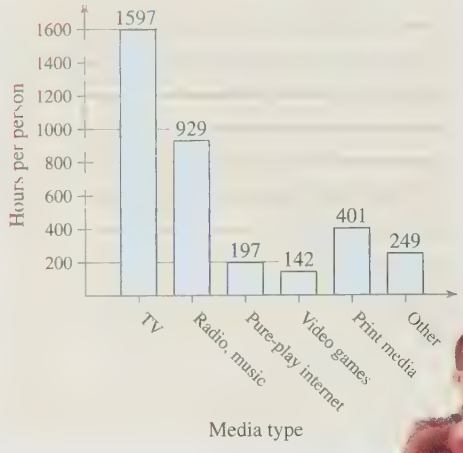


(a) Determine the percent of new motor vehicles that were light trucks.

(b) Determine the numbers of new cars, light trucks, and other types of trucks that were sold or leased in 2009.

(c) Determine the percent of new trucks that were light trucks.

38. Media Usage It is projected that the average person will spend 3515 hours using various media in 2012. The bar graph shows these hours by media type (a person can use more than one type of media at a time). *(Source: Veronis Suhler Stevenson)*





(a) Find the percent of the total media time spent reading print media.

(b) Find the percent of the total media time spent on something other than reading print media.

(c) Find the percent of the total media time spent playing video games and watching TV.

39. Geometry A picture frame (see figure) has a total perimeter of 3 feet. The width of the frame is 0.62 times its length. Find the dimensions of the frame.



40. Geometry A room is 1.5 times as long as it is wide, and its perimeter is 75 feet. Sketch a diagram of the room. Then find the dimensions of the room.

41. Simple Interest You invest $2500 at 4.5% simple interest. How many years will it take for the investment to earn $1000 in interest?

42. Simple Interest An investment earns $1075.20 interest over a seven-year period. What is the rate of simple interest on a $4800 principal investment?

43. Course Grade To get an A in a course, you need an average of 90% or better on four tests that are worth 100 points each. Your scores on the first three tests were 87, 92, and 84. What must you score on the fourth test to get an A for the course?

44. Course Grade To get an A in a course, you need an average of 90% or better on four tests. The first three tests are worth 100 points each and the fourth is worth 200 points. Your scores on the first three tests are 87, 92, and 84. What must you score on the fourth test to get an A for the course?

45. List Price The price of a swimming pool has been discounted 15%. The sale price is $1200. Find the original list price of the swimming pool.

46. List Price The price of a home theater system has been discounted 10%. The sale price is $499. Find the original list price of the system.

47. Wholesale Price A store marks up an electric power drill 60% from its wholesale price. In a clearance sale, the price is discounted by 25%. The sale price is $21.60. What was the wholesale price of the power drill?

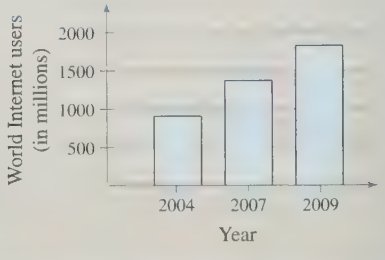
48. Wholesale Price A store marks up a picture frame 80% from its wholesale price. In a clearance sale, the price is discounted by 40%. The sale price is $28.08. What was the wholesale price of the picture frame?

49. Weekly Salary Due to economic factors, your employer has reduced your weekly wage by 15%. Before the reduction, your weekly salary was $425.

(a) What is your reduced salary?

(b) What percent raise must you receive to bring your weekly salary back up to $425? Explain why the percent raise is different from the percent reduction.

50. HOW DO YOU SEE IT? The number of Internet users in the world increased by approximately the same amount from 2004 to 2007 as from 2007 to 2009, as shown in the bar graph. Was the *percent* increase from 2004 to 2007 approximately the same as from 2007 to 2009? Explain. *(Source: ITU World Telecommunication)*

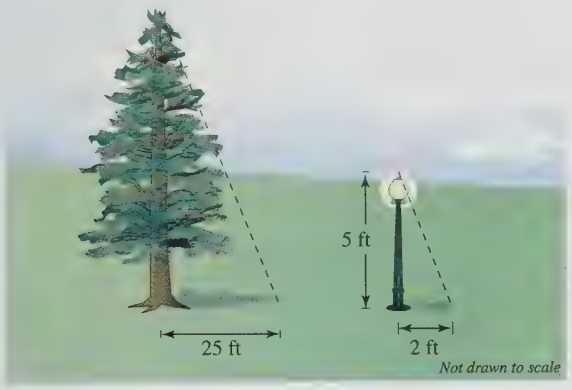


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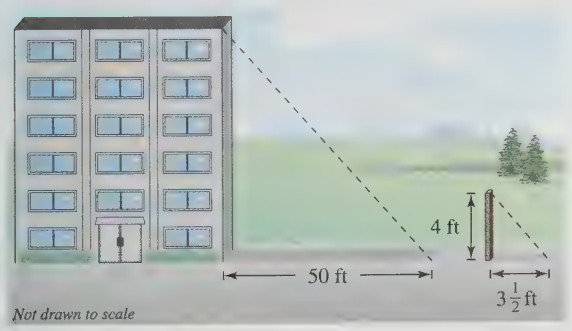
51. Travel Time You are driving to a college 150 miles from home. It takes 28 minutes to travel the first 30 miles. At this rate, how long is your entire trip?

52. Catch-Up Time Students are traveling in two cars to a football game 135 miles away. One car travels at an average speed of 45 miles per hour. The second car starts $\frac{1}{2}$ hour later and travels at an average speed of 55 miles per hour. How long will it take the second car to catch up to the first car?

53. Height of a Tree To determine the height of a tree, you measure its shadow and the shadow of a five-foot lamppost, as shown in the figure. How tall is the tree?



54. Height of a Building To determine the height of a building, you measure the building's shadow and the shadow of a four-foot stake, as shown in the figure. How tall is the building?



55. Radio Waves Radio waves travel at the same speed as light, 3.0×10^8 meters per second. Find the time required for a radio wave to travel from mission control in Houston to NASA astronauts on the surface of the moon 3.84×10^8 meters away.

56. Distance to a Star Find the distance (in miles) to a star that is 50 light years (distance traveled by light in 1 year) away. (Light travels at 186,000 miles per second.)

57. Investment Mix You invest \$15,000 in two funds paying 6.5% and 7.5% simple interest. The total annual interest is \$1020. How much do you invest in each fund?

58. Investment Mix You invest \$30,000 in two funds paying 3% and $4\frac{1}{2}$% simple interest. The total annual interest is \$1230. How much do you invest in each fund?

59. Stock Mix You invest \$5000 in two stocks. In one year, the value of stock A increases by 9.8% and the value of stock B increases by 6.2%. The total value of the stocks is now \$5389.20. How much did you originally invest in each stock?

60. Stock Mix You invest \$4000 in two stocks. In one year, the value of stock A increases by 5.4% and the value of stock B increases by 12.8%. The total value of the stocks is now \$4401. How much did you originally invest in each stock?

Production Limit In Exercises 61 and 62, use the following information. *Variable costs* depend on the number of units produced. *Fixed costs* are the same regardless of how many units are produced. Find the greatest number of units the company can produce each month.

61. The company has fixed monthly costs of \$15,000 and variable monthly costs of \$8.75 per unit. The company has \$90,000 available each month to cover costs.

62. The company has fixed monthly costs of \$10,000 and variable monthly costs of \$9.30 per unit. The company has \$85,000 available each month to cover costs.

Rewriting Formulas In Exercises 63–70, solve for the indicated variable.

63. Area of a Triangle
Solve for h in $A = \frac{1}{2}bh$.

64. Volume of a Rectangular Prism
Solve for l in $V = lwh$

65. Perimeter of a Rectangle
Solve for l in $P = 2l + 2w$.

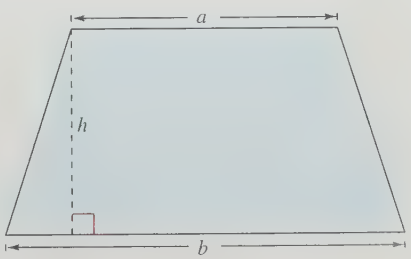
66. Markup
Solve for C in $S = C + RC$.

67. Discount
Solve for L in $S = L - RL$.

68. Investment at Simple Interest
Solve for r in $A = P + Prt$.

69. Investment at Compound Interest
Solve for P in $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

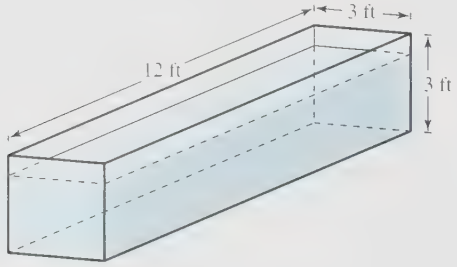
70. Area of a Trapezoid
Solve for b in $A = \frac{1}{2}(a + b)h$.



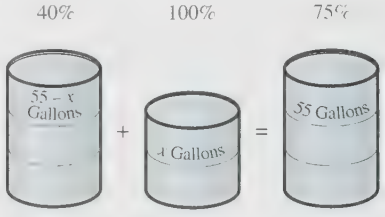
71. **Length of a Tank** The diameter of a cylindrical propane gas tank is 4 feet (see figure). The total volume of the tank is 603.2 cubic feet. Find the length of the tank.



72. **Water Depth** A trough is 12 feet long, 3 feet deep, and 3 feet wide (see figure). Find the depth of the water when the trough contains 70 gallons of water. (1 gallon ≈ 0.13368 cubic foot.)



73. **Mixture** A 55-gallon barrel contains a mixture with a concentration of 40%. How much of this mixture must be withdrawn and replaced by 100% concentrate to bring the mixture up to 75% concentration? (See figure.)



74. **Fuel Mixture** You have two gallons of a fuel mixture that is 32 parts gasoline and 1 part two-cycle engine oil. How much gasoline do you need to add to form a new mixture that is 40 parts gasoline and 1 part oil?

75. **Monthly Sales** The table shows the monthly sales of a sales team for the second quarter of the calendar year. Find the average monthly sales for each salesperson. Then find the team's average sales for each month.

Name	April	May	June
Williams	$25,000	$28,800	$21,000
Gonzalez	$26,200	$27,800	$29,500
Walters	$26,600	$23,400	$26,900
Gilbert	$27,100	$22,200	$29,000
Hart	$23,100	$27,400	$22,800

Courtesy of CitiKitty, Inc

76. **Monthly Sales** The table shows the monthly sales of a sales team for the fourth quarter of the calendar year. Use a spreadsheet software program to find the average monthly sales for each salesperson. Then find the team's average sales for each month.

Name	October	November	December
Williams	$20,000	$25,100	$23,900
Gonzalez	$24,200	$23,600	$18,500
Walters	$31,900	$23,800	$18,400
Gilbert	$24,600	$23,100	$30,700
Hart	$32,400	$19,100	$28,600

77. **Think About It** Textbooks usually give just the information that is needed to solve a problem. In real life, you must often sort through a greater amount of information and discard irrelevant facts, called **red herrings.** Find any red herrings in the following problem.

Beneath the ocean surface, pressure changes at a rate of approximately 4.4 pounds per square inch for every 10-foot change in depth. A diver takes 30 minutes to ascend 25 feet from a depth of 150 feet. What change in pressure does the diver experience?



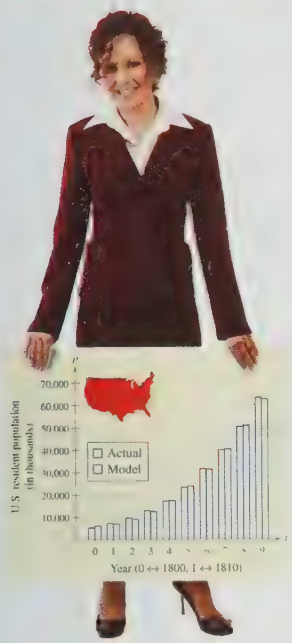
Business Capsule

CitiKitty, Inc. was founded in 2005 by 26-year-old Rebecca Rescate after she moved into a small apartment in New York City with no place to hide her cat's litter box. Finding no easy-to-use cat toilet training kit, she created one, and CitiKitty was born with an initial investment of $20,000. Today the company flourishes with an expanded product line. Revenues in 2010 reached $350,000.

78. **Research Project** Use your school's library, the Internet, or some other reference source to find information about the start-up costs of beginning a business, such as the example above. Write a short paper about the company.

1.3 Quadratic Equations

- Solve a quadratic equation by factoring.
- Solve a quadratic equation by extracting square roots.
- Construct and use a quadratic model to solve an application problem.



In Exercise 87 on page 103, you will use a quadratic model to determine the population of the United States.

Solving Quadratic Equations by Factoring

In the first two sections of this chapter, you studied linear equations in one variable. In this and the next section, you will study quadratic equations.

Definition of a Quadratic Equation

A **quadratic equation** in x is an equation that can be written in the general form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. Another name for a quadratic equation in x is a **second-degree polynomial equation in x.**

There are three basic techniques for solving quadratic equations: factoring, extracting square roots, and the *Quadratic Formula*. (The Quadratic Formula is discussed in the next section.) The factoring technique is based on the following property.

Zero-Factor Property

If $ab = 0$, then $a = 0$ or $b = 0$.

To use this property, rewrite the left side of the general form of a quadratic equation as the product of two linear factors. Then find the solutions of the quadratic equation by setting each linear factor equal to zero.

STUDY TIP

The Zero-Factor Property applies *only* to equations written in general form (in which one side of the equation is zero). So, be sure that all terms are collected on one side *before* factoring. For instance, in the equation

$$(x - 5)(x + 2) = 8$$

it is *incorrect* to set each factor equal to 8. Can you solve this equation correctly?

Example 1 Solving a Quadratic Equation by Factoring

Solve $x^2 - 3x - 10 = 0$.

SOLUTION

$x^2 - 3x - 10 = 0$		Write original equation.
$(x - 5)(x + 2) = 0$		Factor.
$x - 5 = 0$	$\Rightarrow x = 5$	Set 1st factor equal to 0.
$x + 2 = 0$	$\Rightarrow x = -2$	Set 2nd factor equal to 0.

The solutions are $x = 5$ and $x = -2$. Check these in the original equation.

Checkpoint 1

Solve each equation.

a. $x^2 - x - 12 = 0$

b. $x^2 + 10x + 24 = 0$

Karlova Irina/www.shutterstock.com

Example 2 Solving a Quadratic Equation by Factoring

Solve $6x^2 - 3x = 0$.

SOLUTION

$6x^2 - 3x = 0$		Write original equation.
$3x(2x - 1) = 0$		Factor out common factor.
$3x = 0$	$x = 0$	Set 1st factor equal to 0.
$2x - 1 = 0$	$x = \frac{1}{2}$	Set 2nd factor equal to 0.

The solutions are $x = 0$ and $x = \frac{1}{2}$.

CHECK

$6x^2 - 3x = 0$	Write original equation.
$6(0)^2 - 3(0) \stackrel{?}{=} 0$	Substitute 0 for x.
$0 - 0 = 0$	First solution checks. ✓
$6\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) \stackrel{?}{=} 0$	Substitute $\frac{1}{2}$ for x.
$\frac{6}{4} - \frac{3}{2} = 0$	Second solution checks. ✓

✓ Checkpoint 2

Solve $4x^2 - 8x = 0$.

If the two factors of a quadratic expression are the same, the corresponding solution of the quadratic equation is a **double** or **repeated** solution.

Example 3 A Quadratic Equation with a Repeated Solution

Solve $9x^2 - 6x = -1$.

SOLUTION

$9x^2 - 6x = -1$	Write original equation.
$9x^2 - 6x + 1 = 0$	Write in general form.
$(3x - 1)^2 = 0$	Factor.
$3x - 1 = 0$	Set repeated factor equal to 0.
$x = \frac{1}{3}$	Solution

The only solution is $x = \frac{1}{3}$.

CHECK

$9x^2 - 6x = -1$	Write original equation.
$9\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{3}\right) \stackrel{?}{=} -1$	Substitute $\frac{1}{3}$ for x.
$1 - 2 \stackrel{?}{=} -1$	Simplify.
$-1 = -1$	Solution checks. ✓

✓ Checkpoint 3

Solve $x^2 + 4x = -4$.

TECH TUTOR

To check the solution in Example 3 with your graphing utility, you should first write the equation in general form.

$$9x^2 - 6x + 1 = 0$$

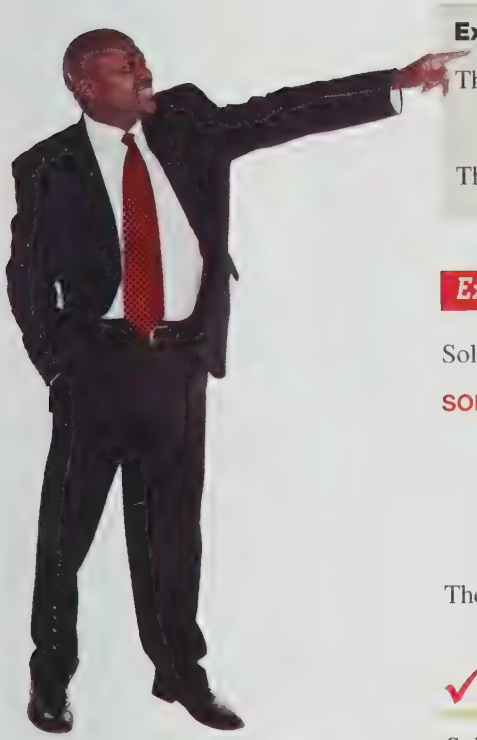
Then enter the expression $9x^2 - 6x + 1$ into y_1 of the equation editor. Now you can use the ASK mode of the *table* feature of your graphing utility to check the solution. For instructions on how to use the *table* feature, see Appendix A.

Extracting Square Roots

There is a shortcut for solving equations of the form $u^2 = d$, where $d > 0$. By factoring, you can see that this equation has two solutions.

$$u^2 = d \qquad \text{Write original equation}$$
$$u^2 - d = 0 \qquad \text{Write in general form}$$
$$(u + \sqrt{d})(u - \sqrt{d}) = 0 \qquad \text{Factor.}$$
$$u + \sqrt{d} = 0 \Rightarrow u = -\sqrt{d} \qquad \text{Set 1st factor equal to 0}$$
$$u - \sqrt{d} = 0 \Rightarrow u = \sqrt{d} \qquad \text{Set 2nd factor equal to 0}$$

Solving an equation of the form $u^2 = d$ without going through the steps of factoring is called **extracting square roots.**



Extracting Square Roots

The equation $u^2 = d$, where $d > 0$, has exactly two solutions:

$$u = \sqrt{d} \quad \text{and} \quad u = -\sqrt{d}.$$

These solutions can also be written as $u = \pm\sqrt{d}$.

Example 4 Extracting Square Roots

Solve $4x^2 = 12$.

SOLUTION

$$4x^2 = 12 \qquad \text{Write original equation.}$$
$$x^2 = 3 \qquad \text{Divide each side by 4.}$$
$$x = \pm\sqrt{3} \qquad \text{Extract square roots.}$$

The solutions are $x = \sqrt{3}$ and $x = -\sqrt{3}$. Check these in the original equation.

✓Checkpoint 4

Solve $2x^2 = 8$.

Example 5 Extracting Square Roots

Solve $(x - 3)^2 = 7$.

SOLUTION

$$(x - 3)^2 = 7 \qquad \text{Write original equation.}$$
$$x - 3 = \pm\sqrt{7} \qquad \text{Extract square roots}$$
$$x = 3 \pm \sqrt{7} \qquad \text{Add 3 to each side}$$

The solutions are $x = 3 \pm \sqrt{7}$. Check these in the original equation.

✓Checkpoint 5

Solve $(x - 1)^2 = 16$.

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Applications

Quadratic equations often occur in problems dealing with area. Here is a simple example.

A square room has an area of 144 square feet. Find the dimensions of the room.

To solve this problem, let x represent the length of each side of the room. Then use the formula for the area of a square to write and solve the equation

$$x^2 = 144$$

to conclude that each side of the room is 12 feet long. Note that although the equation $x^2 = 144$ has two solutions, $x = -12$ and $x = 12$, the negative solution makes no sense (for this problem), so you should choose the positive solution.

Example 6 Finding the Dimensions of a Room

A rectangular sunroom is 3 feet longer than it is wide (see Figure 1.6) and has an area of 154 square feet. Find the dimensions of the room.

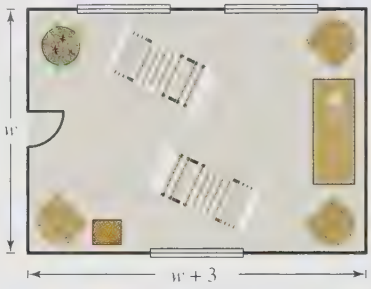


FIGURE 1.6

SOLUTION Use the same type of problem-solving strategy that was presented in Section 1.2.

Verbal Model: Width of room · Length of room = Area of room

Labels:
Area of room = 154 (square feet)
Width of room = w (feet)
Length of room = $w + 3$ (feet)

Equation:

$$w(w + 3) = 154$$

$$w^2 + 3w - 154 = 0$$

$$(w - 11)(w + 14) = 0$$

$$w - 11 = 0 \Rightarrow w = 11$$

$$w + 14 = 0 \Rightarrow w = -14$$

Choosing the positive value, you can conclude that the width is 11 feet and the length is $w + 3 = 14$ feet. Check this in the original statement of the problem, as shown.

CHECK

The length of 14 feet is 3 feet more than the width of 11 feet. ✓

The area of the sunroom is $(11)(14) = 154$ square feet. ✓

✓ Checkpoint 6

A rectangular kitchen is 8 feet longer than it is wide and has an area of 84 square feet. Find the dimensions of the kitchen.

STUDY TIP

The position equation described here ignores air resistance. This implies that it is appropriate to use the position equation only to model falling objects that have little air resistance and that fall over short distances.

Another application of quadratic equations involves an object that is falling (or is vertically projected into the air). The equation that gives the height of such an object is called a **position equation,** and on Earth's surface it has the form

$$s = -16t^2 + v_0t + s_0.$$

In this equation, s represents the height of the object (in feet), v_0 represents the initial velocity of the object (in feet per second), s_0 represents the initial height of the object (in feet), and t represents the time (in seconds).

Example 7 Falling Object

A construction worker accidentally drops a wrench from a height of 235 feet and yells "Look out below!" (see Figure 1.7). Could a person at ground level hear this warning in time to get out of the way of the falling wrench?

SOLUTION Because sound travels at about 1100 feet per second, it follows that a person at ground level hears the warning within 1 second of the time the wrench is dropped. To set up a mathematical model for the height of the wrench, use the position equation

$$s = -16t^2 + v_0t + s_0.$$

Because the object is dropped rather than thrown, the initial velocity is $v_0 = 0$ feet per second. Moreover, because the initial height is $s_0 = 235$ feet, you have the following model.

$$\begin{aligned} s &= -16t^2 + (0)t + 235 \\ &= -16t^2 + 235 \end{aligned}$$

After falling for 1 second, the height of the wrench is

$$-16(1)^2 + 235 = 219 \text{ feet.}$$

After falling for 2 seconds, the height of the wrench is

$$-16(2)^2 + 235 = 171 \text{ feet.}$$

To find the number of seconds it takes the wrench to hit the ground, let the height s be zero and solve the equation for t.

$s = -16t^2 + 235$	Write position equation.
$0 = -16t^2 + 235$	Substitute 0 for s.
$16t^2 = 235$	Add $16t^2$ to each side.
$t^2 = \frac{235}{16}$	Divide each side by 16.
$t = \frac{\sqrt{235}}{4}$	Extract positive square root.
$t \approx 3.83$	Use a calculator.

The wrench will take about 3.83 seconds to hit the ground. If the person hears the warning within 1 second after the wrench is dropped, the person still has almost 3 seconds to get out of the way.

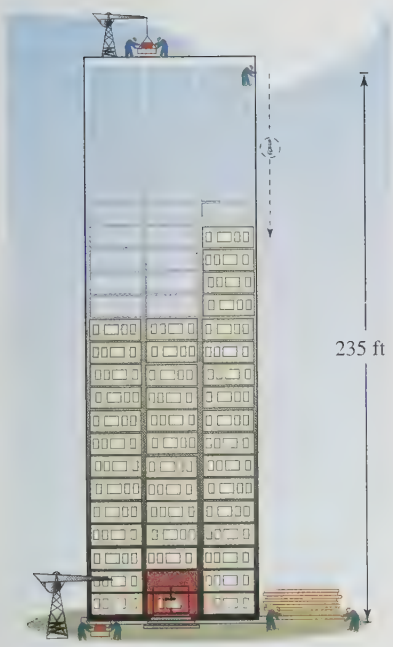


FIGURE 1.7

Checkpoint 7

You drop a rock from a height of 144 feet. How long does it take the rock to hit the ground?

A third type of application using a quadratic equation involves the hypotenuse of a right triangle. Recall from geometry that the sides of a right triangle are related by a formula called the **Pythagorean Theorem.** This theorem states that if a and b are the lengths of the legs of the triangle and c is the length of the hypotenuse (see Figure 1.8),

$$a^2 + b^2 = c^2.$$ Pythagorean Theorem

Notice how this formula is used in the next example.

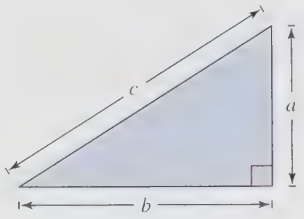


FIGURE 1.8



Example 8 **Cutting Across the Lawn**

An L-shaped sidewalk from the athletic center to the library on a college campus is shown in Figure 1.9. The sidewalk was constructed so that the length of one sidewalk forming the L is twice as long as the other. The length of the diagonal sidewalk that cuts across the grounds between the two buildings is 32 feet. How many feet does a person save by walking on the diagonal sidewalk?

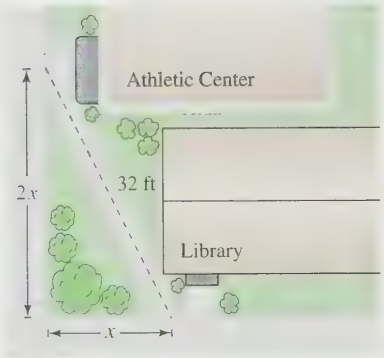


FIGURE 1.9

SOLUTION Use the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$ Pythagorean Theorem

$$x^2 + (2x)^2 = 32^2$$ Substitute for a, b, and c.

$$5x^2 = 1024$$ Combine like terms.

$$x^2 = 204.8$$ Divide each side by 5.

$$x = \pm\sqrt{204.8}$$ Take the square root of each side.

$$x = \sqrt{204.8}$$ Extract positive square root.

The total distance covered by walking on the L-shaped sidewalk is

$$x + 2x = 3x$$
$$= 3\sqrt{204.8}$$
$$\approx 42.9 \text{ feet.}$$

Walking on the diagonal sidewalk saves a person about

$$42.9 - 32 = 10.9 \text{ feet.}$$

Checkpoint 8

In Example 8, suppose the length of one sidewalk forming the L is three times as long as the other. How many feet does a person save by walking on the 32-foot diagonal sidewalk?

A fourth type of application of a quadratic equation is one in which a quantity y is changing over time t according to a quadratic model. In the next example, y is exchanged with R because R is a better descriptor in the model.

Example 9 Electric Power Revenues

From 2003 to 2007, the yearly revenues R (in billions of dollars) for the electric power industry in the United States can be modeled by

$$R = 2.237t^2 + 239.12, \quad 3 \le t \le 7$$

where t represents the year, with $t = 3$ corresponding to 2003 (see Figure 1.10). Use the model to approximate the year that revenues were about \$325,000,000,000. *(Source: Edison Electric Institute)*

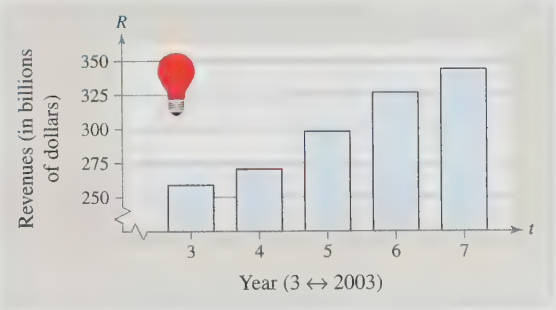


FIGURE 1.10

SOLUTION To solve this problem, let the yearly revenues R be 325 billion dollars and solve the equation for t.

$2.237t^2 + 239.12 = 325$	Substitute 325 for R.
$2.237t^2 = 85.88$	Subtract 239.12 from each side.
$t^2 \approx 38.391$	Divide each side by 2.237.
$t \approx \sqrt{38.391}$	Extract positive square root.
$t \approx 6.196$	Simplify.

The solution is $t \approx 6$. Because $t = 3$ represents 2003, you can conclude that, according to the model, revenues were about \$325 billion in the year 2006.



✓ Checkpoint 9

Use the model in Example 9 to predict the year that revenues will be about \$500 billion.

SUMMARIZE (Section 1.3)

1. Describe how to solve a quadratic equation by factoring *(page 93)*. For examples of solving a quadratic equation by factoring, see Examples 1, 2, and 3.
2. Describe how to solve a quadratic equation by extracting square roots *(page 95)*. For examples of solving a quadratic equation by extracting square roots, see Examples 4 and 5.
3. Describe how a quadratic model can be used to represent data, such as the revenues of a particular industry *(page 99, Example 9)*.

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SKILLS WARM UP 1.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4 and 0.6.

In Exercises 1–4, simplify the expression.

1. $\sqrt{\frac{7}{50}}$
2. $\sqrt{32}$
3. $\sqrt{7^2 + 3 \cdot 7^2}$
4. $\sqrt{\frac{1}{4} + \frac{3}{8}}$

In Exercises 5–10, factor the expression.

5. $3x^2 + 5x$
6. $4x^2 - 25$
7. $16 - (x - 11)^2$
8. $x^2 + 7x - 18$
9. $10x^2 + 13x - 3$
10. $6x^2 - 73x + 12$

Exercises 1.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Writing the General Form In Exercises 1–10, write the quadratic equation in general form.

1. $2x^2 = 3 - 5x$
2. $4x^2 - 2x = 9$
3. $x^2 = 19x$
4. $10x^2 = 70$
5. $(x - 3)^2 = 5$
6. $12 - 3(x + 7)^2 = 0$
7. $x(x + 2) = 3x^2 + 1$
8. $x(x + 5) = 2(x + 5)$
9. $\frac{3x^2 - 10}{5} = 12x$
10. $\frac{x^2 - 7}{3} = 2x$

Solving a Quadratic Equation by Factoring In Exercises 11–22, solve the quadratic equation by factoring. *See Examples 1, 2, and 3.*

11. $x^2 - 2x - 8 = 0$
12. $x^2 - 10x + 9 = 0$
13. $6x^2 + 3x = 0$
14. $9x^2 - 1 = 0$
15. $x^2 + 10x + 25 = 0$
16. $16x^2 + 56x + 49 = 0$
17. $3 + 5x - 2x^2 = 0$
18. $2x^2 = 19x + 33$
19. $x^2 + 4x = 12$
20. $x^2 + 4x = 21$
21. $-x^2 - 7x = 10$
22. $-x^2 + 8x = 12$

Extracting Square Roots In Exercises 23–40, solve the quadratic equation by extracting square roots. When a solution is irrational, list both the exact solution *and* its approximation rounded to two decimal places. *See Examples 4 and 5.*

23. $x^2 = 16$
24. $x^2 = 144$
25. $x^2 = 7$
26. $x^2 = 27$
27. $3x^2 = 36$
28. $9x^2 = 25$
29. $(x - 12)^2 = 144$
30. $(x + 13)^2 = 225$
31. $(x + 2)^2 = 12$
32. $(x + 5)^2 = 20$
33. $12x^2 = 300$
34. $6x^2 = 250$
35. $5x^2 = 190$
36. $15x^2 = 620$
37. $3x^2 + 2(x^2 - 4) = 15$
38. $x^2 + 3(x^2 - 5) = 10$
39. $6x^2 - 3(x^2 + 1) = 23$
40. $2x^2 + 5(x^2 - 2) = 29$

Choosing a Method In Exercises 41–62, solve the quadratic equation using any convenient method. *See Examples 1, 2, 3, 4, and 5.*

41. $x^2 = 64$
42. $7x^2 = 16$
43. $x^2 - 2x + 1 = 0$
44. $x^2 - 6x + 5 = 0$
45. $16x^2 - 9 = 0$
46. $11x^2 + 33x = 0$
47. $4x^2 - 12x + 9 = 0$
48. $x^2 - 14x + 49 = 0$
49. $(x + 4)^2 = 49$
50. $(x - 3)^2 = 36$
51. $4x = 4x^2 - 3$
52. $80 + 6x = 9x^2$
53. $50 + 5x = 3x^2$
54. $144 - 73x + 4x^2 = 0$
55. $12x = x^2 + 27$
56. $26x = 8x^2 + 15$
57. $50x^2 - 60x + 10 = 0$
58. $9x^2 + 12x + 3 = 0$
59. $(x + 3)^2 - 4 = 0$
60. $(x - 2)^2 - 9 = 0$
61. $(x + 1)^2 = x^2$
62. $(x + 1)^2 = 4x^2$

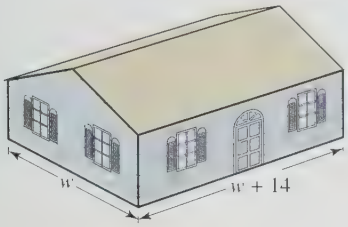


63. **Writing** Consider the expression $(x + 2)^2$. How would you convince someone in your class that $(x + 2)^2 \neq x^2 + 4$? Give an argument based on the rules of algebra. Give an argument using your graphing utility.



64. **Writing** Consider the expression $\sqrt{a^2 + b^2}$. How would you convince someone in your class that $\sqrt{a^2 + b^2} \neq a + b$? Give an argument based on the rules of algebra or geometry. Give an argument using your graphing utility.

65. Geometry A one-story building is 14 feet longer than it is wide (see figure). The building has 1632 square feet of floor space. What are the dimensions of the building?

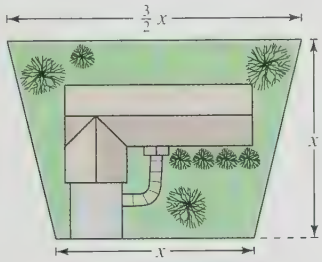


66. Geometry A billboard is 10 feet longer than it is high (see figure). The billboard has 336 square feet of advertising space. What are the dimensions of the billboard?

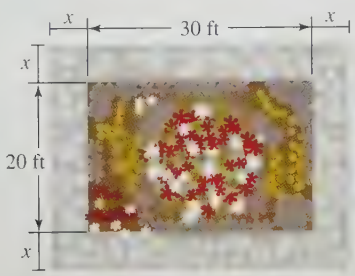


67. Geometry A triangular sign has a height that is equal to its base. The area of the sign is 4 square feet. Find the base and height of the sign.

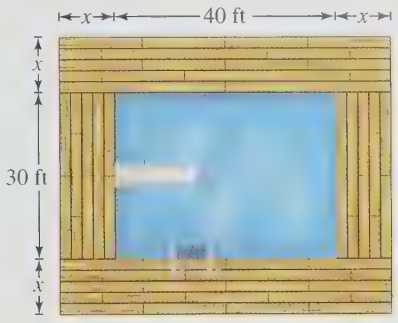
68. Geometry The building lot shown in the figure has an area of 8000 square feet. What are the dimensions of the lot?



69. Geometry A rectangular garden that is 30 feet long and 20 feet wide is surrounded on all four sides by a rock path that is x feet wide. The total area of the garden and the rock path is 1200 square feet. What is the width of the path?



70. Geometry A rectangular pool is 30 feet wide and 40 feet long. It is surrounded on all four sides by a wooden deck that is x feet wide. The total area enclosed within the perimeter of the deck is 3000 square feet. What is the width of the deck?



In Exercises 71–76, assume that air resistance is negligible, which implies that the position equation $s = -16t^2 + v_0t + s_0$ is a reasonable model.

71. Falling Object A rock is dropped from the top of a 200-foot cliff that overlooks the ocean. How long will it take for the rock to hit the water?

72. Royal Gorge Bridge The Royal Gorge Bridge near Canon City, Colorado is one of the highest suspension bridges in the world. The bridge is 1053 feet above the Arkansas river. A rock is dropped from the bridge. How long does it take the rock to hit the water?

73. Olympic Diver The high-dive platform in the Olympics is 10 meters above the water. A diver wants to perform an armstand dive, which means she will drop to the water from a handstand position. How long will the diver be in the air? (*Hint:* 1 meter ≈ 3.2808 feet)

74. The Owl and the Mouse An owl is flying at a height of 100 feet and spots a mouse directly below. The owl folds up its wings and drops into a dive at the mouse. How long does the mouse have to escape?

75. Wind Resistance At the same time a skydiver jumps from an airplane 13,000 feet above the ground, a steel ball is dropped from the plane. Because of air resistance, it takes the skydiver 67 seconds to freefall to a height of 3000 feet, where the parachute opens. The steel ball has relatively no air resistance, so its height can be modeled by the position equation. How much faster does the ball reach a height of 3000 feet than the skydiver?

76. Wind Resistance At the same time a skydiver jumps from an airplane 8900 feet above the ground, a steel ball is dropped from the plane. Because of air resistance, it takes the skydiver 44 seconds to freefall to a height of 2500 feet, where the parachute opens. The steel ball has relatively no air resistance, so its height can be modeled by the position equation. How much faster does the ball reach a height of 2500 feet than the skydiver?

77. **Geometry** The hypotenuse of an isosceles right triangle is 6 centimeters long. How long are the legs? (An isosceles right triangle is one whose two legs are of equal length.)

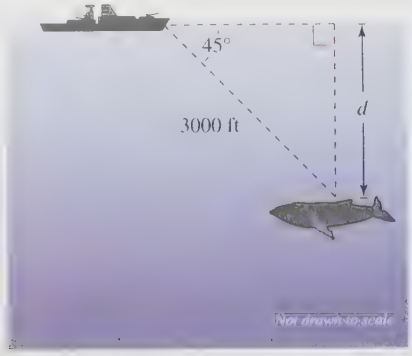
78. **Geometry** An equilateral triangle has a height of 3 feet. How long are each of its legs? (*Hint:* Use the height of the triangle to partition the triangle into two right triangles of the same size.)

79. **Flying Distance** A commercial jet flies to three cities whose locations form the vertices of a right triangle (see figure). The air distance from Atlanta to Buffalo is about 703 miles and the air distance from Atlanta to Chicago is about 583 miles. Approximate the air distance from Atlanta to Buffalo *by way of* Chicago.



In Exercises 80 and 81, use the following information. The sum of the angles of a triangle is 180°. Also, if two angles of a triangle are equal, then the lengths of the sides opposite those angles are equal.

80. **Depth of a Whale** The sonar on a research ship detects a whale that is 3000 feet from the ship. The angle formed by the ocean surface and a line from the ship to the whale is 45° (see figure). How deep is the whale?

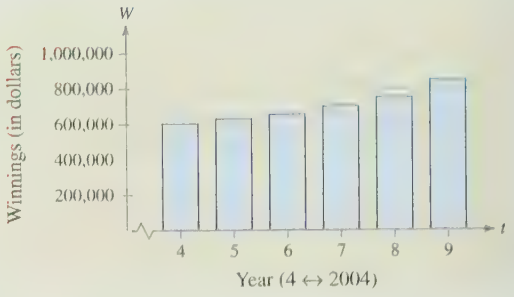


81. **Depth of a Whale Shark** A research ship is tracking the movements of a whale shark that is 700 meters from the ship. The angle formed by the ocean surface and a line from the ship to the whale shark is 45°. How deep is the whale shark?

82. **Tennis** The winnings W (in dollars) of the men's singles champions of the Wimbledon Tennis Tournament from 2004 through 2009 can be approximated by the model

$$W = 8258.93t^2 - 60{,}437.5t + 717{,}732, \quad 4 \le t \le 9$$

where t represents the year, with $t = 4$ corresponding to 2004. *(Source: Wimbledon Championships)*



(a) According to the model, what is the first year the men's singles champion won \$1 million or more?

(b) The Wimbledon men's singles champion won \$1 million in 2010. Based on your answer in part (a), do you think this is a good model for making predictions after 2009? Explain.

83. **Total Revenue** The demand equation for a product is $p = 36 - 0.0003x$, where p is the price per unit and x is the number of units sold. The total revenue R from selling x units is given by

$$R = xp = x(36 - 0.0003x).$$

How many units must be sold to produce a revenue of \$1,080,000?

84. **Total Revenue** The demand equation for a product is $p = 40 - 0.0005x$, where p is the price per unit and x is the number of units sold. The total revenue R from selling x units is given by

$$R = xp = x(40 - 0.0005x).$$

How many units must be sold to produce a revenue of \$800,000?

85. **Production Cost** A company determines that the average monthly cost C (in dollars) of raw materials for manufacturing a product line can be modeled by

$$C = 35.65t^2 + 7205, \quad t \ge 0$$

where t is the year, with $t = 0$ corresponding to 2010. Use the model to estimate the year in which the average monthly cost will be about \$10,000.

86. Blue Oak The blue oak tree, native to California, is known for its slow rate of growth. Fencing enclosures protect seedlings from herbivore damage and promote faster growth. The height H (in inches) of an enclosed blue oak tree is expected to change according to the model

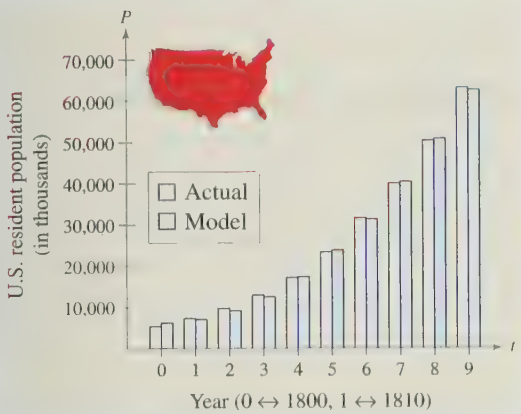
$$H = 0.74t^2 + 25, \quad 0 \le t \le 5$$

where t represents the year, with $t = 0$ corresponding to 2010. Approximate the height of the tree in 2012. Then, use the model to approximate the year in which the height of the tree will be about 37 inches.

87. U.S. Population The resident population P (in thousands) of the United States from 1800 to 1890 can be approximated by the model

$$P = 694.59t^2 + 6179, \quad 0 \le t \le 9$$

where t represents the year, with $t = 0$ corresponding to 1800, $t = 1$ corresponding to 1810, and so on (see figure). *(Source: U.S. Census Bureau)*



(a) Assume this model had continued to be valid up through the present time. In what year would the resident population of the United States have reached 250,000,000?

(b) Judging from the figure, would you say that this model is a good representation of the resident population through 1890?

(c) Use the model to predict the resident population in 2010. How does your answer compare with the actual 2010 resident population of approximately 310,000,000?

(d) The U.S. Census Bureau predicts that the resident population in 2050 will be 439,010,253. Would you say that the model supports this prediction? Explain.

88. HOW DO YOU SEE IT? Each figure shows a person or a ball about to go into motion through the air. Match each figure with the position equation that represents the situation. Without calculating, compare the terms of the equations to determine which situation involves the least amount of time in the air. Explain your reasoning.

(i) 

(ii) 

(iii) 

(a) $s = -16t^2 + 33t$

(b) $s = -16t^2 + 10$

(c) $s = -16t^2 - 20t + 10$

89. Monthly Cost A company determines that the average monthly cost C (in dollars) of staffing temporary positions can be modeled by

$$C = 135.47t^2 + 13{,}702, \quad t \ge 0$$

where t represents the year, with $t = 0$ corresponding to 2010. Use the model to predict the year in which the average monthly cost will be about $25,000.

90. Education The enrollment E in an early childhood development program for a school district from 2000 to 2012 can be approximated by the model

$$E = 1.678t^2 + 1025, \quad 0 \le t \le 12$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to approximate the year in which the early childhood enrollment was about 1160 children. Can you use the model to estimate early childhood enrollment for the year 1980? Explain.

91. Meteorology The temperature T (in degrees Fahrenheit) during a certain day can be approximated by

$$T = 0.31t^2 + 32.9, \quad 7 \le t \le 15$$

where t represents the hour of the day, with $t = 7$ corresponding to 7 A.M. Use the model to approximate the time when the temperature was 85° F. Can you use this model to predict the temperature at 7 P.M.? Explain.

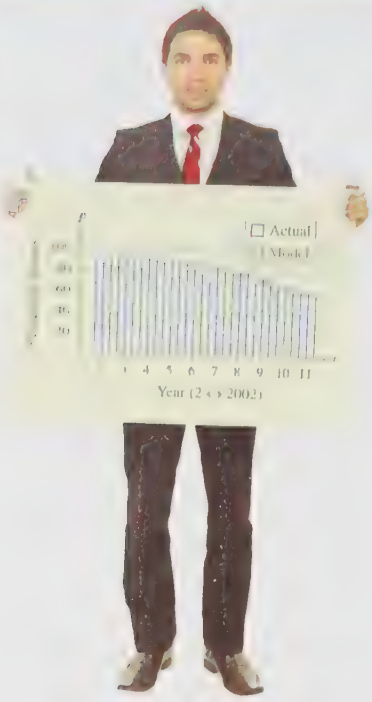
92. Electromedical Exports From 2004 through 2009, the yearly exports E (in billions of dollars) of electromedical equipment from the United States can be approximated by

$$E = 0.119t^2 + 10.7, \quad 4 \le t \le 9$$

where t represents the year, with $t = 4$ corresponding to 2004. Use the model to estimate the year when the value of these exports was about $15 billion. *(Source: AdvaMed)*

1.4 The Quadratic Formula

- Develop the Quadratic Formula by completing the square.
- Use the discriminant to determine the number of real solutions of a quadratic equation.
- Solve a quadratic equation using the Quadratic Formula.
- Use the Quadratic Formula to solve an application problem.



In Exercise 71 on page 112, you will use the Quadratic Formula to determine the year when only 60% of households still had VCRs.

Development of the Quadratic Formula

In Section 1.3, you studied two methods for solving quadratic equations. These two methods are efficient for special quadratic equations that are factorable or that can be solved by extracting square roots. There are, however, many quadratic equations that cannot be solved efficiently by either of these two techniques. Fortunately, there is a general formula that can be used to solve *any* quadratic equation. It is called the **Quadratic Formula.** This formula is derived using a process called **completing the square.**

$$ax^2 + bx + c = 0 \qquad \text{General form, } a \neq 0$$

$$ax^2 + bx = -c \qquad \text{Subtract } c \text{ from each side.}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide each side by } a.$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{Complete the square.}$$

$$\left(\text{half of } \frac{b}{a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Simplify.}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Extract square roots.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Solutions}$$

The Quadratic Formula

The solutions of

$$ax^2 + bx + c = 0, \quad a \neq 0$$

are given by the **Quadratic Formula,**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To help you remember the Quadratic Formula, try to memorize a verbal statement of the rule, such as:

> "The opposite of b, plus or minus the square root of b squared minus $4ac$, all divided by $2a$."

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The Discriminant

In the Quadratic Formula, the quantity under the radical sign, $b^2 - 4ac$, is called the **discriminant** of the quadratic expression $ax^2 + bx + c$.

$b^2 - 4ac$ Discriminant

It can be used to determine the number of real solutions of a quadratic equation.

Solutions of a Quadratic Equation

The solutions of a quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

can be classified by the discriminant, $b^2 - 4ac$, as follows.

1. If $b^2 - 4ac > 0$, then the equation has *two* distinct real solutions.
2. If $b^2 - 4ac = 0$, then the equation has *one* repeated real solution.
3. If $b^2 - 4ac < 0$, then the equation has *no* real solutions.

If the discriminant of a quadratic equation is negative, as in case 3 above, then its square root is imaginary (not a real number) and the Quadratic Formula yields two complex solutions. You will study complex solutions in Section 3.5.

Example 1 Using the Discriminant

Use the discriminant to determine the number of real solutions to each of the following quadratic equations.

a. $4x^2 - 20x + 25 = 0$ **b.** $13x^2 + 7x + 1 = 0$ **c.** $5x^2 = 8x$

SOLUTION

a. Using $a = 4$, $b = -20$, and $c = 25$, the discriminant is

$$\begin{aligned} b^2 - 4ac &= (-20)^2 - 4(4)(25) \\ &= 400 - 400 = 0. \end{aligned}$$

Because $b^2 - 4ac = 0$, there is *one* repeated real solution.

b. Using $a = 13$, $b = 7$, and $c = 1$, the discriminant is

$$\begin{aligned} b^2 - 4ac &= (7)^2 - 4(13)(1) \\ &= 49 - 52 = -3. \end{aligned}$$

Because $b^2 - 4ac < 0$, there are *no* real solutions.

c. In general form, this equation is $5x^2 - 8x = 0$, with $a = 5$, $b = -8$, and $c = 0$, which implies that the discriminant is

$$\begin{aligned} b^2 - 4ac &= (-8)^2 - 4(5)(0) \\ &= 64. \end{aligned}$$

Because $b^2 - 4ac > 0$, there are *two* distinct real solutions.

✓Checkpoint 1

Use the discriminant to determine the number of real solutions of $x^2 + 6x + 9 = 0$.

Using the Quadratic Formula

When using the Quadratic Formula, remember that *before* the formula can be applied, you must first write the quadratic equation in general form.

Example 2 **Two Distinct Solutions**

Solve $x^2 + 3x = 9$.

SOLUTION

$$x^2 + 3x = 9$$ Write original equation.

$$x^2 + 3x - 9 = 0$$ Write in general form.

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-9)}}{2(1)}$$ Quadratic Formula

$$x = \frac{-3 \pm \sqrt{45}}{2}$$ Simplify.

$$x = \frac{-3 \pm 3\sqrt{5}}{2}$$ Simplify.

The solutions are

$$x = \frac{-3 + 3\sqrt{5}}{2} \quad \text{and} \quad x = \frac{-3 - 3\sqrt{5}}{2}.$$

Check these in the original equation.

TECH TUTOR

You can write a program for a graphing utility to solve equations using the Quadratic Formula. Use a program to solve the equation in Example 2.

✓Checkpoint 2

Solve $x^2 + 2x = 2$.

Example 3 **One Repeated Solution**

Solve $8x^2 - 24x + 18 = 0$.

SOLUTION Begin by dividing each side by the common factor 2.

$$8x^2 - 24x + 18 = 0$$ Write original equation.

$$4x^2 - 12x + 9 = 0$$ Divide each side by 2.

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$ Quadratic Formula

$$x = \frac{12 \pm \sqrt{0}}{8}$$ Simplify.

$$x = \frac{3}{2}$$ Repeated solution

The only solution is

$$x = \tfrac{3}{2}.$$

Check this in the original equation.

✓Checkpoint 3

Solve $27x^2 - 18x = -3$.

The discriminant in Example 3 is a perfect square (zero in this case), and you could have factored the quadratic as

$$4x^2 - 12x + 9 = 0$$

$$(2x - 3)^2 = 0$$

and concluded that the solution is $x = \frac{3}{2}$. Because factoring is easier than applying the Quadratic Formula, try factoring first when solving a quadratic equation. If factors cannot easily be found, then use the Quadratic Formula. For instance, try solving the quadratic equation

$$x^2 - x - 12 = 0$$

in two ways—by factoring and by the Quadratic Formula—to see that you get the same solutions either way.

When using a calculator with the Quadratic Formula, you should get in the habit of using the memory key to store intermediate steps. This will save steps and minimize roundoff error.

Example 4 **Using a Calculator with the Quadratic Formula**

Solve $16.3x^2 - 197.6x + 7.042 = 0$.

SOLUTION In this case, $a = 16.3$, $b = -197.6$, $c = 7.042$, and you have

$$x = \frac{-(-197.6) \pm \sqrt{(-197.6)^2 - 4(16.3)(7.042)}}{2(16.3)}.$$

To evaluate these solutions, begin by calculating the square root of the discriminant, as follows.

Scientific Calculator Keystrokes

197.6 [+/−] [x^2] [−] 4 [×] 16.3 [×] 7.042 [=] [$\sqrt{\ }$]

Graphing Calculator Keystrokes

[$\sqrt{\ }$] [(] [(] [(−)] 197.6 [)] [x^2] [−] 4 [×] 16.3 [×] 7.042 [)] [ENTER]

In either case, the result is 196.434777. Storing this result and using the recall key, you can find the following two solutions.

$$x \approx \frac{197.6 + 196.434777}{2(16.3)} \approx 12.087$$ Add stored value.

$$x \approx \frac{197.6 - 196.434777}{2(16.3)} \approx 0.036$$ Subtract stored value.

✓ Checkpoint 4

Solve $4.7x^2 - 3.2x - 5.9 = 0$.



TECH TUTOR

Try to calculate the value of x in Example 4 by using additional parentheses instead of storing the intermediate result, 196.434777, in your calculator.

Applications

In Section 1.3, you studied four basic types of applications involving quadratic equations: area, falling bodies, the Pythagorean Theorem, and quadratic models. The solution to each of these types of problems can involve the Quadratic Formula. For instance, Example 5 shows how the Quadratic Formula can be used to analyze a quadratic model for a patient's blood oxygen level.



A pulse oximeter helps monitor the oxygen saturation of an individual's blood, as well as his or her heart rate.

Example 5 Blood Oxygen Level

Doctors treated a patient at an emergency room from 1:00 p.m. to 5:00 p.m. The patient's blood oxygen level L (in percent) during this time period can be modeled by

$$L = -0.25t^2 + 3.0t + 87, \quad 1 \le t \le 5$$

where t represents the time of day, with $t = 1$ corresponding to 1:00 p.m. Use the model to estimate the time when the patient's blood oxygen level was 95%.

SOLUTION To find the hour when the patient's blood oxygen level was 95%, solve the equation

$$95 = -0.25t^2 + 3.0t + 87.$$

To begin, write the equation in general form.

$$-0.25t^2 + 3.0t - 8 = 0$$

Then apply the Quadratic Formula with $a = -0.25$, $b = 3.0$, and $c = -8$.

$$t = \frac{-3 \pm \sqrt{3^2 - 4(-0.25)(-8)}}{2(-0.25)}$$

$$= \frac{-3 \pm \sqrt{9 - 8}}{-0.5}$$

$$= \frac{-3 \pm \sqrt{1}}{-0.5}$$

$$= 4 \text{ or } 8$$

Of the two possible solutions, only $t = 4$ makes sense in the context of the problem because $t = 8$ is not in the domain of L. Because $t = 1$ corresponds to 1:00 P.M., it follows that $t = 4$ corresponds to 4:00 P.M. So, from the model you can conclude that the patient's blood oxygen level was 95% at about 4:00 P.M. Figure 1.11 shows the patient's oxygen level recorded every 30 minutes.

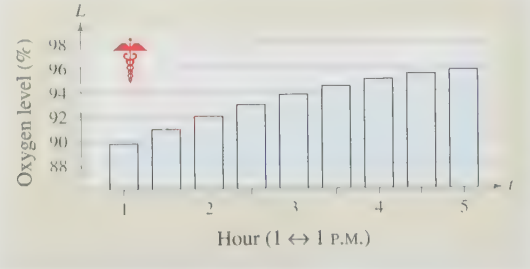


FIGURE 1.11

✓ Checkpoint 5

Use the model in Example 5 to estimate the time when the patient's blood oxygen level was 92%.

Rob Byron/Shutterstock.com

STUDY TIP

Note in the position equation

$$s = -16t^2 + v_0t + s_0$$

that the initial velocity v_0 is positive when an object is rising and negative when an object is falling.

In Section 1.3, you learned that the position equation for a falling object is of the form

$$s = -16t^2 + v_0t + s_0$$

where s is the height (in feet) of the object, v_0 is the initial velocity (in feet per second), t is the time (in seconds), and s_0 is the initial height (in feet). This equation is valid only for free-falling objects near Earth's surface. Because of differences in gravitational force, position equations are different on other planets or moons. The next example looks at a position equation for a falling object on our moon.

Example 6 **Throwing an Object on the Moon**

An astronaut standing on the surface of the moon throws a rock straight up at 27 feet per second from a height of 6 feet. The height s (in feet) of the rock is given by

$$s = -2.7t^2 + 27t + 6$$

where t is the time (in seconds). How much time elapses before the rock strikes the lunar surface?

SOLUTION Because s gives the height of the rock at any time t, you can find the time that the rock hits the surface of the moon by setting s equal to zero and solving for t.

$$-2.7t^2 + 27t + 6 = 0 \qquad \text{Substitute 0 for } s$$

$$t = \frac{-27 \pm \sqrt{(27)^2 - 4(-2.7)(6)}}{2(-2.7)} \qquad \text{Quadratic Formula}$$

$$= \frac{-27 \pm \sqrt{793.8}}{-5.4} \qquad \text{Simplify}$$

$$\approx 10.2 \text{ seconds} \qquad \text{Choose positive solution}$$

So, about 10.2 seconds elapse before the rock hits the lunar surface.

TECH TUTOR

Use the *last entry* feature of your graphing calculator to find the time in the air on Earth for the rock in Example 6. Simply replace -2.7 with -16 in the position equation.

Checkpoint 6

In Example 6, suppose the rock is thrown straight up at 13 feet per second from a height of 4 feet. The height s (in feet) of the rock is given by

$$s = -2.7t^2 + 13t + 4.$$

How much time (in seconds) elapses before the rock strikes the lunar surface?



SUMMARIZE (Section 1.4)

1. State the definition of the discriminant of a quadratic expression and describe how it is used to determine the number of real solutions of a quadratic equation *(page 105)*. For an example of using the discriminant, see Example 1.
2. Describe how to solve a quadratic equation using the Quadratic Formula *(pages 106 and 107)*. For examples that use the Quadratic Formula, see Examples 2, 3, and 4.
3. Describe a real-life example that uses the Quadratic Formula *(pages 108 and 109, Examples 5 and 6)*.

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SKILLS WARM UP 1.4 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4 and 1.3.

In Exercises 1–4, simplify the expression.

1. $\sqrt{25 - 4(3)(-12)}$
2. $\sqrt{49 - 4(2)(3)}$
3. $\sqrt{12^2 - 4(3)(4)}$
4. $\sqrt{15^2 + 4(9)(12)}$

In Exercises 5–10, solve the quadratic equation by factoring.

5. $x^2 - x - 2 = 0$
6. $2x^2 + 3x - 9 = 0$
7. $x^2 - 4x = 5$
8. $2x^2 + 13x = 7$
9. $x(x - 7) = -10$
10. $x(x - 3) = 4$

Exercises 1.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Using the Discriminant In Exercises 1–8, use the discriminant to determine the number of real solutions of the quadratic equation. *See Example 1.*

1. $x^2 - 4x + 4 = 0$
2. $2x^2 - x - 1 = 0$
3. $3x^2 + 4x + 1 = 0$
4. $x^2 + 2x + 4 = 0$
5. $2x^2 - 5x = -5$
6. $3 - 6x = -3x^2$
7. $\frac{1}{5}x^2 + \frac{6}{5}x - 8 = 0$
8. $\frac{1}{3}x^2 - 5x + 25 = 0$

Using the Quadratic Formula In Exercises 9–30, use the Quadratic Formula to solve the quadratic equation. *See Examples 2 and 3.*

9. $2x^2 + x - 1 = 0$
10. $2x^2 - x - 1 = 0$
11. $16x^2 + 8x - 3 = 0$
12. $25x^2 - 20x + 3 = 0$
13. $6x = 4 - x^2$
14. $2 + 2x = x^2$
15. $x^2 - 10x + 22 = 0$
16. $x^2 + 14x + 44 = 0$
17. $x^2 + 8x - 4 = 0$
18. $4x^2 - 4x - 4 = 0$
19. $12x - 9x^2 = -3$
20. $16x^2 + 22 = 40x$
21. $36x^2 + 24x = 7$
22. $3x + x^2 = 1$
23. $4x^2 + 4x = 7$
24. $16x^2 - 40x = -5$
25. $28x - 49x^2 = 4$
26. $9x^2 + 24x = -16$
27. $8t = 5 + 2t^2$
28. $25h^2 + 61 = -80h$
29. $(y - 5)^2 = 2y$
30. $(x + 6)^2 = -2x$

Using a Calculator In Exercises 31–36, use a calculator to solve the quadratic equation. (Round your answer to three decimal places.) *See Example 4.*

31. $5.1x^2 - 1.7x - 3.2 = 0$
32. $-0.067x^2 - 0.852x + 1.277 = 0$
33. $12.67x^2 + 31.55x + 8.09 = 0$
34. $422x^2 - 506x - 347 = 0$
35. $-0.003x^2 + 0.025x - 0.98 = 0$
36. $-0.005x^2 + 0.101x - 0.193 = 0$

Choosing a Method In Exercises 37–46, solve the quadratic equation using any convenient method. *See Examples 2, 3, and 4.*

37. $2x^2 + 7 = 2x^2 - x - 4$
38. $x^2 - 2x + 5 = x^2 - 5$
39. $4x^2 - 15 = 25$
40. $3x^2 - 16 = 38$
41. $x^2 - 2x - 1 = 0$
42. $x^2 + 3x - 4 = 0$
43. $(x - 1)^2 = 9$
44. $(x - 5)^2 = 16$
45. $3x^2 + 5x - 11 = 4(x - 2)$
46. $2x^2 + 4x - 9 = 2(x - 1)$

Writing Real-Life Problems In Exercises 47–50, solve the number problem *and* write a real-life problem that could be represented by this verbal model. For instance, an applied problem that could be represented by Exercise 47 is shown below.

The sum of the length and width of a one-story house is 100 feet. The house has 2500 square feet of floor space. What are the length and width of the house?

47. Find two numbers whose sum is 100 and whose product is 2500.
48. One number is 1 more than another number. The product of the two numbers is 72. Find the numbers.
49. One number is 1 more than another number. The sum of their squares is 113. Find the numbers.
50. One number is 2 more than another number. The product of the two numbers is 440. Find the numbers.

Cost Equation In Exercises 51–54, use the cost equation to find the number of units x that a manufacturer can produce for cost C. (Round your answer to the nearest positive integer.)

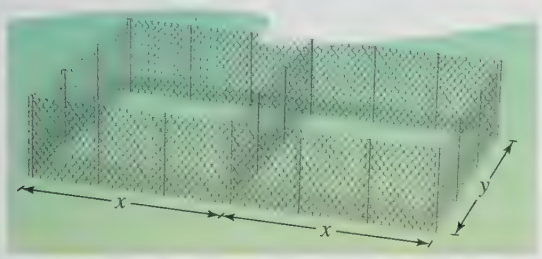
51. $C = 0.125x^2 + 20x + 5000, \quad C = \$14{,}000$
52. $C = 0.5x^2 + 15x + 5000, \quad C = \$11{,}500$

53. $C = 800 + 0.04x + 0.002x^2, \quad C = \$1680$

54. $C = 312.5 - 10x + 0.4x^2, \quad C = \$900$

55. Seating A lecture hall has 108 chairs arranged in rows with the same number of chairs in each row. You eliminate three rows by adding six chairs to each of the other rows. How many rows are there now? How many chairs are in each row?

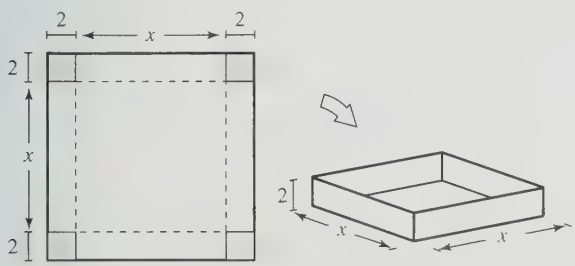
56. Child Care A child care center has 200 feet of fencing to enclose two adjacent rectangular safe play areas (see figure).



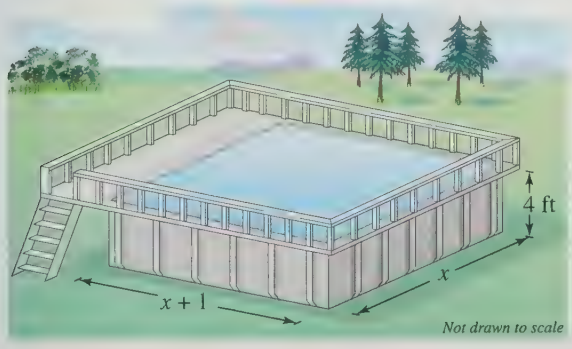
(a) Write an expression for the total area A of the play areas in terms of x.

(b) Find the dimensions for which the enclosed area is 1600 square feet.

57. Geometry An open box is made from a square piece of material by cutting two-inch squares from the corners and turning up the sides (see figure). The volume of the finished box is 200 cubic inches. Find the size of the original piece of material.



58. Geometry The swimming pool shown is 4 feet deep and has a volume of 1024 cubic feet. Find the dimensions of the base of the pool.



59. Stratosphere You are at the Stratosphere Tower in Las Vegas. While riding the X-Scream tower ride, you accidently drop a coin from a height of 866 feet with an initial velocity of −5 feet per second. How long does it take the coin to hit the ground?

60. Sports You throw a baseball straight up into the air at a velocity of 80 feet per second. You release the baseball at a height of 5.5 feet and catch it when it falls back to a height of 5.5 feet.

(a) Use the position equation to write a mathematical model for the height of the baseball.

(b) Find the height of the baseball after 1 second, 2 seconds, and 3 seconds. What must have occurred sometime in the interval $2 \le t \le 3$? Explain.

(c) For how many seconds is the baseball in the air?

61. On the Moon An astronaut on the moon throws a rock straight upward into space. The height s (in feet) of the rock is given by

$$s = -2.7t^2 + 40t + 5$$

where the initial velocity is 40 feet per second, the initial height is 5 feet, and t is the time (in seconds). How long will it take the rock to hit the surface of the moon? Suppose a rock is thrown with the same initial velocity and height on Earth. How long will it take the rock to hit the ground?

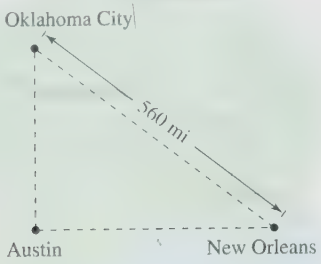
62. Hot Air Balloon Two people are floating in a hot air balloon 200 feet above a lake. One person tosses out a coin with an initial velocity of 20 feet per second. One second later, the balloon is 20 feet higher and the other person drops another coin (see figure). The position equation for the first coin is $s = -16t^2 + 20t + 200$, and the position equation for the second coin is $s = -16t^2 + 220$. Which coin will hit the water first? (*Hint:* Remember that the first coin was tossed one second before the second coin was dropped.)



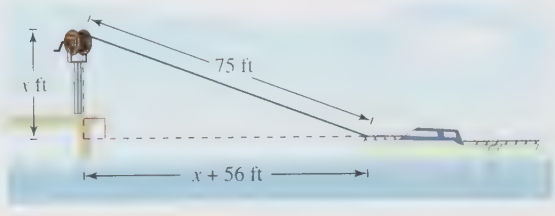
63. Comparing Gravity Does a rock thrown upward from an initial height of 6 feet with an initial velocity of 27 feet per second take longer to reach the ground on Earth or on the moon? (See Example 6.)

64. Comparing Gravity Does a rock thrown downward from an initial height of 6 feet with an initial velocity of −14 feet per second take longer to reach the ground on Earth or on the moon? (See Example 6.)

65. Aviation A commercial jet flies to three cities whose locations form the vertices of a right triangle (see figure). The total flight distance (from Oklahoma City to Austin to New Orleans and back to Oklahoma City) is approximately 1348 miles. It is 560 miles between Oklahoma City and New Orleans. Approximate the other two distances.



66. Distance from a Dock A windlass is used to tow a boat to a dock. The figure shows the boat when it is 75 feet from the windlass. How far is the boat from the dock?



67. Blood Oxygen Level Doctors treated a patient at an emergency room from 2:00 P.M. to 7:00 P.M. The patient's blood oxygen level L (in percent) during this time period can be modeled by

$$L = -0.270t^2 + 3.59t + 83.1, \quad 2 \le t \le 7$$

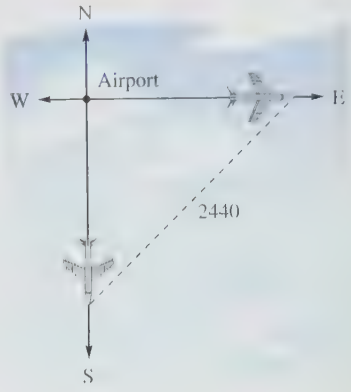
where t represents the time of day, with $t = 2$ corresponding to 2:00 P.M. Use the model to estimate the time (rounded to the nearest hour) when the patient's blood oxygen level was 93%.

68. Green Mountain Coffee Roasters The sales S (in millions of dollars) for Green Mountain Coffee Roasters from 2004 through 2009 can be approximated by

$$S = 31.61t^2 - 283.4t + 776, \quad 4 \le t \le 9$$

where t represents the year, with $t = 4$ corresponding to 2004. Use the model to estimate the year when sales were about $340 million. *(Source: Green Mountain Coffee Roasters, Inc.)*

69. Flying Speed Two planes leave simultaneously from the same airport, one flying due east and the other due south (see figure). The eastbound plane is flying 50 miles per hour faster than the southbound plane. After 3 hours the planes are 2440 miles apart. Find the speed of each plane.

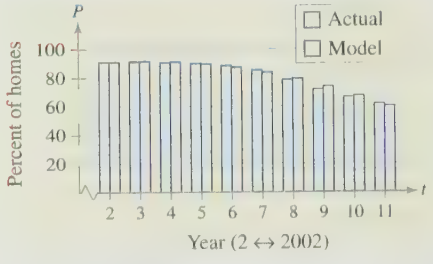


70. Flying Speed Two planes leave simultaneously from the same airport, one flying due east and the other due south. The eastbound plane is flying 100 miles per hour faster than the southbound plane. After 2 hours the planes are 1500 miles apart. Find the speed of each plane.

71. VCRs The percent P of homes in the United States with VCRs from 2002 through 2011 can be approximated by the model

$$P = -0.496t^2 + 3.01t + 87.2, \quad 2 \le t \le 11$$

where t represents the year, with $t = 2$ corresponding to 2002. The bar graph shows the actual percents and the percents represented by the model. *(Source: The Nielsen Company)*



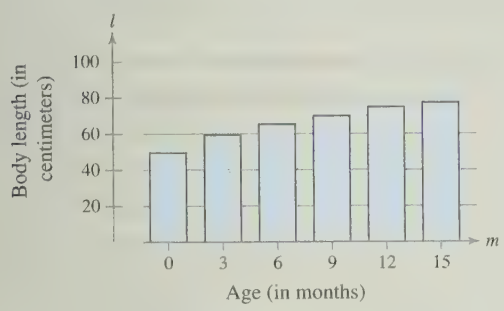
(a) Use the model to estimate the year when the percent of homes with VCRs was about 60%.

(b) Use the model to estimate the year when the percent of homes with VCRs was about 80%.

(c) Use the model to predict the last year that VCRs will exist in U.S. households. Would you say that the model is valid to represent data up to that year?

72. HOW DO YOU SEE IT? The bar graph shows the growth of a baby girl. A model that approximates her body length l (in centimeters) at m months of age is

$$l = -0.0813m^2 + 3.028m + 50.0, \quad 0 \le m \le 15.$$



(a) During which 3-month period shown in the bar graph did the baby girl grow the most?

(b) For $l = 50$ centimeters, the model simplifies to

$$0 = -0.0813m^2 + 3.028m.$$

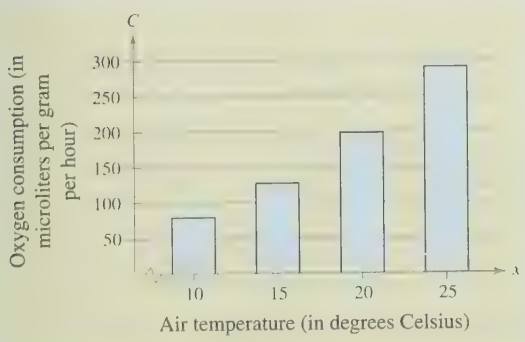
Is the discriminant positive, negative, or zero?

(c) Based on your answer in part (b) and the bar graph, will the original model work well for predicting the girl's height as she gets older? Explain.

73. Biology The metabolic rate of an ectothermic organism increases with increasing temperature within a certain range. The graph shows experimental data for the oxygen consumption C (in microliters per gram per hour) of a beetle at certain temperatures. The data can be approximated by the model

$$C = 0.45x^2 - 1.65x + 50.75, \quad 10 \le x \le 25$$

where x is the temperature in degrees Celsius.



(a) The oxygen consumption is 150 microliters per gram per hour. What is the air temperature?

(b) The temperature is increased from 10°C to 20°C. The oxygen consumption is increased by approximately what factor?

74. Total Revenue The demand equation for a product is

$$p = 60 - 0.0004x$$

where p is the price per unit and x is the number of units sold. The total revenue R from selling x units is given by

$$R = xp.$$

How many units must be sold to produce a revenue of \$220,000?

75. Total Revenue The demand equation for a product is

$$p = 50 - 0.0005x$$

where p is the price per unit and x is the number of units sold. The total revenue R from selling x units is given by

$$R = xp.$$

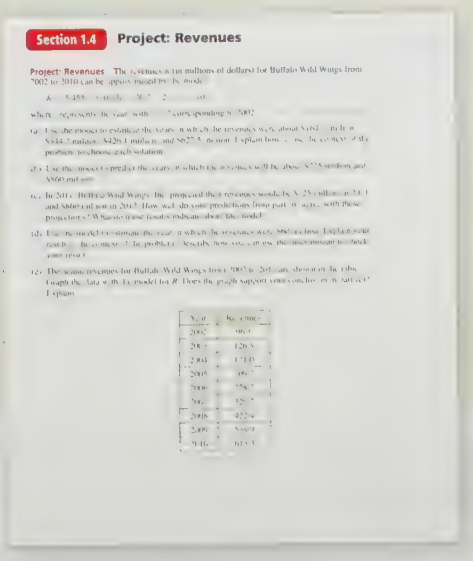
How many units must be sold to produce a revenue of \$250,000?

76. Think About It When the Quadratic Formula is used to solve certain problems, such as the problem in Example 5 on page 108, why is only one solution used?

77. Think About It The discriminants of two quadratic equations are 5 and -10. Can the equations have the same solutions? Explain.

78. Writing List the methods you have learned for solving quadratic equations. Identify the type of quadratic equation that is most easily solved by each method. Give an example showing the solutions for each type.

79. Project: Revenue For a project involving the revenues for Buffalo Wild Wings from 2002 to 2010, visit this text's website at *www.cengagebrain.com.* *(Source: Buffalo Wild Wings, Inc.)*


Section 1.4 Project: Revenues

QUIZ YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, solve the equation and check your solution.

1. $3(x - 2) - 4(2x + 5) = 4$
2. $\dfrac{3x + 3}{5x - 2} = \dfrac{3}{4}$
3. $\dfrac{2}{x(x - 1)} + \dfrac{1}{x} = \dfrac{1}{x - 4}$
4. $(x + 3)^2 - x^2 = 6(x + 2)$
5. Describe how you can check your answers to Exercises 1–4 using your graphing utility.

In Exercises 6 and 7, use a calculator to solve the equation. (Round your solution to three decimal places.)

6. $\dfrac{x}{2.004} - \dfrac{x}{5.128} = 100$
7. $0.378x + 0.757(500 - x) = 215$

In Exercises 8 and 9, write an algebraic equation for the verbal description. Find the solution if possible and check.

8. A company has fixed costs of \$30,000 per month and variable costs of \$8.50 per unit manufactured. The company has \$200,000 available each month to cover monthly costs. How many units can the company manufacture?
9. The demand equation for a product is $p = 75 - 0.0002x$, where p is the price per unit and x is the number of units sold. The total revenue R from selling x units is given by $R = xp$. How many units must be sold to produce a revenue of \$300,000?

In Exercises 10–15, solve the quadratic equation by the indicated method.

10. *Factoring:* $3x^2 + 13x = 10$
11. *Extracting square roots:* $3x^2 = 15$
12. *Extracting square roots:* $(x + 3)^2 = 17$
13. *Quadratic Formula:* $2x + x^2 = 6$
14. *Quadratic Formula:* $3x^2 + 7x - 2 = 0$
15. *Quadratic Formula:* $3x^2 - 4.50x - 0.32 = 0$

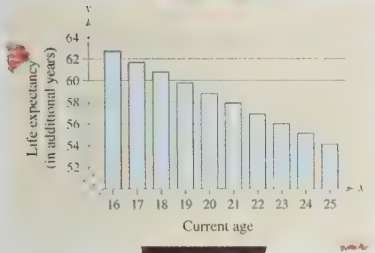
In Exercises 16 and 17, use the discriminant to determine the number of real solutions of the quadratic equation.

16. $2x^2 - 4x + 9 = 0$
17. $9x^2 - 3x + \frac{1}{4} = 0$
18. Describe how you would convince a fellow student that $(x + 3)^2 = x^2 + 6x + 9$ for all real values of x.
19. A rock is dropped from a height of 324 feet. How long will it take the rock to hit the ground?
20. An open box has a square base and a height of 6 inches. The volume of the box is 486 cubic inches. Find the dimensions of the box.

1.5 Other Types of Equations



- Solve a polynomial equation by factoring.
- Rewrite and solve an equation involving radicals or rational exponents.
- Rewrite and solve an equation involving fractions or absolute value.
- Construct and use a nonquadratic model to solve an application problem.





In Exercise 70 on page 124, you will solve an equation involving a radical to determine a person's age given his or her life expectancy.

Polynomial Equations

In this section, you will extend the techniques for solving equations to nonlinear and nonquadratic equations. At this point in the text, you have three basic methods for solving nonlinear equations—*factoring, extracting square roots,* and the *Quadratic Formula*. So, the main goal of this section is to learn to rewrite nonlinear equations in a form to which you can apply one of these methods.

Example 1 Solving a Polynomial Equation by Factoring

Solve $3x^4 = 48x^2$.

SOLUTION The basic approach is first to write the polynomial equation in general form with zero on one side, then to factor the other side, and finally to set each factor equal to zero and solve.

$3x^4 = 48x^2$			Write original equation.
$3x^4 - 48x^2 = 0$			Write in general form.
$3x^2(x^2 - 16) = 0$			Factor out common factor.
$3x^2(x + 4)(x - 4) = 0$			Difference of two squares
$3x^2 = 0$	⇨	$x = 0$	Set 1st factor equal to 0.
$x + 4 = 0$	⇨	$x = -4$	Set 2nd factor equal to 0.
$x - 4 = 0$	⇨	$x = 4$	Set 3rd factor equal to 0.

Check these solutions in the original equation.

CHECK

$3x^4 = 48x^2$	Write original equation.
$3(0)^4 = 48(0)^2$	Substitute 0 for x.
$0 = 0$	0 checks. ✓
$3(-4)^4 = 48(-4)^2$	Substitute -4 for x.
$768 = 768$	-4 checks. ✓
$3(4)^4 = 48(4)^2$	Substitute 4 for x.
$768 = 768$	4 checks. ✓

After checking, you can conclude that the solutions are

$x = 0, x = -4$, and $x = 4$.

STUDY TIP

When solving an equation, avoid dividing each side by a common variable factor to simplify. You may lose solutions. For instance, if you divide each side by x^2 in Example 1, you lose the solution $x = 0$. Also, when solving an equation by factoring, be sure to set each variable factor equal to zero to find all of the possible solutions.

✓Checkpoint 1

Solve $3x^3 = 3x$.

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Example 2 Solving a Polynomial Equation by Factoring

$$x^3 - 3x^2 - 3x + 9 = 0$$ Original equation

$$x^2(x - 3) - 3(x - 3) = 0$$ Group terms and factor.

$$(x - 3)(x^2 - 3) = 0$$ Distributive Property

$$x - 3 = 0 \Rightarrow x = 3$$ Set 1st factor equal to 0.

$$x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$$ Set 2nd factor equal to 0.

The solutions are

$$x = 3, \quad x = \sqrt{3}, \quad \text{and} \quad x = -\sqrt{3}.$$

Check these in the original equation. Notice that this polynomial has a degree of 3 and the equation has three solutions.

Checkpoint 2

Solve $x^3 - x^2 - 2x + 2 = 0$.

Occasionally, mathematical models involve equations that are of **quadratic type.** In general, an equation is of quadratic type if it can be written in the form

$$au^2 + bu + c = 0$$

where $a \neq 0$ and u is an algebraic expression.

Example 3 Solving an Equation of Quadratic Type

Solve $x^4 - 3x^2 + 2 = 0$.

SOLUTION This equation is of quadratic type with $u = x^2$.

$$(x^2)^2 - 3(x^2) + 2 = 0$$

$$u^2 - 3u + 2 = 0$$

To solve, factor as the product of two second-degree polynomials.

$$u^2 - 3u + 2 = 0$$ Write equation.

$$(u - 1)(u - 2) = 0$$ Factor.

$$(x^2 - 1)(x^2 - 2) = 0$$ $u = x^2$

$$(x + 1)(x - 1)(x^2 - 2) = 0$$ Completely factor.

$$x + 1 = 0 \Rightarrow x = -1$$ Set 1st factor equal to 0.

$$x - 1 = 0 \Rightarrow x = 1$$ Set 2nd factor equal to 0.

$$x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$$ Set 3rd factor equal to 0.

STUDY TIP

When using the u-substitution, remember to substitute back to the original variable. If the original equation is in terms of x, the solutions should be in terms of x.

The solutions are

$$x = -1, \quad x = 1, \quad x = \sqrt{2}, \quad \text{and} \quad x = -\sqrt{2}.$$

Check these in the original equation. Notice that this polynomial has a degree of 4 and the equation has four solutions.

Checkpoint 3

Solve $x^4 - 5x^2 + 4 = 0$.

Solving Equations Involving Radicals

The steps involved in solving the remaining equations in this section will often introduce *extraneous solutions*, as discussed in Section 1.1. Operations such as squaring each side of an equation, raising each side of an equation to a rational power, or multiplying each side of an equation by a variable quantity all create this potential danger. So, when you use any of these operations, checking possible solutions is crucial.

Example 4 An Equation Involving a Radical

Solve $\sqrt{2x + 7} - x = 2$.

SOLUTION

$$\sqrt{2x + 7} - x = 2$$ Write original equation.

$$\sqrt{2x + 7} = x + 2$$ Isolate the square root.

$$2x + 7 = x^2 + 4x + 4$$ Square each side.

$$0 = x^2 + 2x - 3$$ Write in general form.

$$0 = (x + 3)(x - 1)$$ Factor.

$$x + 3 = 0 \Rightarrow x = -3$$ Set 1st factor equal to 0.

$$x - 1 = 0 \Rightarrow x = 1$$ Set 2nd factor equal to 0.

By checking these values, you can determine that the only solution is $x = 1$.

✓Checkpoint 4

Solve $\sqrt{3x} - 6 = 0$.

STUDY TIP

The basic technique used in Example 5 is to isolate the factor with the rational exponent and raise each side to the reciprocal power. In Example 4, this is equivalent to isolating the square root and squaring each side.

Example 5 An Equation Involving a Rational Exponent

Solve $4x^{3/2} - 8 = 0$.

SOLUTION

$$4x^{3/2} - 8 = 0$$ Write original equation.

$$4x^{3/2} = 8$$ Add 8 to each side.

$$x^{3/2} = 2$$ Isolate $x^{3/2}$.

$$x = 2^{2/3}$$ Raise each side to the $\frac{2}{3}$ power.

$$x \approx 1.587$$ Round to three decimal places.

CHECK

$$4x^{3/2} - 8 = 0$$ Write original equation.

$$4(2^{2/3})^{3/2} \stackrel{?}{=} 8$$ Substitute $2^{2/3}$ for x.

$$4(2) \stackrel{?}{=} 8$$ Power of a Power Property

$$8 = 8$$ Solution checks. ✓

✓Checkpoint 5

Solve $2x^{3/4} - 54 = 0$.

Equations Involving Fractions or Absolute Value

In Section 1.1, you learned how to solve equations involving fractions. Recall that the first step is to multiply each term of the equation by the least common denominator (LCD).

Example 6 An Equation Involving Fractions

Solve $\frac{2}{x} = \frac{3}{x-2} - 1$.

SOLUTION For this equation, the LCD of the three terms is

$$x(x-2)$$

so begin by multiplying each term of the equation by this expression.

$$\frac{2}{x} = \frac{3}{x-2} - 1$$ Write original equation.

$$x(x-2)\frac{2}{x} = x(x-2)\frac{3}{x-2} - x(x-2)(1)$$ Multiply each term by LCD.

$$2(x-2) = 3x - x(x-2), \quad x \neq 0, 2$$ Simplify.

$$2x - 4 = -x^2 + 5x$$ Simplify.

$$x^2 - 3x - 4 = 0$$ Write in general form.

$$(x-4)(x+1) = 0$$ Factor.

$$x - 4 = 0 \Rightarrow x = 4$$ Set 1st factor equal to 0.

$$x + 1 = 0 \Rightarrow x = -1$$ Set 2nd factor equal to 0.

Notice that the values $x = 0$ and $x = 2$ are excluded from the domains of the fractions because they result in division by zero. So, both

$$x = 4 \quad \text{and} \quad x = -1$$

are possible solutions.

CHECK

$$\frac{2}{x} = \frac{3}{x-2} - 1$$ Write original equation.

$$\frac{2}{4} \stackrel{?}{=} \frac{3}{4-2} - 1$$ Substitute 4 for x.

$$\frac{1}{2} = \frac{3}{2} - 1$$ 4 checks. ✓

$$\frac{2}{-1} \stackrel{?}{=} \frac{3}{-1-2} - 1$$ Substitute -1 for x.

$$-2 = -1 - 1$$ -1 checks. ✓

The solutions are

$$x = 4 \quad \text{and} \quad x = -1.$$

✓ Checkpoint 6

Solve $\frac{3}{x} + \frac{1}{x-2} = 2$.

To solve an equation involving absolute value, remember that the expression inside the absolute value signs can be positive or negative. This results in *two* separate equations, each of which must be solved. For instance, the equation

$$|x - 2| = 3$$

results in the two equations

$$x - 2 = 3 \quad \text{and} \quad -(x - 2) = 3$$

which implies that the original equation has two solutions:

$$x = 5 \quad \text{and} \quad x = 1.$$

When writing the negative expression, it is important to remember to place parentheses around the entire expression that is inside the absolute value signs. After you write the two equations, solve each one independently.

Example 7 **An Equation Involving Absolute Value**

Solve $|x^2 - 3x| = -4x + 6$.

SOLUTION Because the variable expression inside the absolute value signs can be positive or negative, you must solve the following two equations.

First Equation

$$x^2 - 3x = -4x + 6$$ Use positive expression.

$$x^2 + x - 6 = 0$$ Write in general form.

$$(x + 3)(x - 2) = 0$$ Factor.

$$x + 3 = 0 \Rightarrow x = -3$$ Set 1st factor equal to 0

$$x - 2 = 0 \Rightarrow x = 2$$ Set 2nd factor equal to 0

Second Equation

$$-(x^2 - 3x) = -4x + 6$$ Use negative expression

$$x^2 - 7x + 6 = 0$$ Write in general form.

$$(x - 1)(x - 6) = 0$$ Factor.

$$x - 1 = 0 \Rightarrow x = 1$$ Set 1st factor equal to 0.

$$x - 6 = 0 \Rightarrow x = 6$$ Set 2nd factor equal to 0

CHECK

$$|(-3)^2 - 3(-3)| = -4(-3) + 6$$ -3 checks. ✓

$$|(2)^2 - 3(2)| \neq -4(2) + 6$$ 2 does not check.

$$|(1)^2 - 3(1)| = -4(1) + 6$$ 1 checks. ✓

$$|(6)^2 - 3(6)| \neq -4(6) + 6$$ 6 does not check.

The only solutions are

$$x = -3 \quad \text{and} \quad x = 1.$$

✓Checkpoint 7

Solve $|x^2 - 3| = 5x - 3$.

Applications

It would be virtually impossible to categorize all of the many different types of applications that involve nonlinear and nonquadratic models. However, from the few examples and exercises that follow, we hope you will gain some appreciation for the variety of applications that involve such models.



Example 8 Reduced Rates

A group of friends purchase a television sports package at a cost of \$350. To lower the cost per person, the group invites three more people to contribute to the purchase. As a result, the cost per person decreases by \$15. How many friends were in the original group that wanted to purchase the sports package?

SOLUTION Begin the solution by creating a verbal model and assigning labels.

Verbal Model: Cost per person · Total number of people = Cost of package

Labels:

Cost of package = 350 (dollars)

Number of people in original group = x (people)

Total number of people = $x + 3$ (people)

Original cost per person = $\frac{350}{x}$ (dollars per person)

Cost per person = $\frac{350}{x} - 15$ (dollars per person)

Equation:

$$\left(\frac{350}{x} - 15\right)(x + 3) = 350$$ Original equation

$$\left(\frac{350 - 15x}{x}\right)(x + 3) = 350$$ Rewrite first factor.

$$(350 - 15x)(x + 3) = 350x, \quad x \neq 0$$ Multiply each side by x.

$$350x + 1050 - 15x^2 - 45x = 350x$$ Multiply factors.

$$15x^2 - 45x + 1050 = 0$$ Subtract $350x$ from each side.

$$x^2 + 3x - 70 = 0$$ Divide each side by -15.

$$(x + 10)(x - 7) = 0$$ Factor.

$x + 10 = 0 \Rightarrow x = -10$ Set 1st factor equal to 0.

$x - 7 = 0 \Rightarrow x = 7$ Set 2nd factor equal to 0.

Only the positive x-value makes sense in the context of the problem. So, there were 7 friends in the original group that wanted to purchase the sports package.

CHECK

$$\left(\frac{350}{7} - 15\right)(7 + 3) \stackrel{?}{=} 350$$ Substitute 7 for x.

$$(35)(10) \stackrel{?}{=} 350$$ Simplify.

$$350 = 350$$ Solution checks. ✓

✓ Checkpoint 8

In Example 8, suppose the group of friends invites two more people instead of three to contribute to the purchase. As a result, the cost per person decreases by \$20. How many friends were in the original group that wanted to purchase the sports package? ■

Interest earned on a savings account is calculated by one of three basic methods: simple interest, interest compounded n times per year, and interest compounded continuously. The next example uses the formula for interest that is compounded n times per year,

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this formula, A is the balance in the account, P is the principal (or original deposit), r is the annual interest rate (in decimal form), n is the number of compoundings per year, and t is the time in years. In Chapter 4, you will study the derivation of this formula for compound interest.

Example 9 Compound Interest

When you were born, your grandparents deposited \$5000 in a savings account earning interest compounded quarterly. On your 25th birthday, the balance of the account is \$25,062.59. What is the average annual interest rate of the account?



SOLUTION

Formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Labels:

Balance $= A = 25{,}062.59$	(dollars)
Principal $= P = 5000$	(dollars)
Time $= t = 25$	(years)
Compoundings per year $= n = 4$	(compoundings)
Annual interest rate $= r$	(percent in decimal form)

Equation:

$$25{,}062.59 = 5000\left(1 + \frac{r}{4}\right)^{4(25)}$$ Substitute

$$\frac{25{,}062.59}{5000} = \left(1 + \frac{r}{4}\right)^{100}$$ Divide each side by 5000

$$5.0125 \approx \left(1 + \frac{r}{4}\right)^{100}$$ Use a calculator

$$(5.0125)^{1/100} \approx 1 + \frac{r}{4}$$ Raise each side to reciprocal power

$$1.01625 \approx 1 + \frac{r}{4}$$ Use a calculator

$$0.01625 \approx \frac{r}{4}$$ Subtract 1 from each side

$$0.065 \approx r$$ Multiply each side by 4

The average annual interest rate is about

$$0.065 = 6.5\%.$$

Check this in the original statement of the problem.

Checkpoint 9

You placed \$1000 in an account earning interest compounded monthly. After 3 years, the account balance is \$1144.25. What is the annual interest rate of the account?

Stephen Coburn/Shutterstock.com

Example 10 Market Research

The marketing department of a publishing company is asked to determine the price of a book. The department determines that the demand for the book depends on the price of the book according to the model

$$p = 40 - \sqrt{0.0001x + 1}, \quad 0 \le x \le 15{,}990{,}000$$

where p is the price per book (in dollars) and x is the number of books sold at the given price. For instance, in Figure 1.12, note that if the price was \$39, then (according to the model) no one would be willing to buy the book. On the other hand, if the price was \$17.60, 5 million copies could be sold. The marketing department set the price at \$12.95. How many copies can the publisher expect to sell?

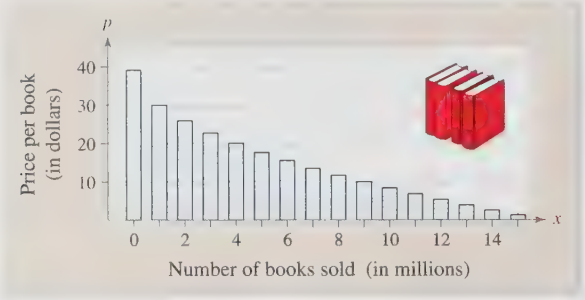


FIGURE 1.12

SOLUTION

$p = 40 - \sqrt{0.0001x + 1}$		Write given model.
$12.95 = 40 - \sqrt{0.0001x + 1}$		Set price at \$12.95.
$\sqrt{0.0001x + 1} = 27.05$		Isolate the radical.
$0.0001x + 1 = 731.7025$		Square each side.
$0.0001x = 730.7025$		Subtract 1 from each side.
$x = 7{,}307{,}025$		Divide each side by 0.0001.

So, by setting the book's price at \$12.95, the publisher can expect to sell about 7.3 million copies.

Checkpoint 10

In Example 10, suppose the marketing department set the price at \$19.95. How many copies can the publisher expect to sell?



SUMMARIZE (Section 1.5)

1. Describe how to solve a polynomial equation by factoring *(pages 115 and 116)*. For examples of solving polynomial equations by factoring, see Examples 1, 2, and 3.
2. Describe how to solve an equation involving absolute value *(page 119)*. For an example of solving an equation involving absolute value, see Example 7.
3. Describe a real-life example that uses a nonlinear or nonquadratic model *(pages 120–122, Examples 8, 9, and 10)*.

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SKILLS WARM UP 1.5 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.3 and 1.4.

In Exercises 1–10, find the real solution(s) of the equation.

1. $x^2 - 22x + 121 = 0$
2. $x(x - 20) + 3(x - 20) = 0$
3. $(x + 20)^2 = 625$
4. $5x^2 + x = 0$
5. $3x^2 + 4x - 4 = 0$
6. $12x^2 + 8x - 55 = 0$
7. $x^2 + 4x - 5 = 0$
8. $4x^2 + 4x - 15 = 0$
9. $x^2 - 3x + 1 = 0$
10. $x^2 - 4x + 2 = 0$

Exercises 1.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Solving a Polynomial Equation In Exercises 1–20, find the real solution(s) of the polynomial equation. Check your solution(s). *See Examples 1, 2, and 3.*

1. $x^3 - 2x^2 - 3x = 0$
2. $20x^3 - 125x = 0$
3. $4x^4 - 36x^2 = 0$
4. $2x^4 - 15x^3 + 18x^2 = 0$
5. $x^4 - 81 = 0$
6. $x^6 - 64 = 0$
7. $5x^3 + 30x^2 + 45x = 0$
8. $9x^4 - 24x^3 + 16x^2 = 0$
9. $x^3 - 7x^2 - 4x + 28 = 0$
10. $x^3 + 2x^2 + 3x + 6 = 0$
11. $x^4 - x^3 + x - 1 = 0$
12. $x^4 + 2x^3 - 8x - 16 = 0$
13. $x^4 - 12x^2 + 11 = 0$
14. $x^4 - 29x^2 + 100 = 0$
15. $x^4 + 5x^2 - 36 = 0$
16. $x^4 - 4x^2 + 3 = 0$
17. $4x^4 - 65x^2 + 16 = 0$
18. $36t^4 + 29t^2 - 7 = 0$
19. $x^6 + 7x^3 - 8 = 0$
20. $x^6 + 3x^3 + 2 = 0$

An Equation Involving a Radical In Exercises 21–34, find the real solution(s) of the radical equation. Check your solution(s). *See Example 4.*

21. $\sqrt{2x} - 10 = 0$
22. $4\sqrt{x} - 3 = 0$
23. $\sqrt{x - 10} - 4 = 0$
24. $\sqrt{5 - x} - 3 = 0$
25. $\sqrt[3]{2x + 5} + 3 = 0$
26. $\sqrt[3]{3x + 1} - 5 = 0$
27. $2x + 9\sqrt{x} - 5 = 0$
28. $6x - 7\sqrt{x} - 3 = 0$
29. $x = \sqrt{11x - 30}$
30. $2x - \sqrt{15 - 4x} = 0$
31. $-\sqrt{26 - 11x} + 4 = x$
32. $x + \sqrt{31 - 9x} = 5$
33. $\sqrt{x + 1} - 3x = 1$
34. $\sqrt{2x + 1} + x = 7$

An Equation Involving a Rational Exponent In Exercises 35–40, find the real solution(s) of the equation involving rational exponents. Check your solution(s). *See Example 5.*

35. $(x - 5)^{2/3} - 16 = 0$
36. $(x + 3)^{4/3} - 16 = 0$
37. $(x + 3)^{3/2} - 8 = 0$
38. $(x^2 + 2)^{2/3} - 9 = 0$
39. $(x^2 - 5)^{2/3} = 16$
40. $(x^2 - x - 22)^{4/3} = 16$

An Equation Involving Fractions In Exercises 41–48, find the real solution(s) of the equation involving fractions. Check your solution(s). *See Example 6.*

41. $\frac{1}{x} = \frac{4}{x - 1} + 1$
42. $\frac{1}{x} - \frac{1}{x + 1} = 3$
43. $\frac{4}{x} - \frac{5}{3} = \frac{x}{6}$
44. $\frac{5}{x} - \frac{3}{2} = \frac{x}{5}$
45. $\frac{4}{x + 1} - \frac{3}{x + 2} = 1$
46. $\frac{x}{x^2 - 4} + \frac{1}{x + 2} = 3$
47. $\frac{x + 2}{x - 2} - \frac{x}{3} = 0$
48. $\frac{x + 1}{3} - \frac{x + 1}{x + 2} = 0$

An Equation Involving Absolute Value In Exercises 49–56, find the real solution(s) of the equation involving absolute value. Check your solution(s). *See Example 7.*

49. $|x + 1| = 2$
50. $|x - 2| = 3$
51. $|2x - 1| - 5 = 0$
52. $|3x + 2| - 7 = 0$
53. $|x| = x^2 + x - 3$
54. $|3x| = x^2 + 3x - 16$
55. $|x - 10| = x^2 - 10x$
56. $|x + 1| = x^2 - 5$

57. Error Analysis Find the error(s) in the solution.

$$\sqrt{3x} = \sqrt{7x + 4}$$
$$3x = 7x + 4$$
$$x = \frac{-7 \pm \sqrt{7^2 - 4(3)(4)}}{2(3)}$$
$$x = -1 \text{ and } x = -\frac{4}{3}$$

58. Error Analysis Find the error(s) in the solution.

$$\sqrt{6 - 2x} - 3 = 0$$
$$6 - 2x + 9 = 0$$
$$-2x = -15$$
$$x = \frac{15}{2}$$

Using a Calculator In Exercises 59–62, use a calculator to find the real solutions of the equation. (Round your answers to three decimal places.)

59. $3.2x^4 - 1.5x^2 - 2.1 = 0$

60. $7.08x^6 + 4.15x^3 - 9.6 = 0$

61. $1.8x - 6\sqrt{x} - 5.6 = 0$

62. $4x + 8\sqrt{x} + 3.6 = 0$

63. Sharing the Cost A college charters a bus for $1700 to take a group of students to the Fiesta Bowl. When six more students join the trip, the cost per student decreases by $7.50. How many students were in the original group?

64. Sharing the Cost Three students plan to share equally in the rent for an apartment. By adding a fourth person, the cost per person decreases by $125 per month. How much is the monthly rent for the apartment?

65. Compound Interest A deposit of $3000 reaches a balance of $3422.91 after 6 years. The interest on the account is compounded monthly. What is the annual interest rate for this investment?

66. Compound Interest A sales representative describes a "guaranteed investment fund" that is offered to new investors. By depositing $15,000 in the fund, you will be guaranteed to receive a total of at least $40,000 after 20 years. Assuming that interest is compounded quarterly, determine the fixed annual rates of interest that yield

(a) the minimum guarantee.

(b) a return of $48,000 after 20 years.

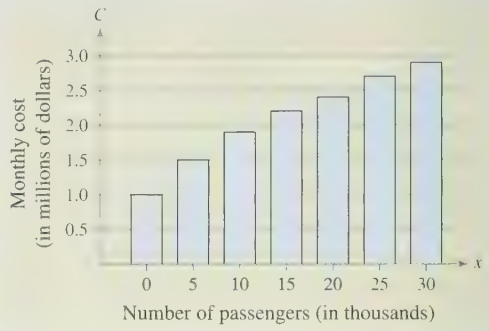
67. Borrowing Money You borrow $300 from a friend and agree to pay the money back, plus $20 in interest, after 6 months. Assuming that the interest is compounded monthly, what annual interest rate are you paying?

68. Cash Advance You take out a cash advance of $1000 on a credit card. After 2 months, you owe $1041.93. The interest is compounded monthly. What is the annual interest rate for this cash advance?

69. Airline Passengers An airline offers daily flights between Chicago and Denver. The total monthly cost C (in millions of dollars) of these flights is modeled by

$$C = \sqrt{0.25x + 1}$$

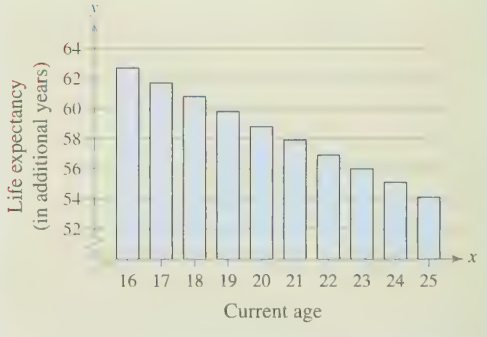
where x is the number of passengers flying that month in thousands (see figure). The total cost of the flights for a month is 3.5 million dollars. Use the model to determine how many passengers flew that month.



70. Life Expectancy The life expectancy of a person who is 16 to 25 years old can be modeled by

$$y = \sqrt{1.216x^2 - 161.12x + 6197.8}, \quad 16 \le x \le 25$$

where y represents the number of additional years the person is expected to live and x represents the person's current age. *(Source: U.S. National Center for Health Statistics)*



(a) Determine the life expectancies of persons who are 18, 20, and 22 years old.

(b) A person's life expectancy is 55 years. Use the model to determine the age of the person.

71. Life Expectancy The life expectancy of a person who is 50 to 65 years old can be modeled by

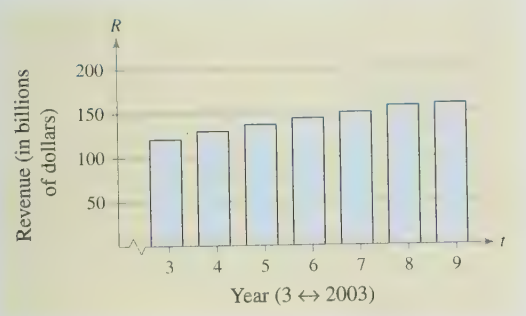
$$y = \sqrt{0.828x^2 - 135.71x + 5672.1}, \quad 50 \le x \le 65$$

where y represents the number of additional years the person is expected to live and x represents the person's current age. A person's life expectancy is 25 years. How old is the person? *(Source: U.S. National Center for Health Statistics)*

72. Fast-Food Industry The yearly revenues R (in billions of dollars) of fast-food restaurants in the United States from 2003 through 2009 can be approximated by

$$R = 125.7 + 4.69t - \frac{58.6}{t}, \quad 3 \le t \le 9$$

where t represents the year, with $t = 3$ corresponding to 2003 (see figure). Use the model to predict the year in which fast-food revenues will be about \$172 billion. *(Source: National Restaurant Association)*

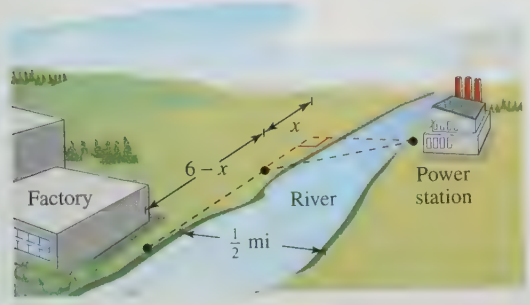


73. Market Research The demand equation for a product is modeled by

$$p = 40 - \sqrt{0.01x + 1}$$

where x is the number of units demanded per day and p is the price per unit. Find the demand when the price is set at \$13.95. Explain why this model is valid only for $0 \le x \le 159{,}900$.

74. Power Line A power station is on one side of a river that is $\frac{1}{2}$ mile wide. A factory is 6 miles downstream on the other side of the river. It costs \$18 per foot to run power lines over land and \$24 per foot to run them under water. The project's cost is \$616,877.27. Find the length x as labeled in the figure.



75. Flour Production A company weighs each 16-ounce bag of flour it produces. After production, any bag that does not weigh within 0.4 ounce of 16 ounces cannot be sold. Solve the equation $|x - 16| = 0.4$ to find the least and greatest acceptable weights of a 16-ounce bag of flour.

76. Sugar Production A company weighs each 80-ounce bag of sugar it produces. After production, any bag that does not weigh within 1.2 ounces of 80 ounces cannot be sold. Solve the equation $|x - 80| = 1.2$ to find the least and greatest acceptable weights of an 80-ounce bag of sugar.

77. Work Rate With only the cold water valve open, it takes 8 minutes to fill the tub of a washing machine. With both the hot and cold water valves open, it takes 5 minutes. The time it takes for the tub to fill with only the hot water valve open can be modeled by the equation

$$\frac{1}{8} + \frac{1}{t} = \frac{1}{5}$$

where t is the time (in minutes) for the tub to fill. How long does it take for the tub of the washing machine to fill with only the hot water valve open?

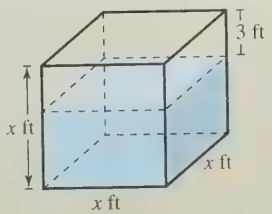
78. Community Service You and a friend volunteer to paint a house as a community service project. Working alone, you can paint the house in 28 hours. Your friend can paint the house in 25 hours working alone. How long will it take both of you, working together, to paint the house?

79. Discovery Look back at your solutions in Exercises 1–20 and recall what you learned in Sections 1.3 and 1.4 about the numbers of solutions of quadratic equations. Make a conjecture about how the number of solutions of a polynomial equation in general form is related to the degree of the polynomial.

80. HOW DO YOU SEE IT? The figure shows a glass cube partially filled with water. The equation

$$x^2(x - 3) = 320$$

represents the situation.



(a) What does the expression $x^2(x - 3)$ represent?

(b) Explain how you can use the given equation to find the capacity of the cube.

1.6 Linear Inequalities



In Exercise 85 on page 136, you will use an inequality to predict when the average National Football League player's salary will exceed $2.2 million.

- Write bounded and unbounded intervals using inequalities or interval notation.
- Develop an understanding of the properties of inequalities.
- Solve and graph a linear inequality.
- Solve an inequality involving absolute value.
- Construct and use a linear inequality to solve an application problem.

Introduction

Simple inequalities are used to *order* real numbers. The inequality symbols $<$, $\leq$, $>$, and $\geq$ are used to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality

$$x \geq 3$$

denotes all real numbers x that are greater than or equal to 3.

In this section, you will expand your work with inequalities to include more involved statements such as

$$5x - 7 > 3x + 9$$

and

$$-3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. Such values are **solutions** and are said to **satisfy** the inequality. The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality. For instance, the solution set of

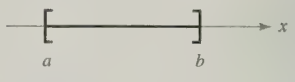
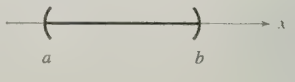
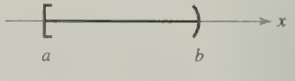
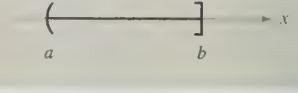
$$x + 3 > 4$$

is all real numbers that are greater than 1.

The set of all points on the real number line that represent the solution set of an inequality is the **graph** of the inequality. Graphs of many types of inequalities consist of intervals on the real number line. The four different types of **bounded** intervals are summarized below.

Bounded Intervals on the Real Number Line

Let a and b be real numbers such that $a < b$. The following intervals on the real number line are **bounded.** The numbers a and b are the **endpoints** of each interval.

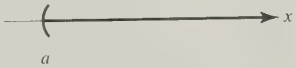
Notation	*Interval Type*	*Inequality*	*Graph*
$[a, b]$	Closed	$a \leq x \leq b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \leq x < b$	
$(a, b]$		$a < x \leq b$	

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Note that a closed interval contains both of its endpoints and an open interval does not contain either of its endpoints. Often, the solution of an inequality is an interval on the real line that is **unbounded.** For instance, the interval consisting of all positive numbers is unbounded. The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$.

Unbounded Intervals on the Real Number Line

Let a and b be real numbers. The following intervals on the real number line are **unbounded.**

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line	$-\infty < x < \infty$	

Example 1 **Intervals and Inequalities**

Write an inequality to represent each interval. Then state whether the interval is bounded or unbounded.

a. $(-3, 5]$

b. $(-4, \infty)$

c. $[0, 2]$

d. $(-\infty, 0)$

SOLUTION

a. $(-3, 5]$ corresponds to $-3 < x \leq 5$. Bounded

b. $(-4, \infty)$ corresponds to $x > -4$. Unbounded

c. $[0, 2]$ corresponds to $0 \leq x \leq 2$. Bounded

d. $(-\infty, 0)$ corresponds to $x < 0$. Unbounded

Checkpoint 1

Write an inequality to represent each interval. Then state whether the interval is bounded or unbounded.

a. $[2, 7)$

b. $(-\infty, 6)$

Properties of Inequalities

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the **properties of inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed. Here is an example.

$2 < 5$ Original inequality

$(-3)(-2) > (-3)(5)$ Multiply each side by -3 and reverse the inequality symbol.

$6 > -15$ Simplify.

Notice in the example above that when the inequality symbol is not reversed, you obtain the false statement

$6 < -15.$ False statement

Two inequalities that have the same solution set are **equivalent.** For instance, the inequalities

$x + 2 < 5$ and $x < 3$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The following list describes operations that can be used to create equivalent inequalities.

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. *Transitive Property*

$a < b$ and $b < c$ $\Rightarrow$ $a < c$

2. *Addition of Inequalities*

$a < b$ and $c < d$ $\Rightarrow$ $a + c < b + d$

3. *Addition of a Constant*

$a < b$ $\Rightarrow$ $a + c < b + c$

4. *Multiplication by a Constant*

For $c > 0$, $a < b$ $\Rightarrow$ $ac < bc$

For $c < 0$, $a < b$ $\Rightarrow$ $ac > bc$ Reverse direction of inequality symbol.

Each of the properties above is true if the symbol $<$ is replaced by $\leq$ and the symbol $>$ is replaced by $\geq$. For instance, another form of the multiplication property would be as shown.

For $c > 0$, $a \leq b$ $\Rightarrow$ $ac \leq bc$

For $c < 0$, $a \leq b$ $\Rightarrow$ $ac \geq bc$

On your own, try to verify each of the properties of inequalities by using several examples with real numbers.

Solving a Linear Inequality

The simplest type of inequality to solve is a **linear inequality** in a single variable. For instance,

$$2x + 3 > 4$$

is a linear inequality in x.

As you read through the following examples, pay special attention to the steps in which the inequality symbol is reversed. Remember that when you multiply or divide by a negative number, you must reverse the inequality symbol.

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For help in solving inequalities similar to the one in Example 2, see the review of solving inequalities on page 149.

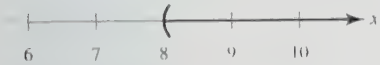
Example 2 Solving a Linear Inequality

Solve $5x - 7 > 3x + 9$.

SOLUTION

$5x - 7 > 3x + 9$	Write original inequality.
$2x - 7 > 9$	Subtract $3x$ from each side.
$x > 8$	Add 7 to each side and then divide each side by 2.

The solution set is all real numbers that are greater than 8, which is denoted by $(8, \infty)$. The graph is shown in Figure 1.13.



Solution Interval: $(8, \infty)$

FIGURE 1.13

Checkpoint 2

Solve $3x < 2x + 1$.

Checking the solution set of an inequality is not as simple as checking the solutions of an equation. You can, however, get an indication of the validity of a solution set by substituting a few convenient values of x to see whether the original inequality is satisfied. For instance, substitute several values of x that are greater than 8 into the original inequality in Example 2 and check to see if the inequality is satisfied.

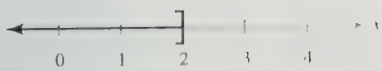
Example 3 Solving a Linear Inequality

Solve $1 - \frac{3x}{2} \geq x - 4$.

SOLUTION

$1 - \frac{3x}{2} \geq x - 4$	Write original inequality.
$2 - 3x \geq 2x - 8$	Multiply each side by 2.
$2 - 5x \geq -8$	Subtract $2x$ from each side.
$-5x \geq -10$	Subtract 2 from each side.
$x \leq 2$	Divide each side by -5 and reverse inequality symbol.

The solution set is all real numbers that are less than or equal to 2, which is denoted by $(-\infty, 2]$. The graph is shown in Figure 1.14.



Solution Interval: $(-\infty, 2]$

FIGURE 1.14

Checkpoint 3

Solve $-\frac{4x}{3} \leq 2 - x$. Then graph the solution set on the real number line.

Sometimes it is convenient to write two inequalities as a **double inequality.** For instance, you can write the two inequalities

$$-4 \le 5x - 2 \quad \text{and} \quad 5x - 2 < 7$$

more simply as

$$-4 \le 5x - 2 < 7.$$

This enables you to solve the two inequalities together, as demonstrated in Example 4.

Example 4 Solving a Double Inequality

Solve $-3 \le 6x - 1 < 3$.

SOLUTION To solve a double inequality, you can isolate x as the middle term.

$$-3 \le 6x - 1 < 3 \qquad \text{Write original inequality.}$$

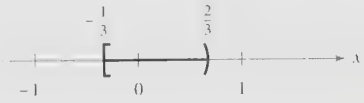
$$-3 + 1 \le 6x - 1 + 1 < 3 + 1 \qquad \text{Add 1 to each part.}$$

$$-2 \le 6x < 4 \qquad \text{Simplify.}$$

$$\frac{-2}{6} \le \frac{6x}{6} < \frac{4}{6} \qquad \text{Divide each part by 6.}$$

$$-\frac{1}{3} \le x < \frac{2}{3} \qquad \text{Simplify.}$$

The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ and less than $\frac{2}{3}$. The interval notation for this solution set is $\left[-\frac{1}{3}, \frac{2}{3}\right)$. The graph of this solution set is shown in Figure 1.15.



Solution Interval: $\left[-\frac{1}{3}, \frac{2}{3}\right)$

FIGURE 1.15

✓Checkpoint 4

Solve $-1 < 3 - 2x \le 5$. Then graph the solution set on the real number line. ■

The double inequality in Example 4 could have been solved in two parts as follows.

$$-3 \le 6x - 1 \quad \text{and} \quad 6x - 1 < 3$$

$$-2 \le 6x \qquad\qquad 6x < 4$$

$$-\frac{1}{3} \le x \qquad\qquad x < \frac{2}{3}$$

The solution set consists of all real numbers that satisfy *both* inequalities. In other words, the solution set is the set of all values of x for which

$$-\tfrac{1}{3} \le x < \tfrac{2}{3}.$$

When combining two inequalities to form a double inequality, be sure that the inequalities satisfy the Transitive Property. For instance, it is *incorrect* to combine the inequalities $3 < x$ and $x \le -1$ as $3 < x \le -1$. This is obviously wrong because 3 is not less than -1.

Inequalities Involving Absolute Value

STUDY TIP

When an absolute value inequality is solved, the solution may consist of a bounded interval on the real number line (Example 5) or two unbounded intervals on the real number line (Example 6).

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that $a \geq 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a.

   $|x| < a$ if and only if $-a < x < a$.

2. The solutions of $|x| > a$ are all values of x that are less than $-a$ or greater than a.

   $|x| > a$ if and only if $x < -a$ or $x > a$.

These rules are also valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Example 5 Solving an Absolute Value Inequality

Solve $|x - 5| < 2$.

SOLUTION

$$|x - 5| < 2$$ Write original inequality.

$$-2 < x - 5 < 2$$ Equivalent inequality

$$-2 + 5 < x - 5 + 5 < 2 + 5$$ Add 5 to each part.

$$3 < x < 7$$ Simplify.

The solution set consists of all real numbers that are greater than 3 and less than 7, which is denoted by $(3, 7)$. The graph is shown in Figure 1.16.

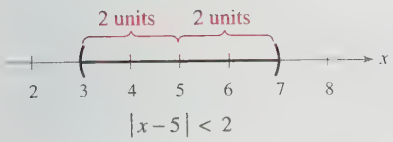


FIGURE 1.16

Checkpoint 5

Solve $|x + 2| \leq 7$. Then graph the solution set on the real number line.

Example 6 Solving an Absolute Value Inequality

Solve $|x + 3| \geq 7$.

SOLUTION

$$|x + 3| \geq 7$$ Write original inequality.

$$x + 3 \leq -7 \quad \text{or} \quad x + 3 \geq 7$$ Equivalent inequalities

$$x + 3 - 3 \leq -7 - 3 \qquad x + 3 - 3 \geq 7 - 3$$ Subtract 3 from each side.

$$x \leq -10 \qquad x \geq 4$$ Simplify.

The solution set is all real numbers that are less than or equal to -10 *or* greater than or equal to 4, which is denoted by

$$(-\infty, -10] \cup [4, \infty)$$

(see Figure 1.17). The symbol $\cup$ (union) means *or*.

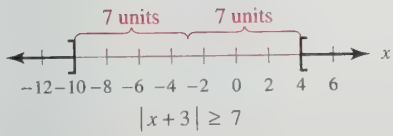


FIGURE 1.17

Checkpoint 6

Solve $|x + 1| > 3$. Then graph the solution set on the real number line.

Applications

Example 7 Comparing Taxi Fares

The cost of traveling by taxi in Las Vegas is \$3.20 for the initial charge and \$2 for each mile driven. The cost in Philadelphia is \$2.30 for the initial charge and \$2.10 for each mile driven. How many miles must a passenger travel by taxi in Philadelphia before the cost is more than the cost for a passenger to travel by taxi in Las Vegas?

SOLUTION

Verbal Model: Cost for taxi in Philadelphia $>$ Cost for taxi in Las Vegas

Labels:
Miles driven $= m$ (miles)
Cost for taxi in Las Vegas $= 3.2 + 2m$ (dollars)
Cost for taxi in Philadelphia $= 2.3 + 2.1m$ (dollars)

Inequality: $2.3 + 2.1m > 3.2 + 2m$

$0.1m > 0.9$

$m > 9$

When a passenger travels more than 9 miles, the cost for a taxi in Philadelphia is more than the cost for a taxi in Las Vegas.

Checkpoint 7

In Example 7, suppose the initial charge for a taxi in Philadelphia is raised to \$2.50 and the cost per mile is raised to \$2.20. How many miles must a passenger travel by taxi in Philadelphia before the cost is more than the cost for a passenger to travel by taxi in Las Vegas?

Example 8 Exercise Program



A 225-pound man begins an exercise and diet program that is designed to reduce his weight by at least 2 pounds per week. Find the maximum number of weeks before the man's weight will reach his goal of 192 pounds.

SOLUTION

Verbal Model: Desired weight $\leq$ Current weight $-$ 2 pounds per week $\cdot$ Number of weeks

Labels:
Desired weight $= 192$ (pounds)
Current weight $= 225$ (pounds)
Number of weeks $= x$ (weeks)

Inequality: $192 \leq 225 - 2x$

$-33 \leq -2x$

$16.5 \geq x$

It will take at most $16\frac{1}{2}$ weeks for the man to reach his goal.

Checkpoint 8

In Example 8, find the maximum number of weeks before the man's weight will reach 200 pounds.

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Example 9 Accuracy of a Measurement

You go to a candy store to buy chocolates that cost \$9.89 per pound. The scale used in the store has a state seal of approval that indicates the scale is accurate to within one-half ounce. According to the scale, your purchase weighs one-half pound and costs \$4.95. How much might you have been undercharged or overcharged due to an error in the scale?

SOLUTION To solve this problem, let x represent the *true* weight of the candy. Because the scale is accurate to within one-half ounce (or $\frac{1}{32}$ pound), you can conclude that the absolute value of the difference of the exact weight (x) and the scale weight ($\frac{1}{2}$ pound) is less than or equal to $\frac{1}{32}$ pound. That is,

$$\left|x - \frac{1}{2}\right| \leq \frac{1}{32}.$$

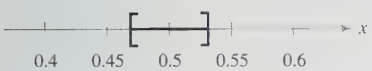
You can solve this inequality as shown.

$$-\frac{1}{32} \leq x - \frac{1}{2} \leq \frac{1}{32}$$

$$\frac{15}{32} \leq x \leq \frac{17}{32}$$

$$0.46875 \leq x \leq 0.53125$$

The graph of this solution set is shown in Figure 1.18. In other words, your "one-half" pound of candy could have weighed as little as 0.46875 pound (which would have cost $0.46875 \cdot \$9.89 \approx \$4.64$) or as much as 0.53125 pound (which would have cost $0.53125 \cdot \$9.89 \approx \$5.25$). So, you could have been undercharged by as much as \$0.30 or overcharged by as much as \$0.31.



Solution Interval: [0.46875, 0.53125]

FIGURE 1.18

Checkpoint 9

You go to a grocery store to buy ground beef that costs \$3.96 per pound. The scale used in the store is accurate to within $\frac{1}{3}$ ounce (or $\frac{1}{48}$ pound). According to the scale, your purchase weighs 7.5 pounds and costs \$29.70. How much might you have been undercharged or overcharged due to an error in the scale?



SUMMARIZE (Section 1.6)

1. Make a list of the different types of bounded and unbounded intervals on the real number line *(pages 126 and 127)*. For an example on bounded and unbounded intervals, see Example 1.
2. Make a list of the properties of inequalities *(page 128)*.
3. Describe how to solve a linear inequality *(pages 128 and 129)*. For examples of solving linear inequalities, see Examples 2 and 3.
4. State the definition of a double inequality *(page 130)*. For an example of solving a double inequality, see Example 4.
5. Describe how to solve an absolute value inequality *(page 131)*. For examples of solving absolute value inequalities, see Examples 5 and 6.
6. Describe a real-life example that uses an inequality *(pages 132 and 133, Examples 7, 8, and 9)*.

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SKILLS WARM UP 1.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.1 and 0.2.

In Exercises 1–4, determine which of the two numbers is greater.

1. $-\frac{1}{2}, -7$
2. $-\frac{1}{3}, -\frac{1}{6}$
3. $\pi, -3$
4. $-6, -\frac{13}{2}$

In Exercises 5–8, use inequality notation to describe the subset of real numbers.

5. x is nonnegative.
6. z is strictly between -3 and 10.
7. P is no more than 2.
8. W is at least 200.

In Exercises 9 and 10, evaluate the expression at each value of x.

9. $|x - 10|, x = 12, x = 3$
10. $|2x - 3|, x = \frac{3}{2}, x = 1$

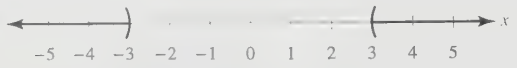
Exercises 1.6

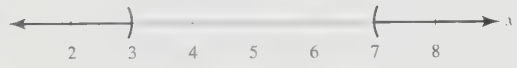
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

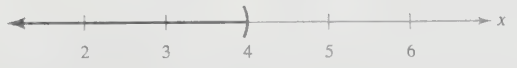
Intervals and Inequalities In Exercises 1–6, write an inequality that represents the interval. Then state whether the interval is bounded or unbounded. *See Example 1.*

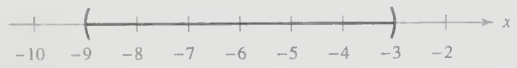
1. $[-1, 5]$
2. $(2, 10]$
3. $(11, \infty)$
4. $[-5, \infty)$
5. $(-\infty, -2)$
6. $(-\infty, 7]$

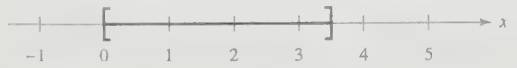
Matching In Exercises 7–14, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

(g) 

(h) 

7. $x < 4$
8. $x \geq 6$
9. $-2 < x \leq 5$
10. $0 \leq x \leq \frac{7}{2}$
11. $|x| < 4$
12. $|x| > 3$
13. $|x - 5| > 2$
14. $|x + 6| < 3$

Checking Solutions In Exercises 15–22, determine whether each value of x is a solution of the inequality.

15. $5x - 12 > 0$

    (a) $x = 3$ (b) $x = -3$ (c) $x = \frac{5}{2}$ (d) $x = \frac{3}{2}$

16. $4x - 5 < 3$

    (a) $x = 0$ (b) $x = 2$ (c) $x = -2$ (d) $x = 4$

17. $0 < \frac{x - 2}{4} < 2$

    (a) $x = 4$
    (b) $x = 10$
    (c) $x = 0$
    (d) $x = \frac{7}{2}$

18. $-1 < \frac{3 - x}{2} \leq 1$

    (a) $x = 0$
    (b) $x = -5$
    (c) $x = 1$
    (d) $x = 5$

19. $|x + 2| \leq 10$

    (a) $x = -15$
    (b) $x = -4$
    (c) $x = 1$
    (d) $x = 8$

20. $|x - 10| \geq 3$

    (a) $x = 13$
    (b) $x = -1$
    (c) $x = 14$
    (d) $x = 9$

21. $|2x - 3| < 15$

    (a) $x = -6$
    (b) $x = 0$
    (c) $x = 12$
    (d) $x = 7$

22. $|3x + 5| > 7$

    (a) $x = -5$
    (b) $x = -2$
    (c) $x = \frac{1}{3}$
    (d) $x = 10$

Solving an Inequality In Exercises 23–30, copy and complete the statement using the correct inequality symbol.

23. If $2x > 6$, then x __?__ 3.

24. If $3x > 9$, then x __?__ 3.

25. If $2x \leq -8$, then x __?__ -4.

26. If $3x \leq -15$, then x __?__ -5.

27. If $2 - 4x > -10$, then x __?__ 3.

28. If $5 - 3x > -7$, then x __?__ 4.

29. If $-\frac{2}{3}x \geq -6$, then x __?__ 9.

30. If $-\frac{3}{4}x \geq -12$, then x __?__ 16.

Solving a Linear Inequality In Exercises 31–44, solve the inequality. Then graph the solution set on the real number line. *See Examples 2 and 3.*

31. $\frac{3}{2}x \geq 9$

32. $\frac{2}{5}x > 7$

33. $-10x < 40$

34. $-6x > 15$

35. $\frac{3}{5}x - 7 < 8$

36. $\frac{5}{4}x + 1 \leq 11$

37. $2x + 7 < 3 + 4x$

38. $6x - 4 \leq 2 + 8x$

39. $2x - 1 \geq 5x + 1$

40. $3x + 1 \geq 2 + x$

41. $3(x + 2) + 7 < 2x - 5$

42. $2(x + 7) - 4 \geq 5(x - 3)$

43. $-3(x - 1) + 7 < 2x + 8$

44. $5 - 3x > -5(x + 4) + 6$

Solving a Double Inequality In Exercises 45–52, solve the inequality. Then graph the solution set on the real number line. *See Example 4.*

45. $3 \leq 2x - 1 < 7$

46. $3 > 1 - \frac{x}{2} > -3$

47. $-11 \leq 1 + 4(x + 1) < 13$

48. $-8 \leq 1 - 3(x - 2) < 13$

49. $-4 < \frac{x - 3}{3} < 4$

50. $0 \leq \frac{x + 3}{2} < 5$

51. $\frac{3}{4} > x + 1 > \frac{1}{4}$

52. $-\frac{1}{2} < x - 1 < \frac{3}{4}$

Solving an Absolute Value Inequality In Exercises 53–70, solve the inequality. Then graph the solution set on the real number line. *See Examples 5 and 6.*

53. $|x| < 6$

54. $|x| > 8$

55. $\left|\frac{x}{2}\right| > 3$

56. $|5x| > 10$

57. $|x + 3| < 5$

58. $|x - 7| < 6$

59. $|x - 5| < 0$

60. $|x - 5| \geq 0$

61. $|2x - 5| > 6$

62. $|2x - 20| \leq 4$

63. $2|x + 10| \geq 9$

64. $3|4 - 5x| \leq 9$

65. $\left|\frac{x - 3}{2}\right| \geq 5$

66. $\left|\frac{2x + 1}{2}\right| < 6$

67. $|9 - x| - 2 < -1$

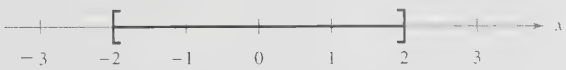
68. $|x + 14| + 3 > 17$

69. $2|2x - 5| - 4 > 6$

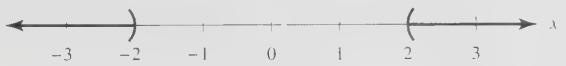
70. $2|5 - 3x| + 7 < 21$

Using Absolute Value In Exercises 71–78, use absolute value notation to define the solution set.

71.



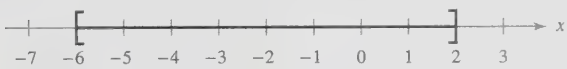
72.



73.



74.



75. All real numbers at most 10 units from 12

76. All real numbers at least 5 units from 8

77. All real numbers whose distances from -3 are more than 5

78. All real numbers whose distances from -6 are no more than 7

79. Annual Operating Cost A utility company has a fleet of vans. The annual operating cost C per van is

$$C = 0.52m + 2500$$

where m is the number of miles traveled by a van in a year. What number of miles will yield an annual operating cost that is less than \$12,000?

80. IQ Scores The admissions office of a college wants to determine whether there is a relationship between IQ scores x and grade-point averages y after the first year of school. An equation that models the data obtained by the admissions office is

$$y = 0.068x - 4.753.$$

Estimate the values of x that predict a grade-point average of at least 3.0.

81. Brand Name Drugs The average prices B (in dollars) of brand name prescription drugs from 2005 through 2009 can be approximated by

$$B = 14.516t + 24.65, \quad 5 \leq t \leq 9$$

where t represents the year, with $t = 5$ corresponding to 2005. Use the model to estimate the first year in which the average price exceeded \$140. *(Source: National Association of Chain Drug Stores)*

82. **Generic Drugs** The average prices G (in dollars) of generic prescription drugs from 2005 through 2009 can be approximated by

$$G = 2.649t + 15.00, \quad 5 \le t \le 9$$

where t represents the year, with $t = 5$ corresponding to 2005. Use the model to estimate the first year in which the average price exceeded \$30. *(Source: National Association of Chain Drug Stores)*

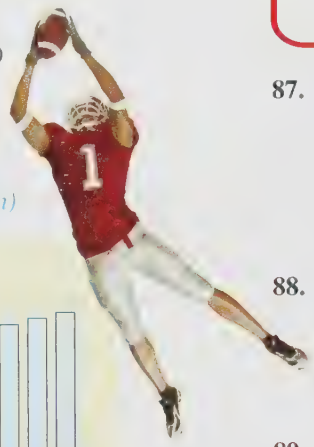
83. **Comparative Shopping** The cost of renting a car from Company A is \$279 per week with no extra charge for mileage. The cost of renting a similar car from Company B is \$199 per week, plus 32 cents for each mile driven. How many miles must you drive in a week to make the rental fee for Company B greater than that for Company A?

84. **Comparative Shopping** Your department sends its copying to a photocopy center, which bills your department \$0.08 per page. You are considering buying a departmental copier for \$2500. With your own copier, the cost per page would be \$0.025. The expected life of the copier is 4 years. How many copies must you make in the four-year period to justify purchasing the copier?

85. **Football Salaries** The average National Football League player's salary S (in millions of dollars) each year from 1998 through 2009 can be approximated by

$$S = 0.1215t - 0.351, \quad 8 \le t \le 19$$

where t represents the year, with $t = 8$ corresponding to 1998 (see figure). *(Source: National Football League Players Association)*





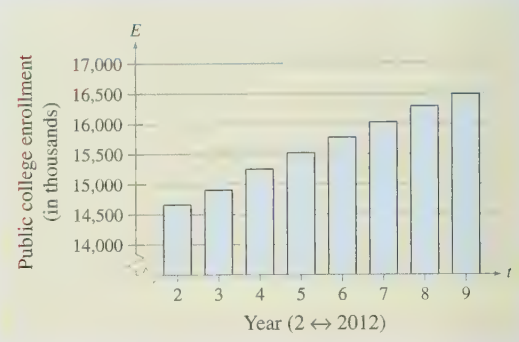
(a) Use the model to estimate the years when the average National Football League player's salary was less than \$1.5 million. Do your results agree with the bar graph? Explain.

(b) Use the model to predict the first year that the average National Football League player's salary will exceed \$2.2 million. Do your results seem reasonable? Explain.

86. **Public College Enrollment** The projected public college enrollment E (in thousands) in the United States from 2012 to 2019 can be approximated by

$$E = 265.3t + 14{,}157, \quad 2 \le t \le 9$$

where t represents the year, with $t = 2$ corresponding to 2012 (see figure). *(Source: U.S. National Center for Education Statistics)*



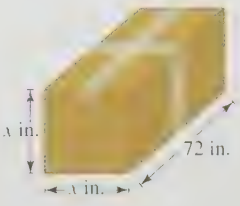
(a) Use the model to predict the first year that the public college enrollment will be greater than 16 million students. Do your results agree with the bar graph? Explain.

(b) Use the model to predict the first year that the public college enrollment will be greater than 17 million students. Do your results seem reasonable? Explain.

87. **Weight Loss Program** A person enrolls in a weight loss program that guarantees a loss of at least $1\frac{1}{2}$ pounds per week. The person's weight at the beginning of the program is 180 pounds. Find the maximum number of weeks it will take for the person to attain a weight of 130 pounds.

88. **Salary Increase** You accept a new job with a starting salary of \$34,800. You are told that you will receive an annual raise of at least \$1500. What is the maximum number of years you must work before your annual salary will be \$45,000?

89. **Maximum Width** An overnight delivery company will not accept any package whose combined length and girth (perimeter of a cross section) exceeds 132 inches. The package shown is 72 inches long and has square cross sections. Describe the lengths of x for which the company will deliver the package.



90. Break-Even Analysis The revenue R from selling x units of a product is

$R = 139.95x.$

The cost C of producing x units is

$C = 97x + 850.$

In order to obtain a profit, the revenue must be greater than the cost.

(a) Complete the table.

x	10	20	30	40	50	60
R						
C						

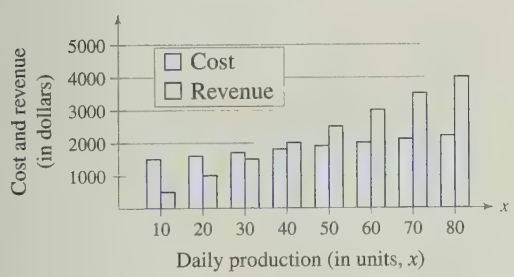
(b) For what values of x will this product return a profit?

91. Break-Even Analysis The revenue R from selling x units of a product is $R = 25.95x$. The cost C of producing x units is

$C = 13.95x + 125{,}000.$

In order to obtain a profit, the revenue must be greater than the cost. For what values of x will the product return a profit?

92. HOW DO YOU SEE IT? A company sells all the units of a product that it produces. The double bar graph shows the daily cost and revenue based on the numbers of units x produced.



(a) Use the double bar graph to estimate the number of units the company must produce in a day to break even, with cost equal to revenue.

(b) Use your answer in part (a) to write an inequality that represents the daily numbers of units x for which the company receives a profit.

(c) Write an inequality that represents the daily numbers of units x for which the company receives a profit of at least \$1000.

93. Simple Interest For \$1500 to grow to more than \$1680 in 3 years, what must the simple interest rate be?

94. Simple Interest For \$2000 to grow to more than \$2180 in 2 years, what must the simple interest rate be?

95. Geometry You measure the side of a square as 10.4 inches with a possible error of $\frac{1}{16}$ inch. Using these measurements, determine the interval containing the possible areas of the square.

96. Geometry You measure the side of a square as 24.2 centimeters with a possible error of 0.25 centimeter. Using these measurements, determine the interval containing the possible areas of the square.

97. Human Height The heights h of two-thirds of a population satisfy the inequality

$|h - 68.5| \leq 2.7$

where h is measured in inches. Determine the interval on the real number line in which these heights lie.

98. Time Study A time study was conducted to determine the length of time required to perform a particular task in a manufacturing process. The times required by approximately two-thirds of the workers in the study satisfied the inequality

$$\left|\frac{t - 15.6}{1.9}\right| < 1$$

where t is time in minutes. Determine the interval on the real number line in which these times lie.

99. Humidity Control The specifications for an electronic device state that it is to be operated in a room with relative humidity h defined by

$|h - 50| \leq 30.$

What are the minimum and maximum relative humidities for the operation of this device?

100. Body Temperature Physicians consider an adult's body temperature x (in degrees Fahrenheit) to be normal when it satisfies the inequality

$|x - 98.6| \leq 1.$

Determine the range of temperatures that are considered to be normal.

101. Accuracy of Measurement You buy six T-bone steaks that cost \$8.99 per pound. The weight listed on the package is 5.72 pounds. The scale that weighed the package is accurate to within $\frac{1}{2}$ ounce. How much money might you have been undercharged or overcharged?

102. Daily Sales A doughnut shop sells a dozen doughnuts for \$7.95. Beyond the fixed costs (rent, utilities, and insurance) of \$165 per day, it costs \$1.45 for enough materials (flour, sugar, and so on) and labor to produce a dozen doughnuts. The daily profit from doughnut sales varies between \$400 and \$1200. Between what numbers of doughnuts (in dozens) do the daily sales vary?

1.7 Other Types of Inequalities

- Use critical numbers to determine test intervals for a polynomial inequality.
- Solve and graph a polynomial inequality.
- Solve and graph a rational inequality.
- Construct and use a polynomial inequality to solve an application problem.
- Determine the domain of an expression involving a radical.



In Exercise 74 on page 147, you will use a quadratic inequality to make a prediction about corporate sales.

Finding Critical Numbers and Test Intervals

To solve a polynomial inequality such as

$$x^2 - 2x - 3 < 0$$

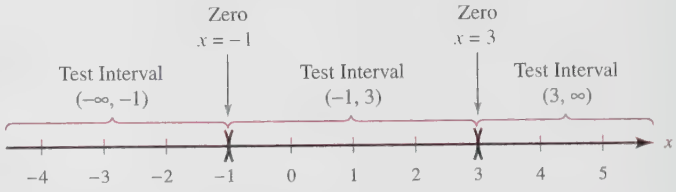
you can use the fact that a polynomial can change signs only at its **zeros.** The zeros are the x-values that make the polynomial equal to zero. Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **critical numbers** of the inequality, and the resulting intervals are the **test intervals** for the inequality. For example, the polynomial above factors as

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

and has two zeros, $x = -1$ and $x = 3$. These zeros divide the real number line into three test intervals:

$$(-\infty, -1), \quad (-1, 3), \quad \text{and} \quad (3, \infty). \qquad \text{(See Figure 1.19.)}$$

So, to solve the inequality $x^2 - 2x - 3 < 0$, you need only test one value from each of these test intervals.



Three Test Intervals for $x^2 - 2x - 3 < 0$

FIGURE 1.19

Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order. These zeros are the critical numbers of the polynomial.
2. Use the critical numbers to determine the test intervals.
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative at the representative x-value, the polynomial will have negative values for *every* x-value in the interval. If the value of the polynomial is positive, the polynomial will have positive values for *every* x-value in the interval.

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Polynomial Inequalities

Example 1 Solving a Polynomial Inequality

Solve $x^2 - x - 6 < 0$.

SOLUTION By factoring the quadratic as

$$x^2 - x - 6 = (x + 2)(x - 3)$$

you can see that the critical numbers are

$$x = -2 \quad \text{and} \quad x = 3.$$ Critical numbers

The boundaries between the numbers that satisfy the inequality and the numbers that do not satisfy the inequality always occur at critical numbers. So, the polynomial's test intervals are

$$(-\infty, -2), \quad (-2, 3), \quad \text{and} \quad (3, \infty).$$ Test intervals

In each test interval, choose a representative x-value and evaluate the polynomial.

Test Interval	x-Value	Polynomial Value	Conclusion
$(-\infty, -2)$	$x = -3$	$(-3)^2 - (-3) - 6 = 6$	Positive
$(-2, 3)$	$x = 0$	$(0)^2 - (0) - 6 = -6$	Negative
$(3, \infty)$	$x = 4$	$(4)^2 - (4) - 6 = 6$	Positive

From this, you can conclude that the polynomial is positive for all x-values in $(-\infty, -2)$ and $(3, \infty)$, and is negative for all x-values in $(-2, 3)$. This implies that the solution of the inequality

$$x^2 - x - 6 < 0$$

is the interval $(-2, 3)$, as shown in Figure 1.20.

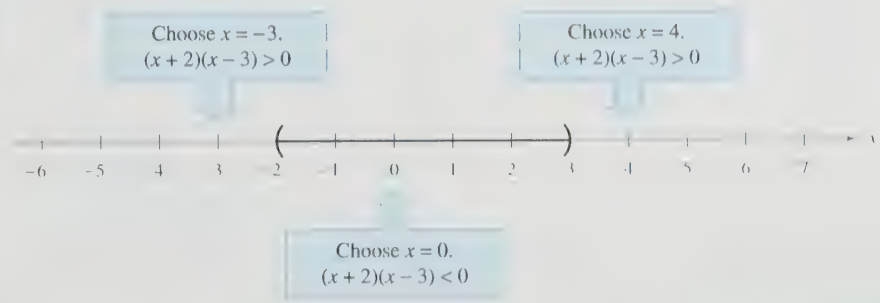
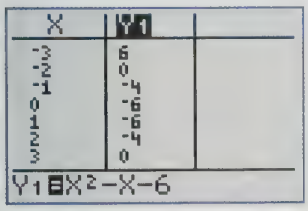


FIGURE 1.20

TECH TUTOR

You can use the *table* feature of your graphing utility to check the sign of the polynomial in each interval.



Checkpoint 1

Solve $x^2 + x - 2 < 0$.

As with linear inequalities, you can check a solution interval of a polynomial inequality by substituting x-values into the original inequality. For instance, to check the solution found in Example 1, try substituting several x-values from the interval $(-2, 3)$ into the inequality

$$x^2 - x - 6 < 0.$$

Regardless of which x-values you choose, the inequality will be satisfied.

In Example 1, the polynomial inequality was given in general form. Whenever this is not the case, begin the solution process by writing the inequality in general form—with the polynomial on one side and zero on the other.

Example 2 Solving a Polynomial Inequality

Solve $x^3 - 3x^2 > 10x$.

SOLUTION

$$x^3 - 3x^2 > 10x \qquad \text{Write original inequality.}$$

$$x^3 - 3x^2 - 10x > 0 \qquad \text{Write in general form.}$$

$$x(x - 5)(x + 2) > 0 \qquad \text{Factor.}$$

You can see that the critical numbers are $x = -2$, $x = 0$, and $x = 5$. So, the test intervals are $(-\infty, -2)$, $(-2, 0)$, $(0, 5)$, and $(5, \infty)$. In each test interval, choose a representative x-value and evaluate the polynomial.

Test Interval	x-Value	Polynomial Value	Conclusion
$(-\infty, -2)$	$x = -3$	$(-3)^3 - 3(-3)^2 - 10(-3) = -24$	Negative
$(-2, 0)$	$x = -1$	$(-1)^3 - 3(-1)^2 - 10(-1) = 6$	Positive
$(0, 5)$	$x = 2$	$(2)^3 - 3(2)^2 - 10(2) = -24$	Negative
$(5, \infty)$	$x = 6$	$(6)^3 - 3(6)^2 - 10(6) = 48$	Positive

From this, you can conclude that the inequality is satisfied on the open intervals $(-2, 0)$ and $(5, \infty)$. So, the solution set consists of all real numbers in the intervals $(-2, 0)$ and $(5, \infty)$, as shown in Figure 1.21.

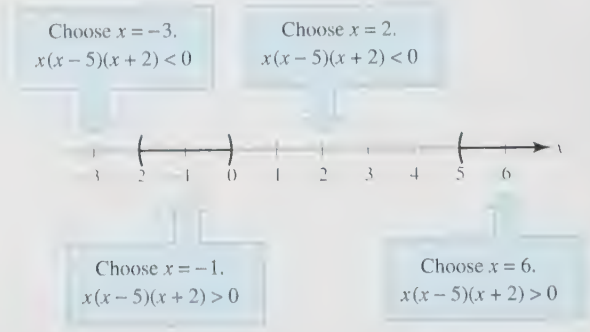


FIGURE 1.21

✓Checkpoint 2

Solve $x^2 - 3x > -2$.

When solving a polynomial inequality, be sure to account for the type of inequality symbol it contains. For instance, in Example 2, note that the solution consisted of two *open* intervals because the original inequality contained a "greater than" symbol. If the original inequality had been

$$x^3 - 3x^2 \geq 10x$$

then the solution would have consisted of the *closed* interval $[-2, 0]$ and the interval $[5, \infty)$.

Each of the polynomial inequalities in Examples 1 and 2 has a solution set that consists of a single interval or the union of two intervals. When solving the exercises for this section, watch for some unusual solution sets, as illustrated in Example 3.

Example 3 Unusual Solution Sets

What is unusual about the solution set for each inequality?

a. $x^2 + 2x + 4 > 0$

The solution set for this inequality consists of the entire set of real numbers, $(-\infty, \infty)$. In other words, the value of the quadratic $x^2 + 2x + 4$ is positive for every real value of x.

b. $x^2 + 2x + 1 \leq 0$

The solution set for this inequality consists of the single real number $\{-1\}$, because the quadratic $x^2 + 2x + 1$ has one critical number, $x = -1$, and it is the only value that satisfies the inequality.

c. $x^2 + 3x + 5 < 0$

The solution set for this inequality is empty. In other words, the quadratic $x^2 + 3x + 5$ is *not* less than zero for any value of x.

d. $x^2 - 4x + 4 > 0$

The solution set for this inequality consists of all real numbers *except* the number 2. In interval notation, this solution can be written as $(-\infty, 2) \cup (2, \infty)$.

✓ Checkpoint 3

What is unusual about the solution set for each inequality?

a. $x^2 + x + 3 \leq 0$ **b.** $x^2 - 2x + 1 > 0$



TECH TUTOR

Most graphing utilities can graph inequalities. Consult your user's guide for specific instructions. Once you know how to graph an inequality, you can check solutions by graphing. (Make sure you use an appropriate viewing window that includes all of the critical numbers.) For example, the solution of

$$x^2 - 5x < 0$$

is the interval $(0, 5)$. When graphed, the solution occurs as an interval above the horizontal axis on the graphing utility, as shown in Figure 1.22. The graph does not indicate whether 0 and/or 5 are part of the solution. You must determine whether the endpoints are part of the solution based on the type of inequality.

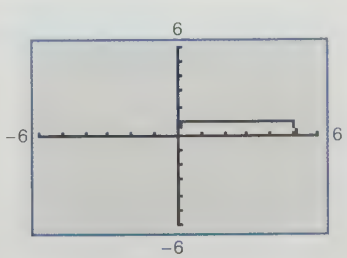


FIGURE 1.22

Rational Inequalities

The concepts of critical numbers and test intervals can be extended to inequalities involving rational expressions. Use the fact that the value of a rational expression can change sign only at its *zeros* (the x-values for which its numerator is zero) and its *undefined values* (the x-values for which its denominator is zero). These two types of numbers make up the **critical numbers** of a rational inequality.

Example 4 **Solving a Rational Inequality**

Solve $\frac{2x - 7}{x - 5} \leq 3$.

SOLUTION

$$\frac{2x - 7}{x - 5} \leq 3 \qquad \text{Write original inequality.}$$

$$\frac{2x - 7}{x - 5} - 3 \leq 0 \qquad \text{Write in general form.}$$

$$\frac{2x - 7 - 3x + 15}{x - 5} \leq 0 \qquad \text{Add fractions.}$$

$$\frac{-x + 8}{x - 5} \leq 0 \qquad \text{Simplify.}$$

Critical numbers: $x = 5, x = 8$

Test intervals: $(-\infty, 5), (5, 8), (8, \infty)$

Test: Is $\frac{-x + 8}{x - 5} \leq 0$?

After testing these intervals, as shown in Figure 1.23, you can see that the inequality is satisfied on the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because

$$\frac{-x + 8}{x - 5} = 0$$

when $x = 8$, you can conclude that the solution set consists of all real numbers in the intervals $(-\infty, 5) \cup [8, \infty)$.

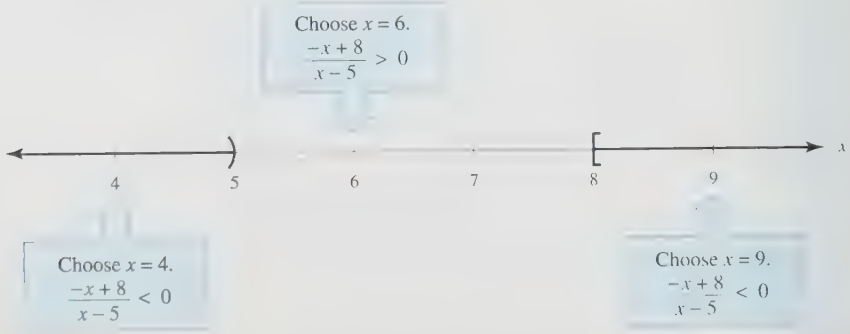


FIGURE 1.23

Checkpoint 4

Solve $\frac{x - 1}{x - 3} \geq -1$.

Application

One common business application of inequalities involves profit, revenue, and cost. The formula that relates these three quantities is

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$P = R - C.$$

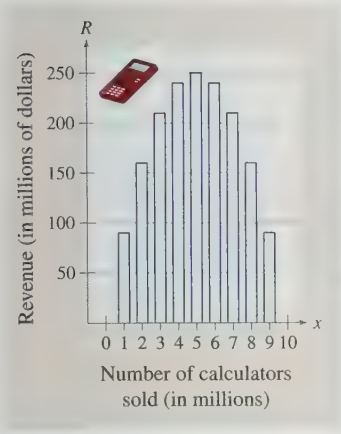


FIGURE 1.24

Example 5 Increasing the Profit for a Product

The marketing department of a calculator manufacturer has determined that the demand for a new model of calculator is given by

$$p = 100 - 10x, \quad 0 \le x \le 10$$ Demand equation

where p is the price per calculator (in dollars) and x is the number of calculators sold, in millions. (If this model is accurate, no one would be willing to pay \$100 for the calculator. At the other extreme, the company couldn't *give* away more than 10 million calculators.) The revenue, in millions of dollars, for selling x million calculators is given by

$$R = xp = x(100 - 10x).$$ Revenue equation

See Figure 1.24. The total cost of producing x million calculators is \$10 per calculator plus a one-time development cost of \$2,500,000. So, the total cost, in millions of dollars, is

$$C = 10x + 2.5.$$ Cost equation

What prices can the company charge per calculator to obtain a profit of at least \$190,000,000?

SOLUTION

Verbal Model: $\text{Profit} = \text{Revenue} - \text{Cost}$

Equation: $P = R - C$

$$P = 100x - 10x^2 - (10x + 2.5)$$

$$P = -10x^2 + 90x - 2.5$$

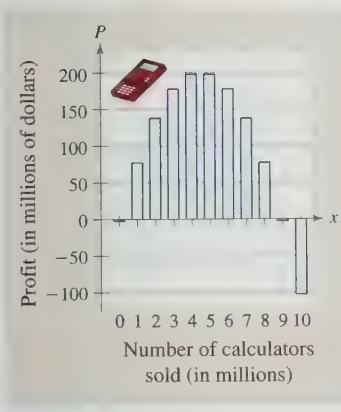


FIGURE 1.25

To answer the question, you must solve the inequality

$$-10x^2 + 90x - 2.5 \ge 190.$$

Using the techniques described in this section, you can find the solution set to be

$$3.5 \le x \le 5.5$$

as shown in Figure 1.25. The prices that correspond to these x-values are given by

$$100 - 10(3.5) \ge p \ge 100 - 10(5.5)$$

$$45 \le p \le 65.$$

The company can obtain a profit of \$190,000,000 or better by charging at least \$45 per calculator and at most \$65 per calculator.

Checkpoint 5

In Example 5, what prices can the company charge per calculator to obtain a profit of at least \$160,000,000?

Domain of a Radical Expression

Recall that the domain of an expression is the set of all x-values for which the expression is defined. You can use inequalities to find the domain of an expression that involves a radical, as shown in Example 6.

Example 6 **Finding the Domain of an Expression**

Find the domain of the expression $\sqrt{64 - 4x^2}$.

SOLUTION Because $\sqrt{64 - 4x^2}$ is defined (has real values) only if $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \geq 0$.

$$64 - 4x^2 \geq 0 \qquad \text{Write in general form.}$$

$$16 - x^2 \geq 0 \qquad \text{Divide each side by 4.}$$

$$(4 - x)(4 + x) \geq 0 \qquad \text{Factor.}$$

So, the inequality has two critical numbers: $x = -4$ and $x = 4$. You can use these two numbers to test the inequality, as follows.

Critical numbers: $x = -4, x = 4$

Test intervals: $(-\infty, -4), (-4, 4), (4, \infty)$

Test: Is $(4 - x)(4 + x) \geq 0$?

A test shows that $64 - 4x^2$ is greater than or equal to 0 in the *closed interval* $[-4, 4]$. So, the domain of the expression $\sqrt{64 - 4x^2}$ is the interval $[-4, 4]$, as shown in Figure 1.26.

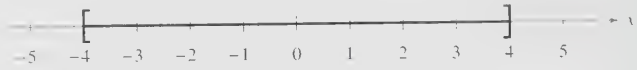


FIGURE 1.26

Checkpoint 6

Find the domain of each expression.

a. $\sqrt{12 - 3x^2}$

b. $\sqrt[3]{x^2 - 2x - 8}$



SUMMARIZE (Section 1.7)

1. Describe how to find the critical numbers and test intervals of a polynomial inequality *(page 138)*.
2. Describe how to solve a polynomial inequality *(pages 139–141)*. For examples of solving polynomial inequalities, see Examples 1, 2, and 3.
3. Describe how to solve a rational inequality *(page 142)*. For an example of solving a rational inequality, see Example 4.
4. Describe a real-life example of how a polynomial inequality can be used to analyze profit, revenue, and cost *(page 143, Example 5)*.
5. Describe how to find the domain of an expression involving a radical *(page 144)*. For an example of finding the domain of an expression involving a radical, see Example 6.

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SKILLS WARM UP 1.7

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 1.6.

In Exercises 1–10, solve the inequality.

1. $-\frac{y}{3} > 2$
2. $-6z < 27$
3. $-3 \le 2x + 3 < 5$
4. $-3x + 5 \ge 20$
5. $10 > 4 - 3(x + 1)$
6. $3 < 1 + 2(x - 4) < 7$
7. $2|x| \le 7$
8. $|x - 3| > 1$
9. $|x + 4| > 2$
10. $|2 - x| \le 4$

Exercises 1.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding Test Intervals In Exercises 1–6, find the test intervals of the inequality.

1. $x^2 - 25 < 0$
2. $x^2 - 6x + 8 > 0$
3. $2x^2 + 7x + 16 \ge 20$
4. $3x^2 - 26x + 25 \le 9$
5. $\frac{x - 3}{x - 1} < 2$
6. $\frac{x - 4}{2x + 3} \ge 1$

Solving a Polynomial Inequality In Exercises 7–34, solve the inequality. Then graph the solution set on the real number line. *See Examples 1, 2, and 3.*

7. $x^2 \le 9$
8. $x^2 < 5$
9. $x^2 > 4$
10. $x^2 \ge 16$
11. $(x + 2)^2 < 25$
12. $(x + 6)^2 \le 8$
13. $x^2 + 4x + 4 \ge 9$
14. $x^2 - 6x + 9 < 16$
15. $x^2 + x < 6$
16. $x^2 + 2x > 3$
17. $3(x - 1)(x + 1) > 0$
18. $6(x + 2)(x - 1) < 0$
19. $x^2 + 2x - 3 < 0$
20. $x^2 - 4x - 1 > 0$
21. $x^2 - 8x + 16 > 0$
22. $x^2 + 6x + 9 > 0$
23. $4x^2 - 4x + 1 \le 0$
24. $4x^2 + 12x + 9 \le 0$
25. $x^2 + 4 > 0$
26. $x^2 + 9 \le 0$
27. $x^2 < -1$
28. $x^2 \ge -7$
29. $4x^3 - 6x^2 < 0$
30. $4x^3 - 12x^2 > 0$
31. $x^3 - 4x \ge 0$
32. $2x^3 - x^4 \le 0$
33. $x^3 - 2x^2 - x + 2 \ge 0$
34. $x^3 + 5x^2 - 4x - 20 \le 0$

Solving a Rational Inequality In Exercises 35–44, solve the inequality. Then graph the solution set on the real number line. *See Example 4.*

35. $\frac{1}{x} > x$
36. $\frac{1}{x} < 4$
37. $\frac{x + 6}{x + 1} < 2$
38. $\frac{x + 12}{x + 2} \ge 3$
39. $\frac{3x - 5}{x - 5} > 4$
40. $\frac{5 + 7x}{1 + 2x} < 4$
41. $\frac{4}{x + 5} > \frac{1}{2x + 3}$
42. $\frac{5}{x - 6} > \frac{3}{x + 2}$
43. $\frac{1}{x - 3} \le \frac{9}{4x + 3}$
44. $\frac{1}{x} \ge \frac{1}{x + 3}$

Finding the Domain of an Expression In Exercises 45–54, find the domain of the expression. *See Example 6.*

45. $\sqrt{x^2 - 9}$
46. $\sqrt{x^2 - 4}$
47. $\sqrt[4]{6 + x^2}$
48. $\sqrt{x^2 + 4}$
49. $\sqrt{81 - 4x^2}$
50. $\sqrt{147 - 3x^2}$
51. $\sqrt{x^2 - 7x + 10}$
52. $\sqrt{12 - x - x^2}$
53. $\sqrt{x^2 - 3x + 3}$
54. $\sqrt[4]{-x^2 + 2x - 2}$

Writing the Solution in Interval Notation In Exercises 55–60, solve the inequality and write the solution set in interval notation.

55. $6x^3 - 10x^2 > 0$
56. $25x^3 - 10x^2 < 0$
57. $x^3 - 9x \le 0$
58. $4x^3 - x^4 \ge 0$
59. $(x - 1)^2(x + 2)^3 \ge 0$
60. $x^4(x - 3) \le 0$

Using a Calculator In Exercises 61–66, use a calculator to solve the inequality. (Round each number in your answer to two decimal places.)

61. $0.4x^2 + 5.26 < 10.2$
62. $-1.3x^2 + 3.78 > 2.12$
63. $-0.5x^2 + 12.5x + 1.6 > 0$
64. $1.2x^2 + 4.8x + 3.1 < 5.3$

65. $\dfrac{1}{2.3x - 5.2} > 3.4$

66. $\dfrac{2}{3.1x - 3.7} > 5.8$

67. **Geometry** A rectangular playing field with a perimeter of 100 meters is to have an area of at least 500 square meters (see figure). Describe the possible lengths l of the field.



68. **Height of a Projectile** A projectile is fired straight upward from ground level with an initial velocity of 160 feet per second. During what time period will its height be less than 384 feet?

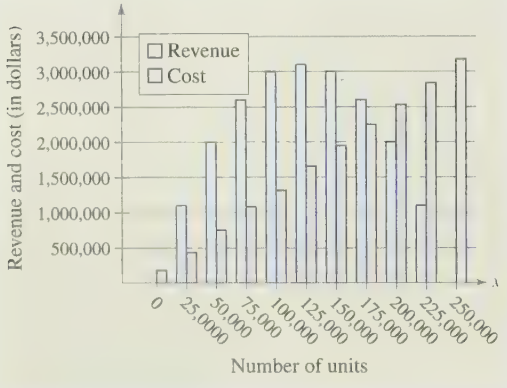
69. **Company Profits** The revenue R and cost C for a product are given by

$R = x(50 - 0.0002x)$

and

$C = 12x + 150{,}000$

where R and C are measured in dollars and x represents the number of units produced and sold (see figure).



(a) How many units must be sold to obtain a profit of at least \$1,650,000?

(b) The demand equation for the product is

$p = 50 - 0.0002x$

where p is the price per unit. What prices will produce a profit of at least \$1,650,000?

(c) As the number of units increases, the revenue eventually decreases. After this point, at what number of units is the revenue approximately equal to the cost? How should this affect the company's decision about the level of production?

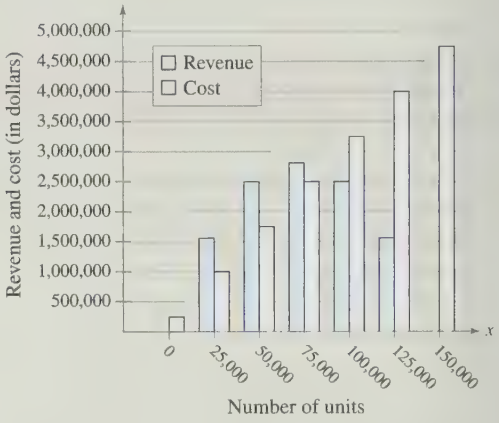
70. **HOW DO YOU SEE IT?** The revenue R and cost C for a product are given by

$R = x(75 - 0.0005x)$

and

$C = 30x + 250{,}000$

where R and C are measured in dollars and x represents the number of units produced and sold. The figure shows the values of R and C at several production levels.



(a) What is the revenue at a production level of zero units?

(b) How many levels of production are there for which the company breaks even, with cost equal to revenue?

(c) Do the maximum revenue and the maximum profit occur at the same level of production? Explain.

(d) Describe the maximum profit shown by the graph and the corresponding level of production.

71. **Compound Interest** P dollars, invested at interest rate r compounded annually, increases to an amount A given by

$A = P(1 + r)^3$

in 3 years. For an investment of \$1000 to increase to an amount greater than \$1150 in 3 years, the interest rate must be greater than what percent?

72. **Compound Interest** P dollars, invested at interest rate r compounded annually, increases to an amount A given by

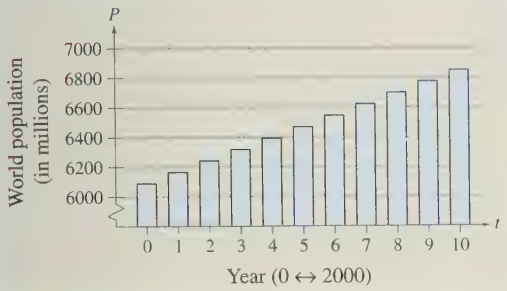
$A = P(1 + r)^2$

in 2 years. For an investment of \$2000 to increase to an amount greater than \$2200 in 2 years, the interest rate must be greater than what percent?

73. World Population The world population P (in millions) from 2000 to 2010 can be approximated by

$$P = 0.06t^2 + 75.9t + 6089, \quad 0 \le t \le 10$$

where t represents the year, with $t = 0$ corresponding to 2000 (see figure). *(Source: U.S. Census Bureau)*



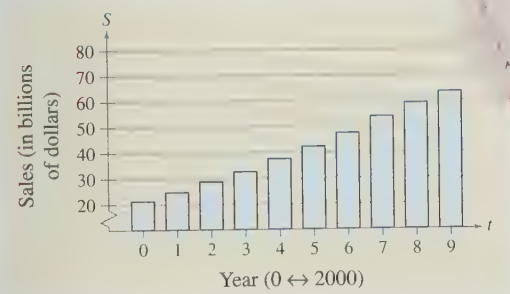
(a) Use the model to estimate the world population in 2004. Compare your answer with the value shown in the bar graph.

(b) Use the model to predict the first year in which the world population will exceed 7 billion. Does your answer seem reasonable?

74. Sales The total sales S (in billions of dollars) for Walgreen Company each year from 2000 through 2009 can be approximated by

$$S = 0.126t^2 + 3.68t + 20.9, \quad 0 \le t \le 9$$

where t represents the year, with $t = 0$ corresponding to 2000 (see figure). *(Source: Walgreen Company)*





(a) Use the model to estimate Walgreen's sales in 2008. How well does your answer compare with the value shown in the bar graph?

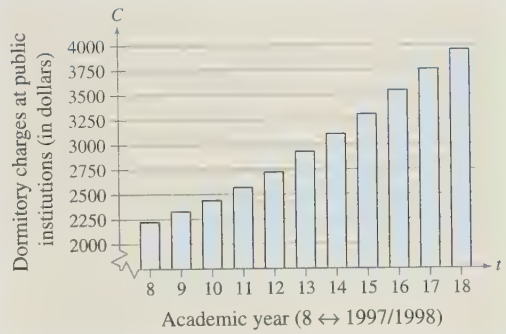
(b) Use the model to predict the first year in which Walgreen's sales will be greater than $75 billion.

(c) Walgreen Company predicts yearly sales exceeding $85 billion between 2013 and 2015. Does the model support this prediction? Explain your reasoning.

75. Higher Education The average yearly dormitory charges C (in dollars) at public institutions of higher learning in the United States for the academic years 1997/1998 through 2007/2008 can be approximated by

$$C = 7.20t^2 - 9.7t + 1828, \quad 8 \le t \le 18$$

where t represents the year, with $t = 8$ corresponding to the academic year 1997/1998 (see figure). Use the model to predict the first academic year in which the average yearly dormitory charges will be greater than $6000. *(Source: U.S. National Center for Education Statistics)*



76. Higher Education The average yearly dormitory charges C (in dollars) at private institutions of higher learning in the United States for the academic years 1996/1997 through 2007/2008 can be approximated by

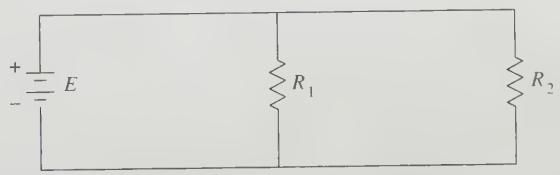
$$C = 6.20t^2 + 26.3t + 2357, \quad 7 \le t \le 18$$

where t represents the year, with $t = 7$ corresponding to the academic year 1996/1997. Use the model to predict the first academic year in which the average yearly dormitory charges will be greater than $6000. *(Source: U.S. National Center for Education Statistics)*

77. Resistors When two resistors of resistances R_1 and R_2 are connected in parallel (see figure), the total resistance R satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Find R_1 for a parallel circuit in which $R_2 = 2$ ohms and R must be at least 1 ohm.



78. Think About It Explain why the domains of

$$\sqrt{x^2 - 7x + 12} \quad \text{and} \quad \sqrt[3]{x^2 - 7x + 12}$$

are different.

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ALGEBRA TUTOR

Solving Equations

Much of the algebra in this chapter involves solving an equation in one variable. To solve a linear equation, you can add or subtract the same quantity from each side of the equation, or multiply or divide each side of the equation by the same *nonzero* quantity.

Example 1 **Solving Equations**

Solve each equation.

a. $x + 8 = 7$ **b.** $-x - 6 = 13$ **c.** $5x = 40$

d. $-10 = -\frac{x}{2}$ **e.** $9 + 4x = 11$ **f.** $-10x - 7 = 33$

SOLUTION

a.
$$x + 8 = 7$$ Write original equation.
$$x + 8 - 8 = 7 - 8$$ Subtract 8 from each side.
$$x = -1$$ Simplify.

b.
$$-x - 6 = 13$$ Write original equation.
$$-x - 6 + 6 = 13 + 6$$ Add 6 to each side.
$$-x = 19$$ Simplify.
$$x = -19$$ Divide each side by -1.

c.
$$5x = 40$$ Write original equation.
$$\frac{5x}{5} = \frac{40}{5}$$ Divide each side by 5.
$$x = 8$$ Simplify.

STUDY TIP

Always remember to check your solution in the *original* equation.

d.
$$-10 = -\frac{x}{2}$$ Write original equation.
$$-10 \cdot (-2) = -\frac{x}{2} \cdot (-2)$$ Multiply each side by -2.
$$20 = x$$ Simplify.

e.
$$9 + 4x = 11$$ Write original equation.
$$9 - 9 + 4x = 11 - 9$$ Subtract 9 from each side.
$$4x = 2$$ Simplify.
$$\frac{4x}{4} = \frac{2}{4}$$ Divide each side by 4.
$$x = \frac{1}{2}$$ Simplify.

f.
$$-10x - 7 = 33$$ Write original equation.
$$-10x - 7 + 7 = 33 + 7$$ Add 7 to each side.
$$-10x = 40$$ Simplify.
$$x = -4$$ Divide each side by -10.

Solving Inequalities

A second algebraic skill used extensively in this chapter is solving an inequality in one variable. This process is similar to solving a linear equation, except when you multiply or divide by a negative number. In this case, you reverse the inequality symbol.

Example 2 Solving Inequalities

Solve each inequality.

a. $x - 3 < 14$ **b.** $-x + 15 > 4$ **c.** $9x \geq 63$

d. $-20 < -\frac{x}{5}$ **e.** $6x - 1 \geq 9$ **f.** $-12x + 5 > 47$

SOLUTION

a.
$$x - 3 < 14$$ Write original inequality.
$$x - 3 + 3 < 14 + 3$$ Add 3 to each side.
$$x < 17$$ Simplify.

b.
$$-x + 15 > 4$$ Write original inequality.
$$-x + 15 - 15 > 4 - 15$$ Subtract 15 from each side.
$$-x > -11$$ Simplify.
$$x < 11$$ Divide each side by -1 and reverse inequality symbol.

c.
$$9x \geq 63$$ Write original inequality.
$$\frac{9x}{9} \geq \frac{63}{9}$$ Divide each side by 9.
$$x \geq 7$$ Simplify.

d.
$$-20 < -\frac{x}{5}$$ Write original inequality.
$$-20 \cdot (-5) > -\frac{x}{5} \cdot (-5)$$ Multiply each side by -5 and reverse inequality symbol.
$$100 > x$$ Simplify.

e.
$$6x - 1 \geq 9$$ Write original inequality.
$$6x - 1 + 1 \geq 9 + 1$$ Add 1 to each side.
$$6x \geq 10$$ Simplify.
$$\frac{6x}{6} \geq \frac{10}{6}$$ Divide each side by 6.
$$x \geq \frac{5}{3}$$ Simplify.

f.
$$-12x + 5 > 47$$ Write original inequality.
$$-12x + 5 - 5 > 47 - 5$$ Subtract 5 from each side.
$$-12x > 42$$ Simplify.
$$x < -\frac{7}{2}$$ Divide each side by -12 and reverse inequality symbol.

SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 152.
Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 1.1

	Review Exercises
■ Classify an equation as an identity or a conditional equation.	*1–4*
■ Determine whether a given value is a solution of an equation.	*5–8*
■ Solve a linear equation in one variable.	*9–12*
A linear equation can be written in the standard form: $ax + b = 0$.	
■ Recognize an equation with no solution or infinitely many solutions.	*13–16*
■ Solve an equation involving fractional expressions.	*17–20*
■ Use a calculator to solve an equation.	*21–24*
■ Use a linear model to solve an application problem.	*25, 26*

Section 1.2

■ Construct a mathematical model from a verbal model.	*27–30*
■ Model and solve percent and distance problems.	*31–36*
■ Model and solve mixture problems.	*37, 38*
■ Use common formulas to solve geometry and simple interest problems.	*39–42*

Square: $A = s^2$, $P = 4s$

Rectangle: $A = lw$, $P = 2l + 2w$

Circle: $A = \pi r^2$, $C = 2\pi r$

Triangle: $A = \frac{1}{2}bh$, $P = a + b + c$

Cube: $V = s^3$

Rectangular Solid: $V = lwh$

Circular Cylinder: $V = \pi r^2 h$

Sphere: $V = \frac{4}{3}\pi r^3$

Temperature: $F = \frac{9}{5}C + 32$

Simple Interest: $I = Prt$

Distance: $d = rt$

Section 1.3

■ Solve a quadratic equation by factoring.	*43–46*
A quadratic equation can be written in the general form: $ax^2 + bx + c = 0$.	
Zero-Factor Property: If $ab = 0$, then $a = 0$ or $b = 0$.	
■ Solve a quadratic equation by extracting square roots.	*47–50*
■ Construct and use a quadratic model to solve area problems, falling-object problems, right triangle problems, and other applications.	*51–54*

* A wide range of valuable study aids are available to help you master the material in this chapter. The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *www.cengagebrain.com* offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

Section 1.4

	Review Exercises
■ Use the discriminant to determine the number of real solutions of a quadratic equation. If $b^2 - 4ac > 0$, then the equation has two distinct real solutions. If $b^2 - 4ac = 0$, then the equation has one repeated real solution. If $b^2 - 4ac < 0$, then the equation has no real solutions.	*55–58*
■ Solve a quadratic equation using the Quadratic Formula. Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	*59–68*
■ Use the Quadratic Formula to solve an application problem.	*69, 70*

Section 1.5

■ Solve a polynomial equation by factoring.	*71, 72*
■ Solve an equation of quadratic type.	*73, 74*
■ Rewrite and solve an equation involving radicals or rational exponents.	*75–80*
■ Rewrite and solve an equation involving fractions or absolute value.	*81–84*
■ Construct and use a nonquadratic model to solve an application problem.	*85–87*
■ Solve a compound interest problem.	*88*

Section 1.6

■ Write bounded and unbounded intervals using inequalities or interval notation.	*89–92*
■ Solve and graph a linear inequality.	*93–96*
■ Solve and graph inequalities involving absolute value. $\lvert x\rvert < a$ if and only if $-a < x < a$ $\lvert x\rvert > a$ if and only if $x < -a$ or $x > a$	*97, 98*
■ Construct and use a linear inequality to solve an application problem.	*99, 100*

Section 1.7

■ Use critical numbers to determine test intervals for a polynomial inequality.	*101–104*
■ Solve and graph a polynomial inequality.	*105–107, 111, 112*
■ Solve and graph a rational inequality.	*108–110, 113, 114*
■ Determine the domain of an expression involving a radical.	*115–118*
■ Construct and use a polynomial inequality to solve an application problem.	*119–126*

Study Strategies

■ **Check Your Answers** Because of the number of steps involved in solving an equation or inequality, there are many ways to make mistakes. So, always check your answers. In some cases, you may even want to check your answers in more than one way, just to be sure.

■ **Using Test Intervals** Make sure that you understand how to use critical numbers to determine test intervals for inequalities. The logic and mathematical reasoning involved in this concept can be applied in many real-life situations.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Classifying Equations In Exercises 1–4, determine whether the equation is an identity or a conditional equation.

1. $5(x - 3) = 2x + 9$
2. $3(x + 2) = 3x + 6$
3. $4(x + 2) = 4x + 8$
4. $2(x + 3) = 3x + 2$

Checking Solutions In Exercises 5–8, determine whether each value of x is a solution of the equation.

5. $5(x - 1) = 3x + 1$

   (a) $x = 0$ (b) $x = -1$ (c) $x = 3$ (d) $x = 2$

6. $3x^2 + 7x + 5 = x^2 + 9$

   (a) $x = 0$ (b) $x = \frac{1}{2}$ (c) $x = -4$ (d) $x = -1$

7. $6 + \frac{3}{x - 4} = 5$

   (a) $x = 5$ (b) $x = 0$ (c) $x = 2$ (d) $x = 7$

8. $\sqrt{x + 4} = 3$

   (a) $x = 0$ (b) $x = -1$ (c) $x = 5$ (d) $x = 12$

Solving an Equation In Exercises 9–16, solve the equation and check your solution. (Some equations have no solution.)

9. $x + 7 = 20$
10. $2x + 15 = 43$
11. $4(x + 3) - 3 = 2(4 - 3x) - 4$
12. $(x + 3) + 2(x - 4) = 5(x + 3)$
13. $5x + x = 6x + 3$
14. $4 - x = 2x - 3x + 1$
15. $8(x - 2) = 8x - 16$
16. $-(3 - 2x) = 2x - 3$

An Equation Involving Fractional Expressions In Exercises 17–20, solve the equation and check your solution.

17. $\frac{3x - 2}{5x - 1} = \frac{3}{4}$
18. $\frac{3}{x - 4} + \frac{8}{2x + 5} = \frac{11}{2x^2 - 3x - 20}$
19. $\frac{x}{x + 3} - \frac{4}{x + 3} + 2 = 0$
20. $7 - \frac{3}{x} = 8 + \frac{5}{x}$

Using a Calculator to Solve an Equation In Exercises 21–24, use a calculator to solve the equation. (Round your solution to three decimal places.)

21. $0.375x - 0.75(300 - x) = 200$
22. $0.235x + 2.6(-x - 4) = 30$
23. $\frac{x}{0.055} + \frac{x}{0.085} = 1$
24. $\frac{x}{0.0645} + \frac{x}{0.098} = 2$

25. **Stores** The numbers N of Safeway stores in operation from 2003 through 2009 can be approximated by the linear equation

$$N = -15.5t + 1859, \quad 3 \le t \le 9$$

where t represents the year, with $t = 3$ corresponding to 2003. Use the model to estimate the year in which the number of Safeway stores decreased to 1720. *(Source: Safeway, Inc.)*

26. **Federal Programs** The numbers of students S (in millions) participating in the school breakfast program from 2001 through 2009 can be approximated by the linear equation

$$S = 0.42t + 7.3, \quad 1 \le t \le 9$$

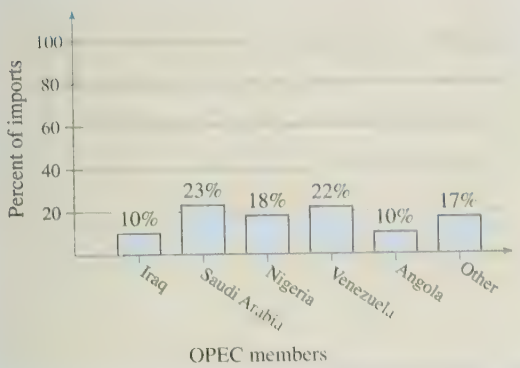
where t represents the year, with $t = 1$ corresponding to 2001. Use the model to estimate the year in which 9.4 million students participated in the school breakfast program. *(Source: U.S. Department of Agriculture, Food and Nutrition Service)*

Constructing a Mathematical Model In Exercises 27–30, use the verbal description to (a) write a verbal model, (b) assign labels to the quantities in the verbal model, (c) use the labels to write a mathematical model, and (d) solve the problem.

27. Three consecutive even integers have a sum of 42. Find the smallest of these integers.
28. One positive number is four times another number. The difference of the numbers is 51. Find the numbers.
29. **Annual Salary** Your annual salary is \$28,900. You receive a 7% raise. What is your new annual salary?
30. **Salary** Your weekly salary increases from \$550 to \$594. What percent is your raise?
31. **List Price** The price of an outdoor barbecue grill is discounted 15%. The sale price is \$139. Find the original list price of the grill.
32. **Discount Rate** The price of a home gym is discounted by \$300. The sale price is \$599.99. What percent of the original price is the discount?
33. **Travel Time** Two cars start at the same time at a given point and travel in the same direction at average speeds of 45 miles per hour and 50 miles per hour. After how long are the cars 10 miles apart?
34. **Exercise** Two bicyclists start at the same time at a given point and travel in the same direction at average speeds of 8 miles per hour and 10 miles per hour. After how long are the bicyclists 5 miles apart?

35. Box Office James Cameron's 1997 movie *Titanic* earned U.S. gross revenue of $600,788,188. Cameron's 2009 movie *Avatar* earned U.S. gross revenue of $760,507,625. Find the percent increase in the U.S. gross revenue from *Titanic* to *Avatar*. *(Source: Nash Information Services)*

36. Oil Imports The United States imported 1594 million barrels of crude oil from members of OPEC (Organization of Petroleum Exporting Countries) in 2009. Use the bar graph to determine the amount imported from each of the five top contributing countries. *(Source: U.S. Energy Information Administration)*



37. Mixture A car radiator contains 10 quarts of a 10% antifreeze solution. The car's owner wishes to create a 10-quart solution that is 30% antifreeze. How many quarts will have to be replaced with pure antifreeze?

38. Mixture A three-gallon acid solution contains 3% boric acid. How many gallons of 20% boric acid solution should be added to make a final solution that is 8% boric acid?

39. Geometry A volleyball court is twice as long as it is wide, and its perimeter is 177 feet. Find the dimensions of the volleyball court.

40. Geometry A room is 1.25 times as long as it is wide, and its perimeter is 90 feet. Find the dimensions of the room.

41. Simple Interest You deposit $500 in a savings account earning 4% simple interest. How much interest will you earn in 1 year?

42. Simple Interest You deposit $800 in a money market account. One year later the account balance is $814. What was the simple interest rate?

Solving a Quadratic Equation by Factoring In Exercises 43–46, solve the quadratic equation by factoring. Check your solutions.

43. $6x^2 = 5x + 4$

44. $-x^2 = 15x + 36$

45. $x^2 - 11x + 24 = 0$

46. $4 - 4x + x^2 = 0$

Extracting Square Roots In Exercises 47–50, solve the quadratic equation by extracting square roots. When a solution is irrational, list both the exact solution and its approximation rounded to two decimal places.

47. $x^2 = 11$

48. $16x^2 = 25$

49. $(x + 4)^2 = 18$

50. $(x - 1)^2 = 5$

51. Geometry A billboard is 12 feet longer than it is high. The billboard has 405 square feet of advertising space. What are the dimensions of the billboard?

52. Grand Canyon The Grand Canyon is 6000 feet deep at its deepest part. A rock is dropped over the deepest part of the canyon. How long does the rock take to hit the water in the Colorado River below?

53. Total Revenue The demand equation for a product is

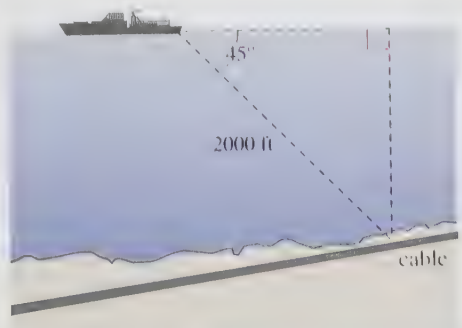
$$p = 60 - 0.0001x$$

where p is the price per unit and x is the number of units sold. The total revenue R from selling x units is given by

$$R = xp = x(60 - 0.0001x).$$

How many units must be sold to produce a revenue of $8,000,000?

54. Depth of an Underwater Cable A ship's sonar locates a cable 2000 feet from the ship (see figure). The angle between the surface of the water and a line from the ship to the cable is 45°. How deep is the cable?



Using the Discriminant In Exercises 55–58, use the discriminant to determine the number of real solutions of the quadratic equation.

55. $x^2 + 11x + 24 = 0$

56. $x^2 + 5x + 12 = 0$

57. $3x^2 - 4x + 2 = 0$

58. $4x^2 + 12x + 9 = 0$

Using the Quadratic Formula In Exercises 59–64, use the Quadratic Formula to solve the quadratic equation. Check your solutions.

59. $x^2 - 12x + 30 = 0$

60. $5x^2 + 16x - 12 = 0$

61. $(y + 7)^2 = -5y$

62. $6x = 7 - 2x^2$

63. $x^2 + 6x - 3 = 0$

64. $10x^2 - 11x = 2$

Using a Calculator In Exercises 65–68, use a calculator to solve the quadratic equation. (Round your answers to three decimal places.)

65. $3.6x^2 - 5.7x - 1.9 = 0$

66. $2.3x^2 + 6.6x - 3.9 = 0$

67. $34x^2 - 296x + 47 = 0$

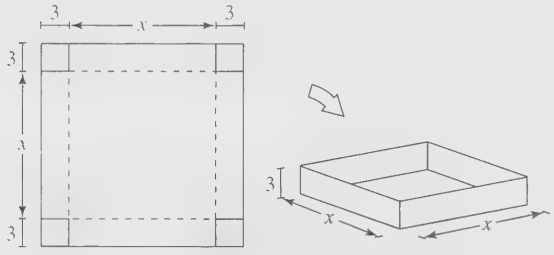
68. $39x^2 + 75x - 21 = 0$

69. On the Moon An astronaut standing on the edge of a cliff on the moon drops a rock over the cliff. The height s of the rock after t seconds is given by

$$s = -2.7t^2 + 200.$$

The rock's initial velocity is 0 feet per second and its initial height is 200 feet. Determine how long it will take the rock to hit the lunar surface. How long does it take a rock dropped from a height of 200 feet on Earth to hit the ground?

70. Geometry An open box is made from a square piece of material by cutting three-inch squares from the corners and turning up the sides (see figure). The volume of the finished box is 363 cubic inches. Find the size of the original piece of material.



Solving Equations In Exercises 71–84, find the real solution(s) of the equation. Check your solution(s).

71. $3x^3 - 9x^2 - 12x = 0$

72. $x^4 + 3x^3 - 5x - 15 = 0$

73. $x^4 - 5x^2 + 4 = 0$

74. $x^6 + 26x^3 - 27 = 0$

75. $2\sqrt{x} - 5 = 0$

76. $\sqrt{3x - 2} + x = 4$

77. $2\sqrt{x - 3} - 4 = 3x$

78. $\sqrt[3]{3x + 5} = 5$

79. $(x^2 - 5)^{2/3} = 9$

80. $(x^2 - 5x - 6)^{4/3} = 16$

81. $|5x + 4| = 11$

82. $|x^2 + 4x| - 2x = 8$

83. $\frac{5}{x + 1} + \frac{3}{x + 3} = 1$

84. $x + \frac{3}{x + 2} = 2$

85. Sharing the Cost Three students are planning to share the expense of renting a condominium at a resort for 1 week. By adding a fourth person to the group, each person could save \$75 in rental fees. How much is the rent for the week?

86. Sharing the Cost A college charters a bus for \$1800 to take a group to a museum. When four more students join the trip, the cost per student decreases by \$5. How many students were in the original group?

87. Market Research The demand equation for a product is given by

$$p = 45 - \sqrt{0.002x + 1}$$

where x is the number of units demanded per day and p is the price per unit. Find the demand when the price is set at \$19.95.

88. Cash Advance You take out a cash advance of \$500 on a credit card. After 3 months, the amount you owe is \$525.29. What is the annual percentage rate for this cash advance? (Assume that the interest is compounded monthly and that you have made no payments.)

Intervals and Inequalities In Exercises 89–92, write an inequality that represents the interval. Then state whether the interval is bounded or unbounded.

89. $[-4, 7]$

90. $(6, 1000)$

91. $(-\infty, 1000]$

92. $(-10, \infty)$

Solving an Inequality In Exercises 93–98, solve the inequality and graph the solution set on the real number line.

93. $3(x - 1) < 2x + 8$

94. $-5 \le 2 - 4(x + 2) \le 6$

95. $-3 < \frac{2x + 1}{4} < 3$

96. $-1 \le -5 - 3x < 4$

97. $|x + 10| + 3 < 5$

98. $|2x - 3| - 4 > 2$

99. Break-Even Analysis The revenue R from selling x units of a product is

$$R = 89.95x.$$

The cost C of producing x units is

$$C = 35x + 2500.$$

For what values of x will this product return a profit?

100. Accuracy of Measurement You buy a 16-inch gold chain that costs \$9.95 per inch. The chain is measured accurately to within $\frac{1}{16}$ inch. How much money might you have been undercharged or overcharged?

Finding Test Intervals In Exercises 101–104, find the test intervals of the inequality.

101. $x^2 - 1 < 0$

102. $x^2 - 3x - 4 \ge 0$

103. $\frac{x + 3}{x - 1} \le 2$

104. $\frac{x + 5}{x + 7} > 3$

Solving an Inequality In Exercises 105–110, solve the inequality and graph the solution set on the real number line.

105. $5(x + 1)(x - 3) < 0$

106. $(x + 4)^2 \le 4$

107. $x^3 - 9x < 0$

108. $\frac{x + 5}{x + 8} \ge 2$

109. $\dfrac{2 + 3x}{4 - x} < 2$

110. $\dfrac{1}{x + 1} \geq \dfrac{1}{x + 5}$

Using a Calculator In Exercises 111–114, use a calculator to solve the inequality. (Round each number in your answer to two decimal places.)

111. $-1.2x^2 + 4.76 > 1.32$

112. $3.5x^2 + 4.9x - 6.1 < 2.4$

113. $\dfrac{1}{3.7x - 6.1} > 2.9$

114. $\dfrac{3}{5.4x - 2.7} < 8.9$

Finding the Domain of an Expression In Exercises 115–118, find the domain of the expression.

115. $\sqrt{x - 10}$

116. $\sqrt[4]{2x + 5}$

117. $\sqrt{x^2 - 15x + 54}$

118. $\sqrt{81 - 4x^2}$

119. Height of a Flare A flare is fired straight upward from ground level with an initial velocity of 100 feet per second. During what time period will its height exceed 150 feet?

120. Path of a Soccer Ball A soccer player kicks the ball toward the center of the goal. The path of the ball can be modeled by

$$y = -0.054x^2 + 1.43x$$

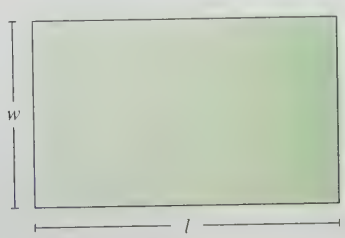
where x is the horizontal distance (in feet) from where the ball is kicked and y is the corresponding height (in feet).

(a) The soccer goal is 8 feet high. Write an inequality to determine the values of x for which the ball is low enough to go into the goal.

(b) Solve the inequality from part (a).

(c) The player kicks the ball 15 feet from the goal and the ball is not blocked. Does the player score a goal? Explain your reasoning.

121. Geometry A rectangular field with a perimeter of 80 meters is to have an area of at least 380 square meters (see figure). Describe the possible lengths l of the field.



122. Compound Interest P dollars, invested at interest rate r compounded annually, increases to an amount A given by

$$A = P(1 + r)^5$$

in 5 years. An investment of \$1000 increases to an amount greater than \$1400 in 5 years. The interest rate must be greater than what percent?

123. Compound Interest P dollars, invested at an interest rate r compounded semiannually, increases to an amount

$$A = P(1 + r/2)^{2 \cdot 8}$$

in 8 years. An investment of \$2000 increases to an amount greater than \$2800 in 8 years. The interest rate must be greater than what percent?

124. Company Profits The revenue R and cost C for a product are given by

$$R = x(75 - 0.0005x)$$

and

$$C = 25x + 100{,}000$$

where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$500,000?

125. Price of a Product In Exercise 124, the revenue equation

$$R = x(75 - 0.0005x)$$

implies that the demand equation is

$$p = 75 - 0.0005x$$

where p is the price per unit. What prices per unit can the company set to obtain a profit of at least \$1,000,000?

126. Income The total investment income I (in millions of dollars) of Aflac each year from 2001 through 2009 can be approximated by the model

$$I = 6.07t^2 + 90.3t + 1448, \quad 1 \leq t \leq 9$$

where t represents the year, with $t = 1$ corresponding to 2001. *(Source: Aflac Incorporated)*

(a) Complete the table. Round each value of I to the nearest whole number.

t	1	3	5	9
I				

(b) Aflac expected to earn investment income of greater than \$3 billion in 2010. Does the model support this? Explain.

(c) Aflac predicts that the yearly investment income will be at least \$3.5 billion by 2015. Does the model support this prediction? Explain.

TEST YOURSELF

See www.CalcChat.com for worked-out solutions to odd numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

1. Solve the equation $3(x + 2) - 8 = 4(2 - 5x) + 7$.
2. Find the domain of (a) $\sqrt[3]{2x + 3}$ and (b) $\sqrt{9 - x^2}$.
3. In May, the total profit for a company was 8% less than it was in April. The total profit for the 2 months was \$625,509.12. Find the profit for each month.

In Exercises 4–13, solve the equation. Check your solution(s).

4. *Factoring:* $6x^2 + 7x = 5$
5. *Factoring:* $12 + 5x - 2x^2 = 0$
6. *Extracting square roots:* $x^2 - 5 = 10$
7. *Quadratic Formula:* $(x + 5)^2 = -3x$
8. *Quadratic Formula:* $3x^2 - 11x = 2$
9. *Quadratic Formula:* $5.4x^2 - 3.2x - 2.5 = 0$
10. $|2x - 3| = 10$
11. $\sqrt{x - 3} + x = 5$
12. $x^4 - 10x^2 + 9 = 0$
13. $(x^2 - 9)^2 + 9 = 9$
14. The demand equation for a product is $p = 40 - 0.0001x$, where p is the price per unit and x is the number of units sold. The total revenue R from selling x units is given by $R = xp$. How many units must be sold to produce a revenue of \$2,000,000? Explain your reasoning.

In Exercises 15–18, solve the inequality and graph the solution set on the real number line.

15. $\dfrac{3x + 1}{5} < 2$
16. $|4 - 5x| \geq 24$
17. $\dfrac{x + 3}{x + 7} > 2$
18. $3x^4 - 12x^2 \leq 0$
19. The revenue R and cost C for a product are given by

    $R = x(90 - 0.0004x)$

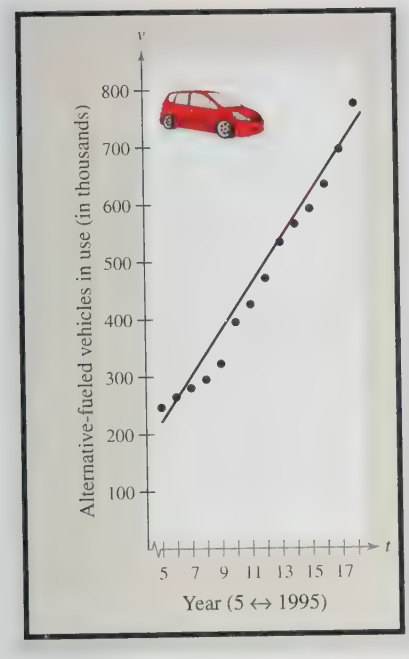
    and

    $C = 25x + 300{,}000$

    where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$800,000?
20. The total payroll P (in billions of dollars) for federal civilian employees each year from 1996 through 2009 can be approximated by the model

    $P = 0.145t^2 + 0.79t + 107.4, \quad 6 \leq t \leq 19$

    where t represents the year, with $t = 6$ corresponding to 1996. Use the model to predict the year when the payroll will exceed \$190 billion. *(Source: U.S. Office of Personnel Management)*



Example 8 on page 189 shows how a scatter plot can be used to find a linear model that approximates the number of alternative-fueled vehicles in use in the United States for a given year.

2 Functions and Graphs

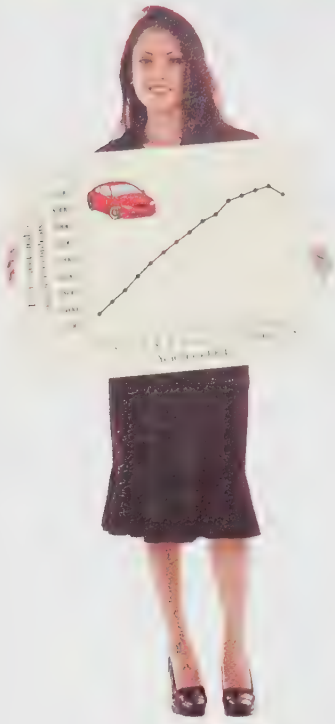
2.1 Graphs of Equations
2.2 Lines in the Plane
2.3 Linear Modeling and Direct Variation
2.4 Functions
2.5 Graphs of Functions
2.6 Transformations of Functions
2.7 The Algebra of Functions



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2.1 Graphs of Equations

- Plot points in the Cartesian plane.
- Use the Distance Formula to find the distance between two points in the coordinate plane, and use the Midpoint Formula to find the midpoint of a line segment joining two points.
- Determine whether a point is a solution of an equation, and sketch the graph of an equation.
- Find the x- and y-intercepts of the graph of an equation.
- Determine the symmetry of a graph.
- Write the equation of a circle in standard form.



In Exercise 105 on page 170, you will use points plotted in a coordinate plane to determine the percent increase in miles traveled by vehicles over a specified period of time.

The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane. This plane is called the **rectangular coordinate system,** or the **Cartesian plane,** named after the French mathematician René Descartes (1596–1650).

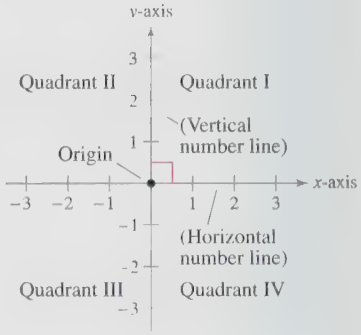


FIGURE 2.1

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure 2.1. The horizontal real number line is usually called the ***x*-axis,** and the vertical real number line is usually called the ***y*-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y, called the **coordinates** of the point. The ***x*-coordinate** represents the directed distance from the y-axis to the point, and the ***y*-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure 2.2.

The notation (x, y) denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

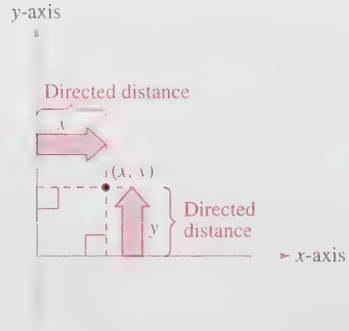


FIGURE 2.2

Example 1 Plotting Points in the Cartesian Plane

To plot the point $(-1, 2)$ in the Cartesian plane, imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. Other points, such as $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$, can be plotted in a similar way, as shown in Figure 2.3.

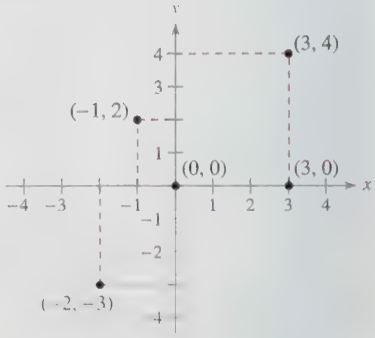


FIGURE 2.3

Checkpoint 1

Plot the points $(0, -2)$ and $(-4, 1)$.

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The Distance and Midpoint Formulas

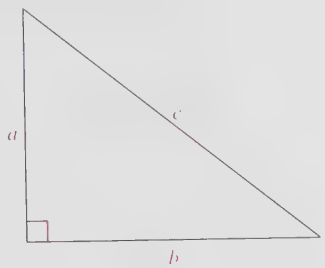
ALGEBRA TUTOR

For help with identifying and plotting points in the Cartesian plane, see the *Chapter 2 Algebra Tutor* on pages 240 and 241.

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length c and legs of lengths a and b, you have

$$a^2 + b^2 = c^2$$

as shown in Figure 2.4. Note that the converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.



$a^2 + b^2 = c^2$

FIGURE 2.4

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) that do not lie on the same horizontal or vertical line. With these two points, a right triangle can be formed, as shown in Figure 2.5.

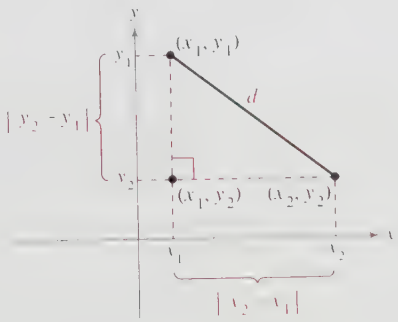


FIGURE 2.5

The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, you can write

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$ Choose positive square root.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The result is the **Distance Formula.**

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

When using the Distance Formula, remember that assigning points to (x_1, y_1) and (x_2, y_2) is arbitrary.

The formula below shows how to find the *midpoint* of the line segment that joins two points.

The Midpoint Formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 2 Using the Distance and Midpoint Formulas

Find (a) the distance between, and (b) the midpoint of the line segment joining, the points $(-2, 1)$ and $(3, 4)$.

SOLUTION

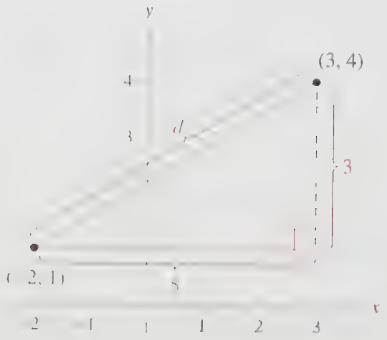
Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$.

a. Apply the Distance Formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Substitute for } x_1, x_2, y_1, \text{ and } y_2. \\ &= \sqrt{5^2 + 3^2} && \text{Simplify.} \\ &= \sqrt{34} \approx 5.83 && \text{Simplify. See Figure 2.6} \end{aligned}$$

b. By the Midpoint Formula, you have

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) && \text{Midpoint Formula} \\ &= \left(\frac{-2 + 3}{2}, \frac{1 + 4}{2}\right) && \text{Substitute for } x_1, x_2, y_1, \text{ and } y_2. \\ &= \left(\frac{1}{2}, \frac{5}{2}\right). && \text{Simplify. See Figure 2.7.} \end{aligned}$$



$d = \sqrt{34} \approx 5.83$

FIGURE 2.6

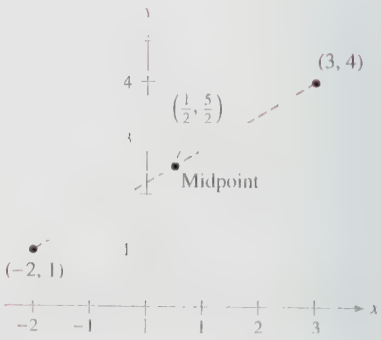


FIGURE 2.7

Checkpoint 2

Find (a) the distance between, and (b) the midpoint of the line segment joining, the points $(-7, 5)$ and $(1, -1)$.

The Graph of an Equation

Frequently, a relationship between two quantities is written in the form of an equation. In the remainder of this section, you will study a procedure for sketching the graph of an equation. For an equation in the variables x and y, a point (a, b) is a **solution** if the substitutions $x = a$ and $y = b$ satisfy the equation.

Example 3 Solution of an Equation

Determine whether $(-1, 0)$ is a solution of the equation $y = 2x^2 - 4x - 6$.

SOLUTION

$y = 2x^2 - 4x - 6$	Write original equation
$0 \stackrel{?}{=} 2(-1)^2 - 4(-1) - 6$	Substitute -1 for x and 0 for y
$0 = 0$	Simplify

Both sides of the equation are equivalent. So, the point $(-1, 0)$ is a solution.

✓Checkpoint 3

Determine whether $(-1, 3)$ is a solution of the equation $y = x + 4$.

Most equations have *infinitely* many solutions. The **graph of an equation** is the set of all points that are solutions of the equation. To sketch the graph of an equation, you can use the point-plotting method.

The Point-Plotting Method of Graphing

1. If possible, isolate one of the variables.
2. Construct a table of values showing several solution points.
3. Plot these points in a rectangular coordinate system.
4. Connect the points with a smooth curve or a line.

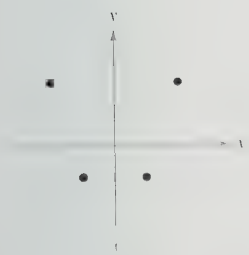


FIGURE 2.8

Step 4 of the point-plotting method can be difficult. For instance, how would you connect the four points in Figure 2.8? Without further information about the equation, any one of the three graphs in Figure 2.9 would be reasonable. These graphs show that with too few solution points, you can misrepresent the graph of an equation. Throughout this course, you will study many ways to improve your graphing techniques. For now, you should plot enough points to reveal the essential behavior of the graph. It is important to use negative values, zero, and positive values for x when constructing a table.



FIGURE 2.9

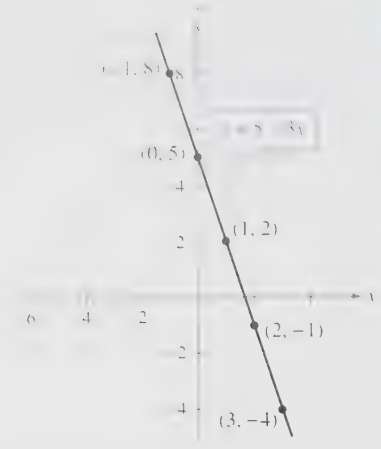


FIGURE 2.10

Example 4 Sketching the Graph of an Equation

Sketch the graph of $3x + y = 5$.

SOLUTION First, rewrite the equation as $y = 5 - 3x$ with y isolated on the left. Next, construct a table of values by choosing several values of x and calculating the corresponding values of y.

x	-1	0	1	2	3
$y = 5 - 3x$	8	5	2	-1	-4

From the table, it follows that $(-1, 8)$, $(0, 5)$, $(1, 2)$, $(2, -1)$, and $(3, -4)$ are solution points of the equation. After plotting these points and connecting them, you can see that they appear to lie on a line, as shown in Figure 2.10.

Checkpoint 4

Sketch the graph of each equation.

a. $y - x = 3$ **b.** $2x + y = -1$

Example 5 Sketching the Graph of an Equation

Sketch the graph of $y = x^2 - 2$.

SOLUTION First, construct a table of values by choosing several convenient values of x and calculating the corresponding values of y.

x	-3	-2	-1	0	1	2	3
$y = x^2 - 2$	7	2	-1	-2	-1	2	7

Next, plot the corresponding solution points. Finally, connect the points with a smooth curve, as shown in Figure 2.11.

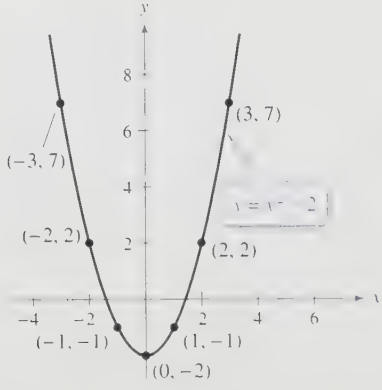


FIGURE 2.11

Checkpoint 5

Sketch the graph of each equation.

a. $y = \frac{1}{4}x^2$ **b.** $y = 5 - x^2$

Intercepts of a Graph

When you are sketching a graph, points for which either the y-coordinate or the x-coordinate is zero are especially useful.

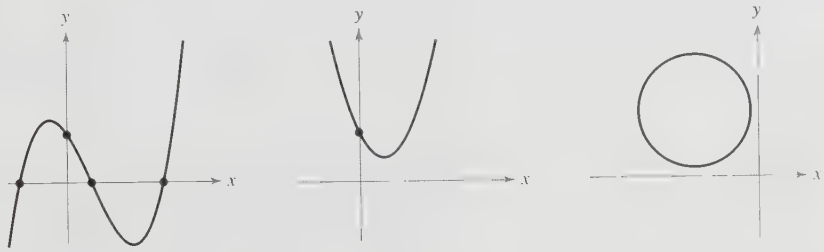


Definition of Intercepts

1. The ***x*-intercepts** of a graph are the points at which the graph intersects the x-axis. To find the x-intercepts, let y equal zero and solve for x.
2. The ***y*-intercepts** of a graph are the points at which the graph intersects the y-axis. To find the y-intercepts, let x equal zero and solve for y.

Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, the term *intercept* will be used to mean either the point or the coordinate.

A graph may have no intercepts, one intercept, or several intercepts. For instance, consider the three graphs in Figure 2.12.



Three x-intercepts
One y-intercept

No x-intercept
One y-intercept

No intercepts

FIGURE 2.12

Example 6 Finding *x*- and *y*-Intercepts

Find the x- and y-intercepts of the graph of

$$y^2 - 3 = x.$$

SOLUTION To find the x-intercept, let $y = 0$. This produces

$$-3 = x$$

which implies that the graph has one x-intercept, which occurs at

$$(-3, 0).$$ x-intercept

To find the y-intercept, let $x = 0$. This produces

$$y^2 - 3 = 0$$

which has two solutions: $y = \pm\sqrt{3}$. So, the graph has two y-intercepts, which occur at

$$(0, \sqrt{3}) \quad \text{and} \quad (0, -\sqrt{3}).$$ y-intercepts

See Figure 2.13.

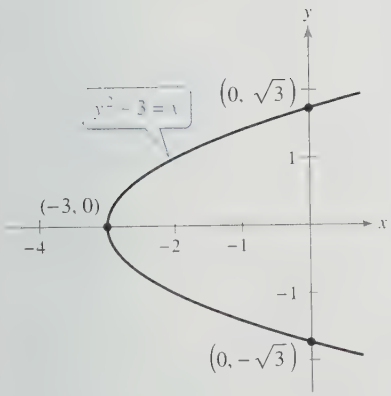


FIGURE 2.13

Checkpoint 6

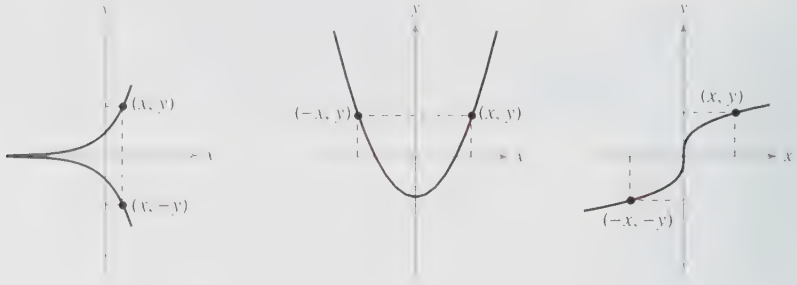
Find the x- and y-intercepts of the graph of each equation.

a. $y = 4x + 7$ **b.** $y = x^2 - 1$

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Symmetry

Symmetry with respect to the x-axis means that if the Cartesian plane is folded along the x-axis, then the portion of the graph above the x-axis would coincide with the portion below the x-axis. Symmetry with respect to the y-axis can be described in a similar manner. Symmetry with respect to the origin means that the graph rotates 180° about the origin, it looks the same. (See Figure 2.14.)



x-Axis symmetry y-Axis symmetry Origin symmetry

FIGURE 2.14

Knowing the symmetry of a graph *before* attempting to sketch it is helpful, because then you need only half as many solution points to sketch the graph. The three basic types of symmetry are defined below.

Definition of Symmetry

1. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is on the graph, $(x, -y)$ is also on the graph.
2. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is on the graph, $(-x, y)$ is also on the graph.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is on the graph, $(-x, -y)$ is also on the graph.

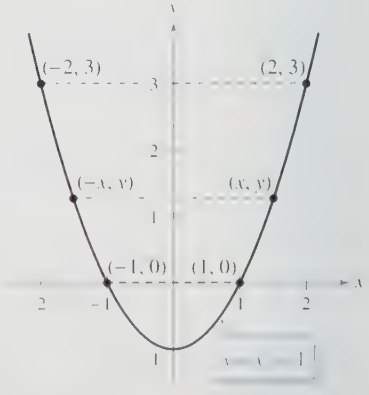
You can apply this definition of symmetry to the graph of the equation $y = x^2 - 1$. Replacing x with $-x$ produces

$y = x^2 - 1$ Write original equation.

$y = (-x)^2 - 1$ Replace x with $-x$.

$y = x^2 - 1.$ Replacement yields equivalent equation.

Because the substitution did not change the equation, it follows that if (x, y) is a solution of the equation, then $(-x, y)$ must also be a solution. So, the graph of $y = x^2 - 1$ is symmetric with respect to the y-axis. By plotting the points in the table below, you can confirm that the graph is symmetric with respect to the y-axis, as shown in Figure 2.15.



y-Axis Symmetry

FIGURE 2.15

x	−2	−1	1	2
y	3	0	0	3

Tests for Symmetry

1. The graph of an equation is symmetric with respect to the x-axis if replacing y with $-y$ yields an equivalent equation.
2. The graph of an equation is symmetric with respect to the y-axis if replacing x with $-x$ yields an equivalent equation.
3. The graph of an equation is symmetric with respect to the origin if replacing x with $-x$ *and* y with $-y$ yields an equivalent equation.

Example 7 Testing for Symmetry

Test $y = 2x^3$ for symmetry with respect to both axes and the origin.

SOLUTION

x-Axis:	$y = 2x^3$	Write original equation.
	$-y = 2x^3$	Replace y with $-y$. Result is *not* an equivalent equation.
y-Axis:	$y = 2x^3$	Write original equation.
	$y = 2(-x)^3$	Replace x with $-x$.
	$y = -2x^3$	Simplify. Result is *not* an equivalent equation.
Origin:	$y = 2x^3$	Write original equation.
	$-y = 2(-x)^3$	Replace y with $-y$ and x with $-x$.
	$-y = -2x^3$	Simplify.
	$y = 2x^3$	Equivalent equation

Of the three tests for symmetry, the only one that is satisfied is the test for origin symmetry (see Figure 2.16).

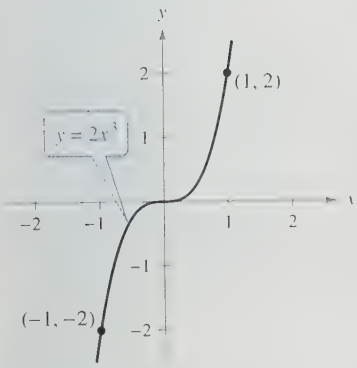


FIGURE 2.16

Checkpoint 7

Test $y = -4x^2$ for symmetry with respect to both axes and the origin.

Example 8 Using Symmetry as a Sketching Aid

Use symmetry to sketch the graph of $x - y^2 = 1$.

SOLUTION Of the three tests for symmetry, the only one that is satisfied is the test for x-axis symmetry.

$x - y^2 = 1$	Write original equation.
$x - (-y)^2 = 1$	Replace y with $-y$.
$x - y^2 = 1$	Equivalent equation

So, the graph is symmetric with respect to the x-axis. To sketch the graph, plot the points above the x-axis and use symmetry to complete the graph, as shown in Figure 2.17.

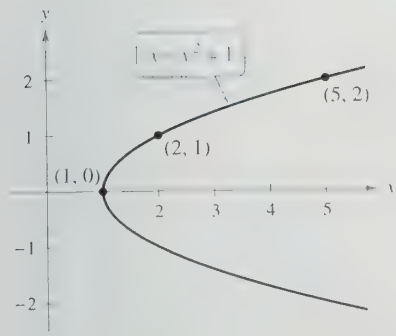


FIGURE 2.17

Checkpoint 8

Use symmetry to sketch the graph of each equation.

a. $y = |x|$ **b.** $y = x^3 + x$

The Equation of a Circle

In this section, you have studied the point-plotting method and two additional concepts (intercepts and symmetry) that can be used to streamline the graphing procedure. Another graphing aid is *equation recognition*, which is the ability to recognize the general shape of a graph simply by looking at its equation.

Figure 2.18 shows a circle of *radius* r with *center* at the point (h, k). The point (x, y) is on this circle if and only if its distance from the center (h, k) is r. This means that a **circle** in the plane consists of all points (x, y) that are a given positive distance r from a fixed point (h, k). Using the Distance Formula, you can conclude that the point (x, y) lies on the circle if and only if

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

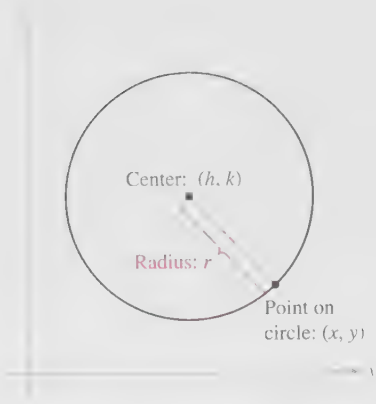


FIGURE 2.18

By squaring each side of this equation, you obtain the **standard form of the equation of a circle.** For example, a circle with its center at the origin, $(h, k) = (0, 0)$, and radius $r = 4$ is given by

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 4 \qquad \text{Substitute for } h, k, \text{ and } r.$$

$$\sqrt{x^2 + y^2} = 4 \qquad \text{Simplify.}$$

$$x^2 + y^2 = 16. \qquad \text{Square each side.}$$

STUDY TIP

The standard form of the equation of a circle with radius r whose center is at the *origin* is

$$x^2 + y^2 = r^2.$$

Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** is

$$(x - h)^2 + (y - k)^2 = r^2.$$

The point (h, k) is called the **center** of the circle, and the positive number r is called the **radius** of the circle.

Example 9 Finding the Equation of a Circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure 2.19. Write the standard form of the equation of this circle.

SOLUTION The radius of the circle is the distance between the center $(-1, 2)$ and the point $(3, 4)$.

$$r = \sqrt{(x - h)^2 + (y - k)^2} \qquad \text{Distance Formula}$$

$$r = \sqrt{[3 - (-1)]^2 + (4 - 2)^2} \qquad \text{Substitute for } x, y, h, \text{ and } k.$$

$$= \sqrt{4^2 + 2^2} \qquad \text{Simplify.}$$

$$= \sqrt{16 + 4} \qquad \text{Simplify.}$$

$$= \sqrt{20} \qquad \text{Radius}$$

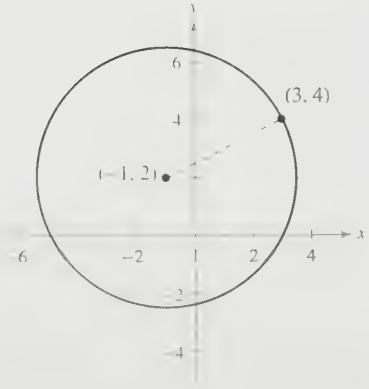


FIGURE 2.19

Using $(h, k) = (-1, 2)$ and $r = \sqrt{20}$, the equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Equation of circle}$$

$$[x - (-1)]^2 + (y - 2)^2 = \left(\sqrt{20}\right)^2 \qquad \text{Substitute for } h, k, \text{ and } r.$$

$$(x + 1)^2 + (y - 2)^2 = 20. \qquad \text{Standard form}$$

✓Checkpoint 9

The point $(4, 4)$ lies on a circle whose center is at $(0, 1)$. Write the standard form of the equation of this circle.

When you expand the standard equation in Example 9, you obtain the result shown.

$$(x + 1)^2 + (y - 2)^2 = 20 \quad \text{Standard form}$$

$$x^2 + 2x + 1 + y^2 - 4y + 4 = 20 \quad \text{Expand terms.}$$

$$x^2 + y^2 + 2x - 4y - 15 = 0 \quad \text{General form}$$

The last equation is in the **general form of the equation of a circle,**

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \quad A \neq 0.$$

STUDY TIP

Recall that to complete the square, you add the square of half the coefficient of the linear term to each side.

The general form is less useful than the standard form. For instance, it is not immediately apparent from the general form shown above that the center is $(-1, 2)$ and the radius is $\sqrt{20}$. To graph the equation of a circle, write the equation in standard form. You can do this by **completing the square,** as shown in Example 10.

Example 10 **Completing the Square to Sketch a Circle**

Sketch the circle given by $4x^2 + 4y^2 + 20x - 16y + 37 = 0$.

SOLUTION Begin by writing the original equation in standard form by completing the square for both the x-terms *and* the y-terms.

$$4x^2 + 4y^2 + 20x - 16y + 37 = 0 \quad \text{Write original equation.}$$

$$x^2 + y^2 + 5x - 4y + \frac{37}{4} = 0 \quad \text{Divide by 4.}$$

$$\left(x^2 + 5x + \quad\right) + \left(y^2 - 4y + \quad\right) = -\frac{37}{4} \quad \text{Group terms.}$$

$$\left[x^2 + 5x + \left(\frac{5}{2}\right)^2\right] + (y^2 - 4y + 2^2) = -\frac{37}{4} + \frac{25}{4} + 4 \quad \text{Complete the square.}$$

$$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = 1 \quad \text{Standard form}$$

So, the center of the circle is $\left(-\frac{5}{2}, 2\right)$ and the radius of the circle is 1. Using this information, you can sketch the circle, as shown in Figure 2.20.

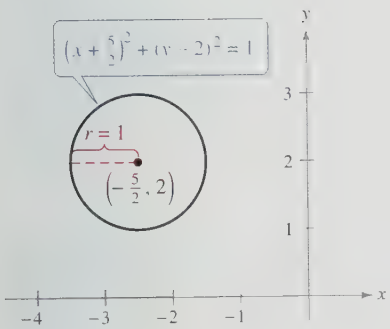
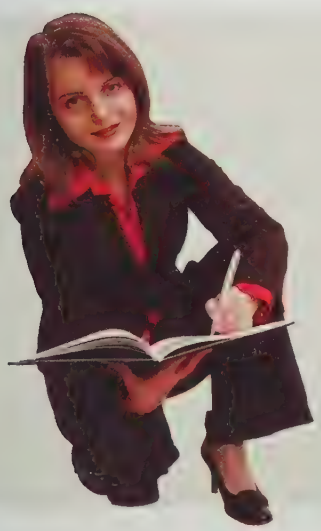


FIGURE 2.20

✓Checkpoint 10

Sketch the circle given by $x^2 + y^2 - 2x - 4y + 1 = 0$.



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SUMMARIZE (Section 2.1)

1. Describe how to plot a point in the Cartesian plane *(page 158)*. For an example of plotting points, see Example 1.
2. State the Distance Formula and the Midpoint Formula *(pages 159 and 160)*. For an example that uses these formulas, see Example 2.
3. Describe how to sketch the graph of an equation *(pages 161 and 162)*. For examples of graphing equations, see Examples 4 and 5.
4. State the definitions of x-intercept and y-intercept *(page 163)*. For an example of finding x- and y-intercepts, see Example 6.
5. State the standard form of the equation of a circle *(page 166)*. For an example of finding the standard form of the equation of a circle, see Example 9.

SKILLS WARM UP 2.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.4, and 1.5.

In Exercises 1–6, simplify the expression.

1. $\sqrt{(2-6)^2 + [1-(-2)]^2}$
2. $\sqrt{(1-4)^2 + (-2-1)^2}$
3. $\dfrac{4+(-2)}{2}$
4. $\dfrac{-1+(-3)}{2}$
5. $\sqrt{18} + \sqrt{45}$
6. $\sqrt{12} + \sqrt{44}$

In Exercises 7–10, solve the equation.

7. $\sqrt{(4-x)^2 + (5-2)^2} = \sqrt{58}$
8. $\sqrt{(8-6)^2 + (y-5)^2} = 2\sqrt{5}$
9. $x^3 - 9x = 0$
10. $x^4 - 8x^2 + 16 = 0$

Exercises 2.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Plotting Points in the Cartesian Plane In Exercises 1 and 2, plot the points in the Cartesian plane. *See Example 1.*

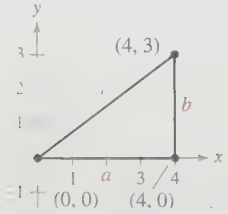
1. $(-5, 3), (1, -1), (-2, -4), (2, 0), (1, -6)$
2. $(0, -4), (5, 1), (-3, 5), (2, -2), (-6, -1)$

Using the Distance and Midpoint Formulas In Exercises 3–12, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points. *See Examples 1 and 2.*

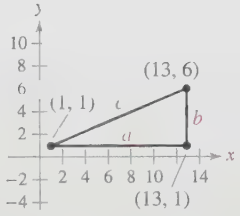
3. $(2, -5), (-6, 1)$
4. $(1, 12), (6, 0)$
5. $(3, -11), (-12, -3)$
6. $(-7, 3), (2, -9)$
7. $(-1, 2), (5, 4)$
8. $(2, 10), (10, 2)$
9. $\left(\frac{1}{2}, 1\right), \left(-\frac{3}{2}, -5\right)$
10. $\left(-\frac{1}{3}, -\frac{1}{3}\right), \left(-\frac{1}{6}, -\frac{1}{2}\right)$
11. $(1.8, 7.5), (-2.5, 2.1)$
12. $(5.2, 6.4), (-2.7, 1.8)$

Finding the Hypotenuse In Exercises 13–16, find the length of the hypotenuse in two ways: (a) using the Pythagorean Theorem and (b) using the Distance Formula.

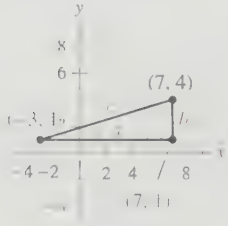
13.



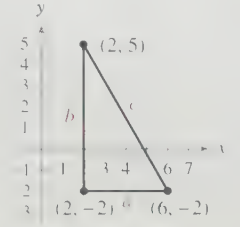
14.



15.



16.



Using the Distance Formula In Exercises 17 and 18, find the values of x for which the distance between the points is 15.

17. $(3, -4), (x, 5)$
18. $(x, 8), (-9, -4)$

Using the Distance Formula In Exercises 19 and 20, find the values of y for which the distance between the points is 20.

19. $(-15, y), (-3, -7)$
20. $(6, -1), (-10, y)$

Solution of an Equation In Exercises 21–24, determine whether each point is a solution of the equation. *See Example 3.*

	Equation	*Points*	
21.	$2x - 3y + 11 = 0$	(a) $(2, 5)$	(b) $(3, 2)$
22.	$y = 2x^2 - 7x + 3$	(a) $(1, -1)$	(b) $(3, 0)$
23.	$y = \sqrt{x - 5}$	(a) $(9, 2)$	(b) $(21, 4)$
24.	$y = \dfrac{x+1}{5-x}$	(a) $\left(1, \frac{1}{2}\right)$	(b) $(0, 1)$

Sketching the Graph of an Equation In Exercises 25–28, complete the table below for the given equation. Use the resulting solution points to sketch the graph of the equation. *See Examples 4 and 5.*

x	-3	-2	-1	0	1	2	3
y							

25. $y = \frac{3}{4}x - 1$
26. $\frac{3}{2}x + y = 2$
27. $y = x^2 - 4$
28. $y = \frac{1}{2}x^2$

Finding *x*- and *y*-Intercepts In Exercises 29–36, find the *x*- and *y*-intercepts of the graph of the equation. *See Example 6.*

29. $y = 2x - 1$

30. $2x = -y - 6$

31. $y = x^2 + x - 2$

32. $y = 4 - x^2$

33. $y = \sqrt{4 - x^2}$

34. $y = \sqrt{x^2 + 9}$

35. $2y - xy + 3x = 4$

36. $x^2y - x^2 + 4y = 0$

37. Think About It Use your knowledge of the Cartesian plane and intercepts to explain why you let y equal zero when you are finding the x-intercepts of the graph of an equation, and why you let x equal zero when you are finding the y-intercepts of the graph of an equation.

38. Think About It Is it possible for a graph to have no x-intercepts? no y-intercepts? no x-intercepts and no y-intercepts? Give examples to support your answers.

Testing for Symmetry In Exercises 39–50, test for symmetry with respect to both axes and the origin. *See Example 7.*

39. $x^4 - 2y = 0$

40. $y = x^4 - x^2 + 3$

41. $x - y^2 = 0$

42. $y^2 = x + 2$

43. $y = \sqrt{16 - x^2}$

44. $y = \sqrt{4 - x^2}$

45. $xy = 2$

46. $x^3y = 1$

47. $y = \dfrac{x}{x^2 - 4}$

48. $y = \dfrac{x}{x^2 + 1}$

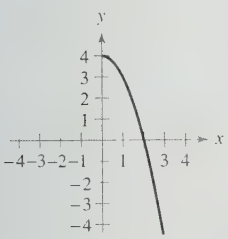
49. $x^2 + y^2 = 25$

50. $x^2 + y^2 = 9$

Using Symmetry as a Sketching Aid In Exercises 51–54, use symmetry to complete the graph of the equation. *See Example 8.*

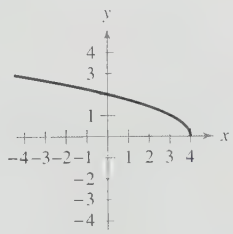
51. *y*-axis symmetry

$y = -x^2 + 4$



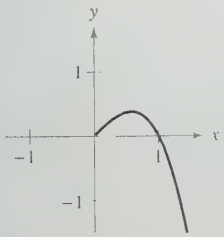
52. *x*-axis symmetry

$y^2 = -x + 4$



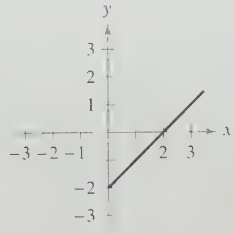
53. Origin symmetry

$y = -x^3 + x$



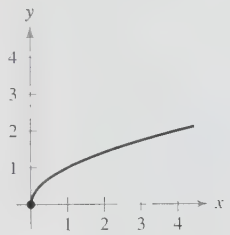
54. *y*-axis symmetry

$y = |x| - 2$

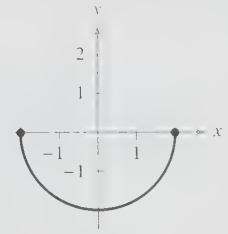


Matching In Exercises 55–60, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

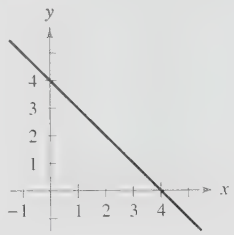
(a)



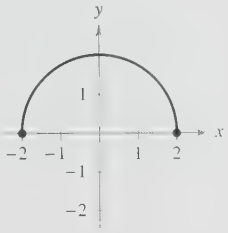
(b)



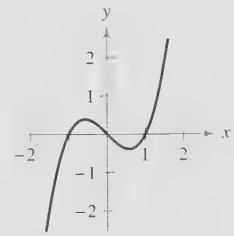
(c)



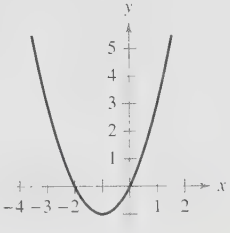
(d)



(e)



(f)



55. $y = 4 - x$

56. $y = \sqrt{4 - x^2}$

57. $y = x^2 + 2x$

58. $y = \sqrt{x}$

59. $y = x^3 - x$

60. $y = -\sqrt{4 - x^2}$

Sketching the Graph of an Equation In Exercises 61–80, sketch the graph of the equation. Identify any intercepts and test for symmetry.

61. $y = 5 - 3x$

62. $y = 2x - 3$

63. $y = 1 - x^2$

64. $y = x^2 - 1$

65. $y = x^2 - 4x + 3$

66. $y = -x^2 - 4x$

67. $y = x^3 + 2$

68. $y = x^3 - 1$

69. $y = \dfrac{8}{x^2 + 4}$

70. $y = \dfrac{4}{x^2 + 1}$

71. $y = \sqrt{x + 1}$

72. $y = \sqrt{1 - x}$

73. $y = \sqrt[3]{x}$

74. $y = \sqrt[3]{x + 1}$

75. $y = |x - 4|$

76. $y = |x| - 3$

77. $x = y^2 - 1$

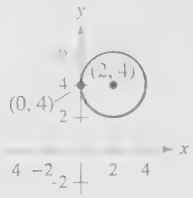
78. $x = y^2 - 4$

79. $x^2 + y^2 = 4$

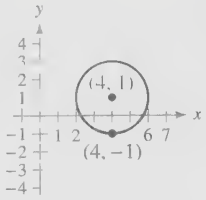
80. $x^2 + y^2 = 16$

Finding the Radius In Exercises 81–84, the center of a circle and a point on the circle are shown. Find the radius of the circle.

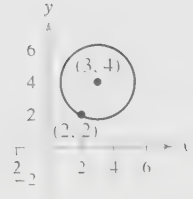
81.



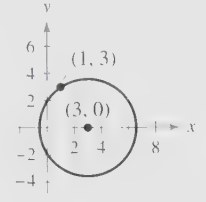
82.



83.

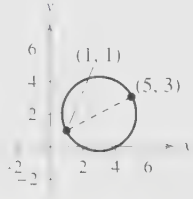


84.

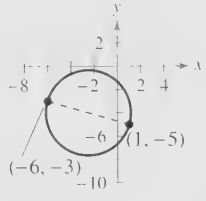


Finding the Center and Radius In Exercises 85 and 86, the endpoints of a diameter of a circle are shown. Find the center and radius of the circle.

85.



86.



Finding the Equation of a Circle In Exercises 87–94, find the standard form of the equation of the circle. *See Example 9.*

87. Center: (0, 0); radius: 3

88. Center: (0, 0); radius: 5

89. Center: (−4, 1); radius: $\sqrt{2}$

90. Center: $\left(0, \frac{1}{2}\right)$; radius: $\frac{2}{3}$

91. Center: (−1, 2); point on circle: (0, 0)

92. Center: (3, −2); point on circle: (−1, 1)

93. Endpoints of a diameter: (−3, 4), (5, −2)

94. Endpoints of a diameter: (−4, −1), (4, 1)

Completing the Square to Sketch a Circle In Exercises 95–102, write the equation of the circle in standard form. Then sketch the circle. *See Example 10.*

95. $x^2 + y^2 - 6x + 4y - 3 = 0$

96. $x^2 + y^2 - 2x + 6y - 15 = 0$

97. $x^2 + y^2 - 4x + 6y + 9 = 0$

98. $5x^2 + 5y^2 + 10x + 1 = 0$

99. $2x^2 + 2y^2 - 2x - 2y - 3 = 0$

100. $4x^2 + 4y^2 - 4x + 2y - 1 = 0$

101. $16x^2 + 16y^2 + 16x + 40y - 7 = 0$

102. $x^2 + y^2 - 4x + 2y + 3 = 0$

Using the Standard Form In Exercises 103 and 104, an equation of a circle is written in standard form. Indicate the coordinates of the center of the circle and determine the radius of the circle. Rewrite the equation of the circle in general form.

103. $(x - 3)^2 + (y + 1)^2 = 25$

104. $\left(x - \frac{1}{2}\right)^2 + (y - 2)^2 = 7$

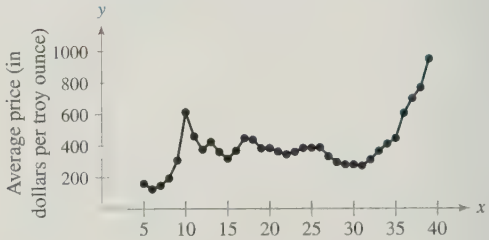
105. Vehicle Distance The graph shows the total number of miles traveled by vehicles in the United States each year from 1994 through 2008. *(Source: U.S. Federal Highway Administration)*



(a) Estimate the percent increase in miles traveled by vehicles from 1994 to 2000.

(b) Estimate the percent increase in miles traveled by vehicles from 2000 to 2008.

106. HOW DO YOU SEE IT? The graph shows the average prices of gold for the years 1975 through 2009. *(Sources: U.S. Bureau of Mines; U.S. Geological Survey)*



(a) What is the highest price shown in the graph? When did this occur?

(b) What is the lowest price shown in the graph? When did this occur?

(c) What is the highest price shown in the graph for the years 1975 through 2000? Does this represent the highest price that gold ever reached in those years? Explain your reasoning.

107. Population The population y (in millions of people) of North America from 1980 through 2050 can be modeled by

$$y = 5.3x + 482, \quad -20 \le x \le 50$$

where x represents the year, with $x = 50$ corresponding to 2050. *(Source: U.S. Census Bureau)*

(a) Find the y-intercept of the graph of the model. What does it represent in the given situation?

(b) Construct a table of values for $x = -20, -10, 0, 10, 20, 30, 40$, and 50.

(c) Plot the solution points given by the table in part (b) and use the points to sketch the graph of the model.

108. Population The population y (in millions of people) of South America from 1980 through 2050 can be modeled by

$$y = -0.03x^2 + 5.0x + 351, \quad -20 \le x \le 50$$

where x represents the year, with $x = 50$ corresponding to 2050. *(Source: U.S. Census Bureau)*

(a) Find the y-intercept of the graph of the model. What does it represent in the given situation?

(b) Construct a table of values for $x = -20, -10, 0, 10, 20, 30, 40$, and 50.

(c) Plot the solution points given by the table in part (b) and use the points to sketch the graph of the model.

109. Retail Stores The number y of stores operated by Family Dollar Stores from 2000 through 2010 can be approximated by the model

$$y = -24.91t^2 + 566.6t + 3624, \quad 0 \le t \le 10$$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: Family Dollar Stores, Inc.)*

(a) Sketch a graph of the model.

(b) Family Dollar Stores planned to have 7050 stores in operation by the end of 2011. Use the model to predict the number of stores in 2011. How well does the model support the company's plans?

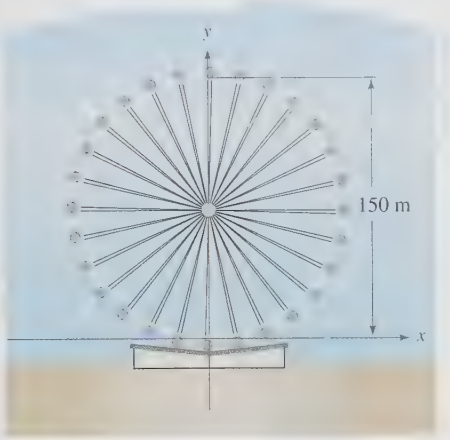
(c) Complete the table by using the model to predict the number of stores in each year from 2012 through 2015.

Year, t	12	13	14	15
Number of stores				

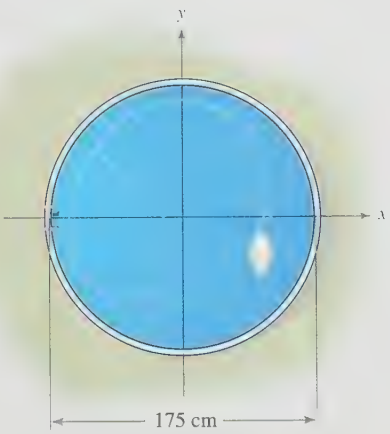
(d) Extend the graph of the model in part (a) to include the years 2011 through 2015.

(e) Family Dollar Stores expects to have 7550 stores in operation sometime during the years 2013 through 2015. Based on your results in parts (c) and (d), what can you conclude about the model?

110. Observation Wheel The Singapore Flyer opened in Singapore in 2008 as the largest observation wheel in the world. Use the diagram to write an equation that models the circular shape of the wheel.



111. Spatial Learning The Morris water maze is an experiment for studying spatial learning in rats. A rat is repeatedly placed in a circular pool of water with a hidden platform that the rat uses to escape the water. The paths taken by the rat are studied as the rat uses visual cues to learn the location of the platform (see figure).



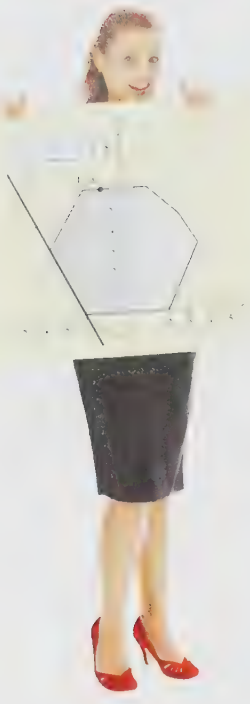
(a) Use the figure to write an equation that models the circular shape of the pool, using centimeters as units.

(b) Researchers track the time that a rat spends in each quadrant. The white dashed line shows the path taken by the rat. Determine the sequence of quadrants that the rat passed through.

(c) The platform is located at the point $(-40, 0)$ and the rat is at $(52, -40)$. How far is the rat from the platform? Round your answer to the nearest centimeter.

2.2 Lines in the Plane

- Find the slope of a line passing through two points.
- Use the point-slope form to find the equation of a line.
- Use the slope-intercept form to sketch the graph of a line.
- Use slope to determine whether lines are parallel or perpendicular, and write the equation of a line parallel or perpendicular to a given line.



In Exercise 96 on page 182, you will use perpendicular lines to write an equation representing the line of a cut in making a memorial stone.

The Slope of a Line

The **slope** of a nonvertical line is a measure of the steepness of the line. The slope represents the number of units the line rises or falls vertically for each unit of horizontal change from left to right. For instance, consider the two points (x_1, y_1) and (x_2, y_2) on the line shown in Figure 2.21. As you move from left to right along this line, a change of $y_2 - y_1$ units in the vertical direction corresponds to a change of $x_2 - x_1$ units in the horizontal direction. That is,

$$y_2 - y_1 = \text{the change in } y$$

and

$$x_2 - x_1 = \text{the change in } x.$$

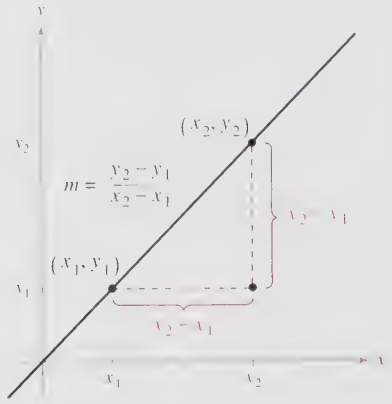


FIGURE 2.21

The slope of the line is defined as the quotient of these two changes.

Definition of the Slope of a Line

The **slope** m of the nonvertical line passing through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

where $x_1 \neq x_2$.

The change in x is sometimes called the *run* and the change in y is sometimes called the *rise*.

In real-life problems, such as finding the steepness of a ramp or the increase in the value of a product, the slope of a line can be interpreted as either a *ratio* or a *rate*. If the x-axis and the y-axis have the same units of measure, then the slope has no units and is a *ratio*. If the x-axis and the y-axis have different units of measure, then the slope is a *rate* or *rate of change*. You will learn more about rates of change in Section 2.3.

STUDY TIP

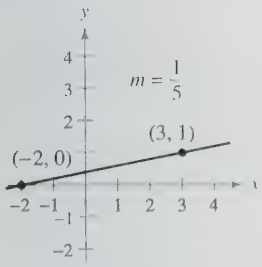
When using the formula for slope, the *order of subtraction* is important. Given two points on a line, you can label either one of them as (x_1, y_1) and the other as (x_2, y_2). Once this is done, you must form the numerator and denominator using the same order of subtraction.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Correct

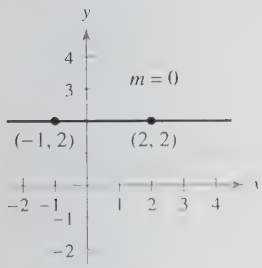
$$m = \frac{y_1 - y_2}{x_1 - x_2}$$ Correct

$$m = \frac{y_2 - y_1}{x_1 - x_2}$$ Incorrect

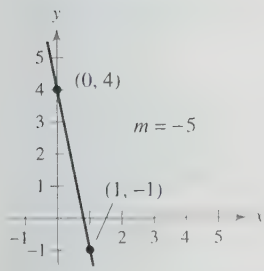
Yuri Arcurs /Shutterstock.com



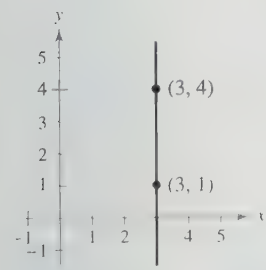
(a) If m is positive, the line rises from left to right.



(b) If m is zero, the line is horizontal.



(c) If m is negative, the line falls from left to right.



(d) If the line is vertical, the slope is undefined.

FIGURE 2.22

Example 1 Finding the Slope of a Line Through Two Points

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$ **b.** $(-1, 2)$ and $(2, 2)$

c. $(0, 4)$ and $(1, -1)$ **d.** $(3, 4)$ and $(3, 1)$

SOLUTION

a. Letting $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (3, 1)$, you obtain a slope of

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \frac{\text{Difference in } y\text{-values}}{\text{Difference in } x\text{-values}}$$

$$= \frac{1 - 0}{3 - (-2)}$$

$$= \frac{1}{5}.$$

b. The slope of the line passing through $(-1, 2)$ and $(2, 2)$ is

$$m = \frac{2 - 2}{2 - (-1)}$$

$$= \frac{0}{3}$$

$$= 0.$$

c. The slope of the line passing through $(0, 4)$ and $(1, -1)$ is

$$m = \frac{-1 - 4}{1 - 0}$$

$$= \frac{-5}{1}$$

$$= -5.$$

d. The slope of the line passing through $(3, 4)$ and $(3, 1)$ is undefined. Applying the formula for slope, you have

$$m = \frac{1 - 4}{3 - 3} = \frac{-3}{0}.$$ Division by zero is undefined.

Because division by zero is not defined, the slope of a vertical line is not defined.

The graphs of the four lines are shown in Figure 2.22. ■

✓Checkpoint 1

Find the slope of the line passing through the points $(-1, 2)$ and $(3, 4)$. ■

From Example 1, you can make the following generalizations about the slope of a line.

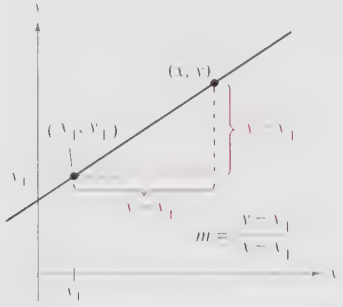
Slope of a Line

1. A line with positive slope ($m > 0$) *rises* from left to right.
2. A line with negative slope ($m < 0$) *falls* from left to right.
3. A line with zero slope ($m = 0$) is *horizontal*.
4. A line with undefined slope is *vertical*.

The Point-Slope Form

When you know the slope of a line and the coordinates of one point on the line, you can find an equation of the line. For instance, in Figure 2.23, let (x_1, y_1) be a given point on the line whose slope is m. If (x, y) is *any other* point on the line, then it follows that

$$\frac{y - y_1}{x - x_1} = m.$$



Any two points on a line can be used to determine the slope of the line.

FIGURE 2.23

ALGEBRA TUTOR

For help with identifying and plotting points in the Cartesian plane, see the *Chapter 2 Algebra Tutor* on pages 240 and 241.

This equation in the variables x and y can be rewritten to produce the following **point-slope form** of the equation of a line.

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

Example 2 The Point-Slope Form of the Equation of a Line

Find an equation of the line that passes through $(1, -2)$ and has a slope of 3.

SOLUTION Use the point-slope form with $(x_1, y_1) = (1, -2)$ and $m = 3$.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute $y_1 = -2$, $x_1 = 1$, and $m = 3$.
$y + 2 = 3x - 3$	Simplify
$y = 3x - 5$	Equation of line

The graph of this line is shown in Figure 2.24.

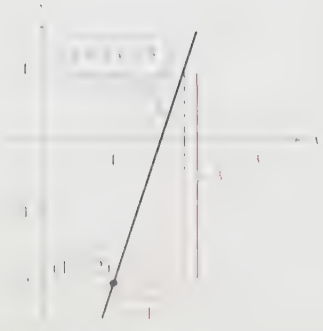


FIGURE 2.24

Checkpoint 2

Find an equation of the line that passes through the given point and has the given slope.

a. $(2, 4)$, $m = -2$

b. $(-8, -3)$, $m = \frac{3}{2}$

c. $(5, 1)$, $m = 0$

TECH TUTOR

Several models of graphing utilities have programs that use the two-point form to find an equation of a line. After you enter the coordinates of two points, the program outputs the slope and y-intercept of the line that passes through the points. Try using one of these programs to find the linear equation in Example 3.

The point-slope form can be used to find the equation of a line passing through two points (x_1, y_1) and (x_2, y_2). First, use the formula for the slope of a line passing through two points. Then, use the point-slope form to obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is sometimes called the **two-point form** of the equation of a line.

Example 3 A Linear Model for Sales Prediction

During the first two quarters of the calendar year, Apple Inc. had sales of \$13.5 billion and \$15.7 billion, respectively.

a. Write a linear equation giving the sales y in terms of the quarter x.

b. Use the equation to predict the sales during the fourth quarter. Can you assume that sales will follow this linear pattern? *(Source: Apple Inc.)*

SOLUTION

a. Let $(x_1, y_1) = (1, 13.5)$ and $(x_2, y_2) = (2, 15.7)$ be two points on the line. Use the two-point form to find an equation of the line.

$$y - 13.5 = \frac{15.7 - 13.5}{2 - 1}(x - 1)$$ Substitute for x_1, y_1, x_2, and y_2 in two-point form.

$$y - 13.5 = 2.2(x - 1)$$ Simplify quotient.

$$y = 2.2x + 11.3$$ Equation of line

b. Using the equation from part (a), the fourth-quarter sales ($x = 4$) should be

$$y = 2.2(4) + 11.3$$ Substitute 4 for x.

$$= \$20.1 \text{ billion.}$$ Simplify.

See Figure 2.25. Without more data, you cannot assume that the sales pattern will be linear. Many factors, such as seasonal demand and past sales history, help to determine the sales pattern.

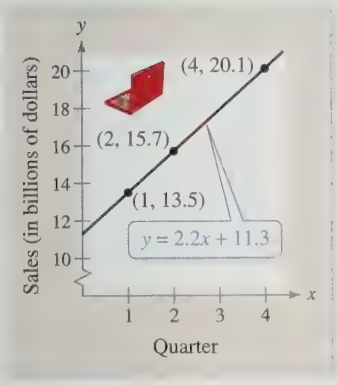
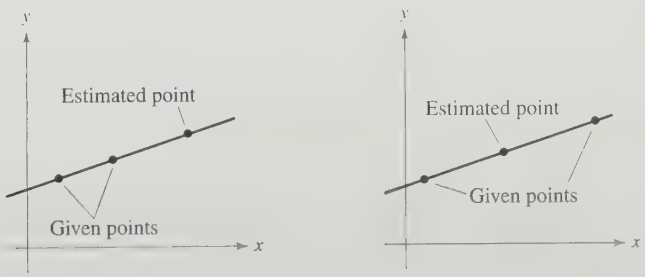


FIGURE 2.25

Checkpoint 3

A company has sales of \$1.2 million and \$1.4 million in its first two years. Write a linear equation giving the sales y in terms of the year x.

The estimation method illustrated in Example 3 is called **linear extrapolation.** Note in Figure 2.26(a) that for linear extrapolation, the estimated point lies to the *right* of the given points. When the estimated point lies *between* two given points, the procedure is called **linear interpolation,** as shown in Figure 2.26(b).



(a) Linear extrapolation (b) Linear interpolation

FIGURE 2.26



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Sketching Graphs of Lines

You have seen that for *finding the equation of a line* it is convenient to use the point-slope form. This formula, however, is not particularly useful for *sketching the graph of a line*. The form that is better suited to graphing linear equations is the **slope-intercept form** of the equation of a line. You can derive the slope-intercept form as shown.

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y = mx - mx_1 + y_1 \qquad \text{Solve for } y.$$
$$y = mx + (y_1 - mx_1) \qquad \text{Commutative Property of Addition}$$
$$y = mx + b \qquad \text{Slope-intercept form } (b = y_1 - mx_1)$$

Slope-Intercept Form of the Equation of a Line

The graph of the equation

$$y = mx + b$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

Example 4 Sketching the Graphs of Linear Equations

Sketch the graph of each linear equation.

a. $y = 2x + 1$ **b.** $y = 2$ **c.** $x + y = 2$

SOLUTION

a. Because $b = 1$, the y-intercept is $(0, 1)$. Moreover, because the slope is $m = 2$, this line *rises* two units for each unit it moves to the right, as shown in Figure 2.27(a).

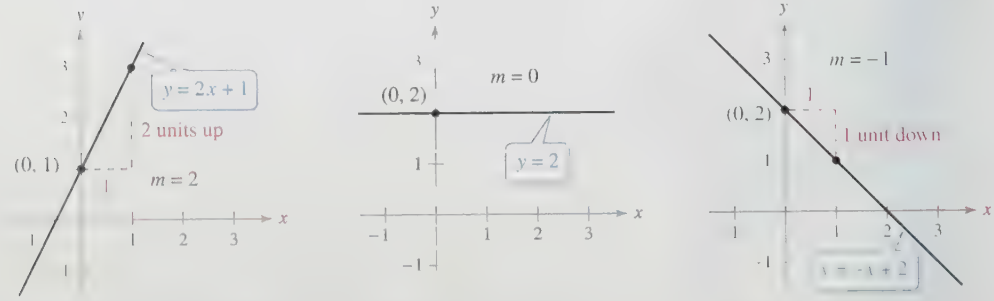
b. By writing the equation $y = 2$ in the form $y = (0)x + 2$ you can see that the y-intercept is $(0, 2)$ and the slope is zero. A zero slope implies that the line is horizontal, as shown in Figure 2.27(b).

c. By writing the equation $x + y = 2$ in slope-intercept form $y = -x + 2$ you can see that the y-intercept is $(0, 2)$. Moreover, because the slope is $m = -1$, this line *falls* one unit for each unit it moves to the right, as shown in Figure 2.27(c).

Checkpoint 4

Sketch the graph of each linear equation.

a. $y = 4x + 1$ **b.** $y - 2x = -3$



(a) When m is positive, the line rises from left to right.

(b) When m is zero, the line is horizontal.

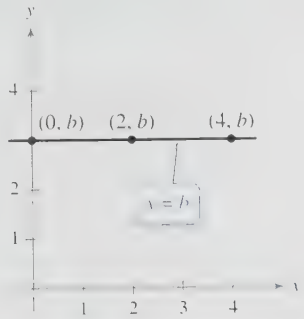
(c) When m is negative, the line falls from left to right.

FIGURE 2.27

From the slope-intercept form of the equation of a line, you can see that a horizontal line ($m = 0$) has an equation of the form

$$y = (0)x + b \quad \text{or} \quad y = b. \qquad \text{Horizontal line}$$

This is consistent with the fact that each point on a horizontal line through $(0, b)$ has a y-coordinate of b, as shown in Figure 2.28.



Horizontal Line

FIGURE 2.28

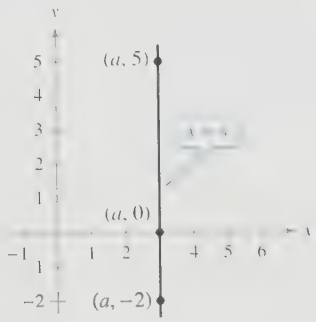
Similarly, each point on a vertical line through $(a, 0)$ has an x-coordinate of a, as shown in Figure 2.29. So, a vertical line has an equation of the form

$$x = a. \qquad \text{Vertical line}$$

This equation cannot be written in slope-intercept form because the slope of a vertical line is undefined. However, *every* line has an equation that can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form}$$

where A and B are not *both* zero. If $A = 0$ (and $B \neq 0$), then the general equation can be reduced to the form $y = b$, which represents a horizontal line. If $B = 0$ (and $A \neq 0$), then the general equation can be reduced to the form $x = a$, which represents a vertical line.



Vertical Line

FIGURE 2.29

Summary of Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

Make sure you understand that one form of the equation of a line may be more efficient than another form for different problem situations. For instance, the point-slope form (or two-point form) is more efficient for problems in which two points are given. Note that another form of an equation of a line is the *two-intercept* or *intercept form*. This form is introduced in Exercises 59–64 in this section.

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel, perpendicular, or neither.

Parallel Lines

Two distinct nonvertical lines are **parallel** if and only if their slopes are equal.

Example 5 Equations of Parallel Lines

Find an equation of the line that passes through the point $(2, -1)$ and is parallel to the line $2x - 3y = 5$, as shown in Figure 2.30.

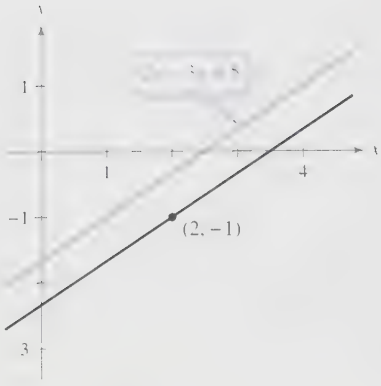


FIGURE 2.30

SOLUTION Start by rewriting the given equation in slope-intercept form.

$$2x - 3y = 5 \qquad \text{Write original equation.}$$

$$-3y = -2x + 5 \qquad \text{Subtract } 2x \text{ from each side.}$$

$$y = \frac{2}{3}x - \frac{5}{3} \qquad \text{Write in slope-intercept form.}$$

So, the given line has a slope of

$$m = \tfrac{2}{3}.$$

Because any line parallel to the given line must also have a slope of $\frac{2}{3}$, the required line through $(2, -1)$ has the following equation.

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - (-1) = \frac{2}{3}(x - 2) \qquad \text{Substitute for } y_1, x_1, \text{ and } m.$$

$$y + 1 = \frac{2}{3}x - \frac{4}{3} \qquad \text{Simplify.}$$

$$y = \frac{2}{3}x - \frac{4}{3} - 1 \qquad \text{Solve for } y.$$

$$y = \frac{2}{3}x - \frac{7}{3} \qquad \text{Write in slope-intercept form.}$$

Notice the similarity between the slope-intercept form of the original equation and the slope-intercept form of the parallel equation.

✓Checkpoint 5

Find an equation of the line that passes through the given point and is parallel to the line $2y - 6x = 2$.

a. $(2, 4)$

b. $(-2, 0)$

You have seen that two nonvertical lines are parallel if and only if they have the same slope. Two nonvertical lines are *perpendicular* if and only if their slopes are negative reciprocals of each other. For instance, the lines

$$y = 2x \quad \text{and} \quad y = -\tfrac{1}{2}x$$

are perpendicular because one has a slope of $2 = \frac{2}{1}$ and the other has a slope of $-\frac{1}{2}$.

Perpendicular Lines

Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,

$$m_1 = -\frac{1}{m_2}.$$

Example 6 Equations of Perpendicular Lines

Find an equation of the line that passes through the point $(2, -1)$ and is perpendicular to the line $2x - 3y = 5$, as shown in Figure 2.31.

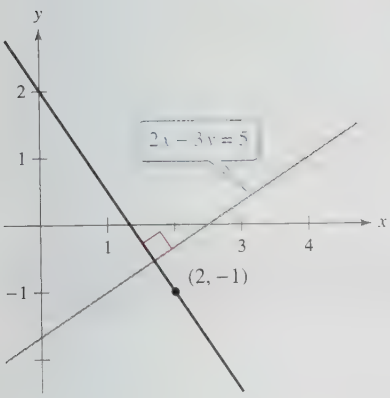


FIGURE 2.31

SOLUTION By rewriting the given equation in slope-intercept form

$$y = \frac{2}{3}x - \frac{5}{3}$$

you can see that the line has a slope of $\frac{2}{3}$. So, any line that is perpendicular to this line must have a slope of $-\frac{3}{2}$ because $-\frac{3}{2}$ is the negative reciprocal of $\frac{2}{3}$. The required line through the point $(2, -1)$ has the following equation.

$$y - y_1 = m(x - x_1)$$ Point-slope form

$$y - (-1) = -\frac{3}{2}(x - 2)$$ Substitute for y_1, x_1, and m.

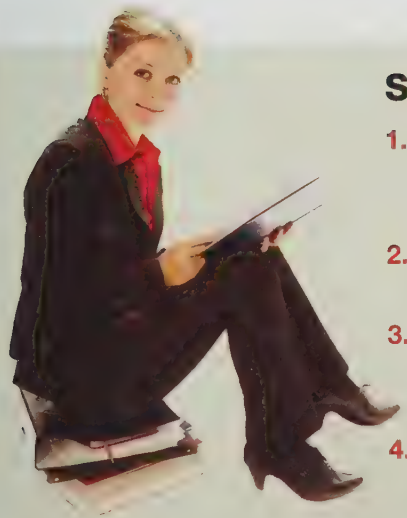
$$y + 1 = -\frac{3}{2}x + 3$$ Simplify.

$$y = -\frac{3}{2}x + 3 - 1$$ Solve for y.

$$y = -\frac{3}{2}x + 2$$ Write in slope-intercept form.

Checkpoint 6

Find an equation of the line that passes through the point $(-2, 12)$ and is perpendicular to the line $y = \frac{1}{4}x - 2$.



SUMMARIZE (Section 2.2)

1. State the definition of the slope of a line and describe how to find the slope of a line passing through two points *(page 172)*. For an example of finding the slope of a line through two points, see Example 1.
2. State the point-slope form of the equation of a line *(page 174)*. For examples of using the point-slope form of the equation of a line, see Examples 2 and 3.
3. State the slope-intercept form of the equation of a line and describe how it is used to sketch the graph of a line *(page 176)*. For an example of using the slope-intercept form to sketch the graphs of lines, see Example 4.
4. Describe the slopes of two nonvertical parallel lines and the slopes of two nonvertical perpendicular lines *(pages 178 and 179)*. For examples of finding equations of parallel and perpendicular lines, see Examples 5 and 6.

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SKILLS WARM UP 2.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2 and 1.2.

In Exercises 1 and 2, perform the indicated operations.

1. $\dfrac{4-(-4)}{-3-(-1)}$

2. $\dfrac{-5-8}{0-(-3)}$

3. Evaluate $-1/m$ when $m = 4/5$.

4. Evaluate $-1/m$ when $m = -3$.

In Exercises 5–10, solve for y in terms of x.

5. $2x - 3y = 6$

6. $4x + 2y = 0$

7. $y - (-4) = 3[x - (-1)]$

8. $y - 7 = \dfrac{2}{3}(x - 3)$

9. $y - (-1) = \dfrac{3-(-1)}{2-4}(x - 4)$

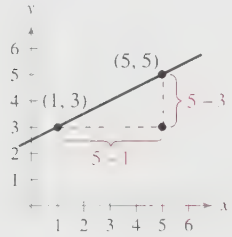
10. $y - 5 = \dfrac{3-5}{0-2}(x - 2)$

Exercises 2.2

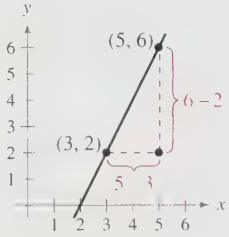
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding the Slope of a Line In Exercises 1–4, use the change in y and the change in x to find the slope of the line.

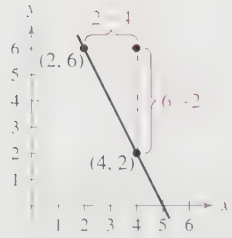
1.



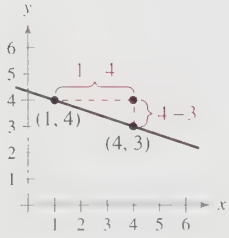
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3.

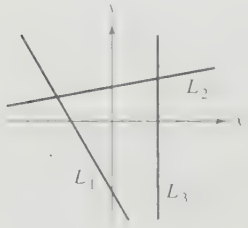


4.

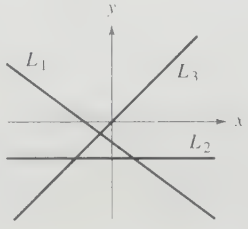


Using Slope In Exercises 5 and 6, identify the line that has the indicated slope.

5. (a) $m = \frac{1}{3}$
   (b) m is undefined.
   (c) $m = -2$



6. (a) $m = 0$
   (b) $m = -\frac{3}{4}$
   (c) $m = 1$



Sketching the Graphs of Lines In Exercises 7 and 8, sketch the graphs of the lines through the point with the indicated slopes in the same coordinate plane.

	Point	*Slopes*	
7.	$(-3, 4)$	(a) -2	(b) $\frac{2}{3}$
		(c) 0	(d) Undefined
8.	$(-2, -5)$	(a) -1	(b) $\frac{3}{4}$
		(c) 0	(d) Undefined

Finding the Slope of a Line Through Two Points In Exercises 9–14, plot the points and find the slope of the line passing through the points. *See Example 1.*

9. $(6, 9), (-4, -1)$
10. $(2, 4), (4, -4)$
11. $(-6, -1), (-6, 4)$
12. $(0, -10), (-4, 0)$
13. $\left(-\frac{1}{3}, 1\right), \left(-\frac{2}{3}, \frac{5}{6}\right)$
14. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

Using Slope In Exercises 15–22, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)

	Point	*Slope*
15.	$(5, -2)$	$m = 0$
16.	$(-3, 4)$	$m = 0$
17.	$(2, -5)$	m is undefined.
18.	$(-1, 3)$	m is undefined.
19.	$(5, -6)$	$m = 1$
20.	$(10, -6)$	$m = -1$
21.	$(-6, -1)$	$m = \frac{1}{2}$
22.	$(7, -5)$	$m = -\frac{2}{3}$

The Point-Slope Form of the Equation of a Line In Exercises 23–32, find an equation of the line that passes through the point and has the indicated slope. Then sketch the graph of the line. *See Example 2.*

	Point	*Slope*
23.	$(7, 0)$	$m = 1$
24.	$(0, -4)$	$m = -1$
25.	$(-2, 0)$	$m = -4$
26.	$(1, 3)$	$m = 3$
27.	$(4, 0)$	$m = -\frac{1}{3}$
28.	$(-8, 3)$	$m = -\frac{1}{2}$
29.	$(-2, -7)$	$m = 0$
30.	$(-10, 4)$	$m = 0$
31.	$\left(4, \frac{5}{2}\right)$	$m = \frac{4}{3}$
32.	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$m = -3$

Writing an Equation of a Line In Exercises 33–44, find an equation of the line passing through the points.

33. $(0, 0), (-1, -4)$
34. $(6, -1), (0, 0)$
35. $(7, -4), (-7, 3)$
36. $(4, 3), (-4, -4)$
37. $(-9, 11), (-9, 14)$
38. $(3, 5), (3, -2)$
39. $(-1, 7), (3, 7)$
40. $(3, -2), (-8, -2)$
41. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$
42. $(1, 1), \left(6, -\frac{2}{3}\right)$
43. $(1, 0.6), (-2, -0.6)$
44. $(-8, 0.6), (2, -2.4)$



45. Discovery Use a graphing utility to graph each equation in the same viewing window. What is the y-intercept of each graph?

$y_1 = x + 2$ $\quad y_2 = \frac{1}{4}x + 2$ $\quad y_3 = -x + 2$

46. Discovery In Exercise 45, what can you determine about each graph from the coefficient of x in its equation?

Sketching the Graph of a Linear Equation In Exercises 47–56, find the slope and y-intercept (if possible) of the line specified by the equation. Then sketch the graph of the line. *See Example 4.*

47. $y = 2x - 1$
48. $y = 3 - x$
49. $4x - y - 6 = 0$
50. $2x + 3y - 9 = 0$
51. $8 - 3x = 0$
52. $2x + 5 = 0$
53. $7x + 6y - 30 = 0$
54. $x - y - 10 = 0$
55. $2y - 7 = 0$
56. $8 - 5y = 0$

57. Writing A student is confused about why the slope of a vertical line is undefined. Describe how you would help this student understand the concept of undefined slope.

58. Writing A student makes the comment, "I do not understand why a horizontal line has zero slope and how that is different from undefined or no slope." Describe how you would explain the concepts of zero slope and undefined slope and how they are different from each other.

The Intercept Form of the Equation of a Line In Exercises 59–64, use the *intercept form* to find an equation of the line with the given intercepts. The intercept form of the equation of a line with intercepts $(a, 0)$ and $(0, b)$ is

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, \quad b \neq 0.$$

59. x-intercept: $(1, 0)$; y-intercept: $(0, -4)$
60. x-intercept: $(-3, 0)$; y-intercept: $(0, 4)$
61. x-intercept: $(-2, 0)$; y-intercept: $(0, -2)$
62. x-intercept: $(5, 0)$; y-intercept: $(0, 1)$
63. x-intercept: $\left(-\frac{1}{6}, 0\right)$; y-intercept: $\left(0, -\frac{2}{3}\right)$
64. x-intercept: $\left(-\frac{2}{3}, 0\right)$; y-intercept: $\left(0, \frac{1}{2}\right)$



65. Discovery Use a graphing utility to graph the equation $y = x$ in a viewing window with a *square* setting, in which the horizontal and vertical tick marks have equal spacing. Then adjust the viewing window several times by increasing and decreasing the number of units on the y-axis. How does changing the viewing window affect the graph? Which setting best represents the slope of the line?



66. Discovery Use a graphing utility to graph each equation in the same viewing window with a *square* setting (see Exercise 65).

$y_1 = \frac{4}{7}x - 1$ $\quad y_2 = \frac{4}{7}x + 2$ $\quad y_3 = -\frac{7}{4}x + 8$

(a) What is the geometric relationship between the two lines that have the same slope?

(b) In each pair of perpendicular lines, what is the numerical relationship between their slopes?

Parallel and Perpendicular Lines In Exercises 67–74, the equations of two lines are given. Determine whether the lines L_1 and L_2 are parallel, perpendicular, or neither.

67. L_1: $y = 3x + 4$; L_2: $y = x - \frac{1}{4}$
68. L_1: $y = \frac{3}{4}x + 1$; L_2: $y = -\frac{4}{3}x + 3$
69. L_1: $2x - y = 1$; L_2: $x + 2y = -1$
70. L_1: $x - 5y = -2$; L_2: $-3x + 15y = 6$
71. L_1: $x - 3y = -3$; L_2: $2x - 6y = 6$
72. L_1: $4x - y = -2$; L_2: $8x - 2y = 6$
73. L_1: $2x - 3y - 15 = 0$; L_2: $3x + 2y + 8 = 0$
74. L_1: $x - 4y - 12 = 0$; L_2: $3x - 4y - 8 = 0$

Parallel and Perpendicular Lines In Exercises 75–82, determine whether the lines L_1 and L_2 passing through the indicated pairs of points are parallel, perpendicular, or neither.

75. L_1: $(-5, 0), (-2, 1)$; L_2: $(0, 1), (3, 2)$
76. L_1: $(-1, 6), (1, 4)$; L_2: $(3, -3), (6, -9)$
77. L_1: $(0, -1), (5, 9)$; L_2: $(0, 3), (4, 1)$
78. L_1: $(3, 6), (-6, 0)$; L_2: $(0, -1), \left(5, \frac{7}{3}\right)$
79. L_1: $(-2, -1), (1, 5)$; L_2: $(1, 3), (5, -5)$

80. L_1: (4, 8), (−4, 2); L_2: (3, −5), $\left(-1, \frac{1}{3}\right)$

81. L_1: (−1, 7), (−6, 4); L_2: (0, 1), (5, 4)

82. L_1: (−1, 3), (2, −5); L_2: (3, 0), (2, −7)

Equations of Parallel and Perpendicular Lines In Exercises 83–88, write equations of the lines passing through the given point that are (a) parallel to the given line and (b) perpendicular to the given line. *See Examples 5 and 6.*

	Point	*Line*
83.	(6, 2)	$y = 2x - 1$
84.	(−5, 4)	$y = -x + 8$
85.	$\left(\frac{1}{4}, -\frac{2}{3}\right)$	$2x - 3y = 5$
86.	$\left(\frac{7}{8}, \frac{3}{4}\right)$	$5x + 3y = 0$
87.	(−1, 0)	$y = -3$
88.	(2, 5)	$x = 4$

89. Wheelchair Ramp The maximum recommended slope of a wheelchair ramp is $\frac{1}{12}$. A business is installing a wheelchair ramp that rises 34 inches over a horizontal length of 30 feet. Is the ramp steeper than recommended? *(Source: Americans with Disabilities Act Handbook)*

90. HOW DO YOU SEE IT? Match the description of the situation with its graph. Then write the equation of the line.

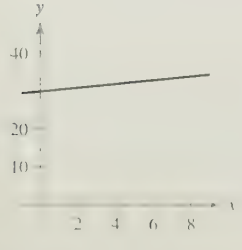
(a) You are paying \$10 per week to repay a \$100 loan.

(b) An employee is paid \$12.50 per hour plus \$1.50 for each unit produced per hour.

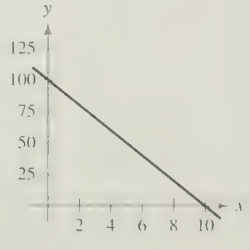
(c) A sales representative receives \$30 per day for food plus \$0.51 for each mile traveled.

(d) A computer that was purchased for \$800 depreciates \$100 per year.

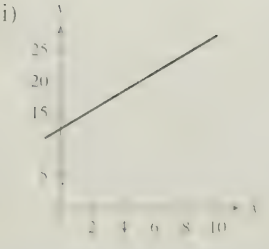
(i)



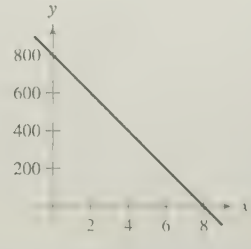
(ii)



(iii)



(iv)



91. Temperature Find an equation of the line that gives the relationship between the temperature in degrees Celsius C and the temperature in degrees Fahrenheit F. Remember that water freezes at 0° Celsius (32° Fahrenheit) and boils at 100° Celsius (212° Fahrenheit).

92. Temperature Use the result of Exercise 91 to complete the table. Is there a temperature for which the Fahrenheit reading is the same as the Celsius reading? If so, what is it?

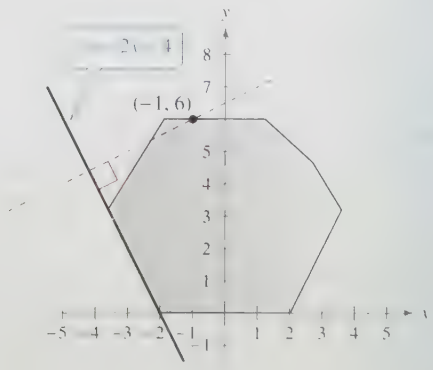
C		−10°	10°			177°
F	0°			68°	90°	

93. Sales During the first two quarters of the calendar year, a business had sales of \$158,000 and \$165,000, respectively. Write a linear equation giving the sales y in terms of the quarter x. Use the equation to predict the fourth quarter sales. Can you assume that sales will follow this linear pattern?

94. Retail Stores AutoZone operated 3483 stores in 2004. By 2007, AutoZone operated 4056 stores. Assuming the number of stores followed a linear pattern, how many stores would AutoZone have operated in 2010? The actual number of stores operated by AutoZone in 2010 was 4627. Was the actual increase in the number of stores approximately linear? *(Source: AutoZone)*

95. Scuba Diving The pressure (in atmospheres) exerted on a scuba diver's body has a linear relationship with the diver's depth. At sea level (or a depth of 0 feet), the pressure exerted on a diver is 1 atmosphere. At a depth of 99 feet, the pressure exerted on a diver is 4 atmospheres. Write a linear equation to describe the pressure p (in atmospheres) in terms of the depth d (in feet) below the surface of the sea. What is the rate of change in pressure with respect to depth? *(Source: PADI Open Water Diver Manual)*

96. Stone Cutting A stone cutter is making a 6-foot-tall memorial stone. The diagram shows coordinates labeled in feet. The stone cutter plans to cut through the point (−1, 6) along a line that is perpendicular to one side of the stone, as indicated by the dashed line in the figure. Find an equation of the line of the cut.



2.3 Linear Modeling and Direct Variation

- Use a mathematical model to approximate a set of data points.
- Construct a linear model to relate quantities that vary directly.
- Construct and use a linear model with slope as the rate of change.
- Use a scatter plot to find a linear model that fits a set of data.



In Exercise 41 on page 192, you will use data points from a scatter plot to create a linear model describing the number of autistic children receiving disability services.

Mathematical Models

The primary objective of applied mathematics is to find equations or **mathematical models** that describe real-world situations. In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals—accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results.

You have already studied some techniques for fitting models to data. For instance, in Section 2.2, you learned how to find the equation of a line that passes through two points. In this section, you will study other techniques for fitting models to data: *direct variation, rates of change,* and *linear regression.*

Example 1 A Mathematical Model

The weight of a puppy recorded every two months is shown in the table.

Age (in months)	2	4	6	8	10	12
Weight (in pounds)	24	45	67	93	117	130

A linear model that approximates the puppy's weight w (in pounds) in month t is

$$w = 11.03t + 2.1, \quad 2 \le t \le 12.$$

How closely does the model represent the data?

SOLUTION By graphing the data points with the linear model (see Figure 2.32), you can see that the model is a "good fit" for the actual data. The table shows how each actual weight w compares with the weight w^* given by the model.

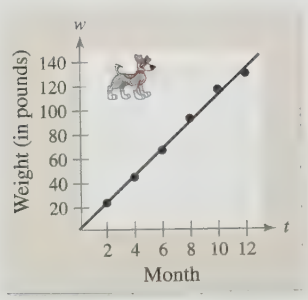


FIGURE 2.32

t	w	w^*
2	24	24.16
4	45	46.22
6	67	68.28
8	93	90.34
10	117	112.4
12	130	134.46

Checkpoint 1

In Example 1, what are the best and worst approximations given by the model?

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Direct Variation

There are two basic types of linear models in x and y. The more general model has a y-intercept that is nonzero:

$$y = mx + b, \quad b \neq 0.$$

The simpler model,

$$y = mx$$

has a y-intercept that is zero. In the simpler model, y is said to **vary directly** as x, or to be **directly proportional** to x.

Direct Variation

The following statements are equivalent.

1. y **varies directly** as x.
2. y is **directly proportional** to x.
3. $y = mx$ for some nonzero constant m, where m is the **constant of variation** or the **constant of proportionality.**

Example 2 State Income Tax

In Pennsylvania, state income tax is directly proportional to *taxable income.* For a taxable income of \$40,000, the Pennsylvania state income tax is \$1228. Find a mathematical model that gives the Pennsylvania state income tax in terms of taxable income.

SOLUTION

Verbal Model: State income tax $=$ m $\cdot$ Taxable income

Labels:
State income tax $= y$ (dollars)
Taxable income $= x$ (dollars)
Income tax rate $= m$ (percent in decimal form)

Equation: $y = mx$

Find m by substituting the given information into the equation $y = mx$.

$y = mx$ Direct variation model

$1228 = m(40{,}000)$ Substitute $y = 1228$ and $x = 40{,}000$.

$\frac{1228}{40{,}000} = m$ Divide each side by 40,000.

$0.0307 = m$ Income tax rate

An equation (or model) for state income tax in Pennsylvania is

$$y = 0.0307x.$$

So, Pennsylvania has a state income tax rate of 3.07% of taxable income. The graph of this equation is shown in Figure 2.33.

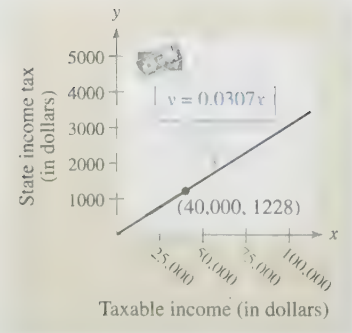


FIGURE 2.33

Checkpoint 2

You buy a flash drive for \$14.50 and pay sales tax of \$0.87. The sales tax is directly proportional to the price. Find a mathematical model that gives the sales tax in terms of the price.

Most measurements in the English system and the metric system are directly proportional. The next example shows how to use a direct proportion to convert between centimeters per day and inches per day.

Example 3 The English and Metric Systems

Bamboos are some of the fastest growing plants in the world. One type of bamboo has been measured to grow 47.6 inches, or 120.9 centimeters, per day. Use this information to find a mathematical model that relates inches per day to centimeters per day. *(Source: American Bamboo Society)*

SOLUTION Let y represent the growth rate in inches per day and let x represent the growth rate in centimeters per day. Then y and x are related by the equation

$$y = mx.$$

Use the fact that $y = 47.6$ when $x = 120.9$ to find the value of m.

$$y = mx \qquad \text{Direct variation model}$$

$$47.6 = m(120.9) \qquad \text{Substitute } y = 47.6 \text{ and } x = 120.9.$$

$$\frac{47.6}{120.9} = m \qquad \text{Divide each side by } 120.9.$$

$$0.39371 \approx m \qquad \text{Use a calculator.}$$

So, the conversion factor from centimeters per day to inches per day is approximately 0.39371, and the model is

$$y = 0.39371x.$$

The graph of this equation is shown in Figure 2.34.

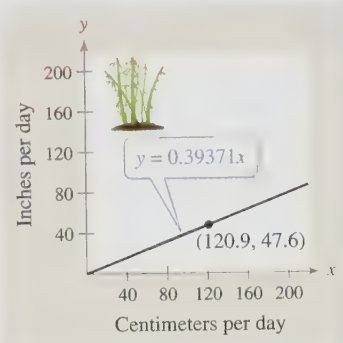


FIGURE 2.34

Checkpoint 3

You buy an ice bucket with a capacity of 44 ounces, or 1.3 liters. Write a mathematical model that relates ounces to liters. ■

You can use the model from Example 3 to convert any growth rate in centimeters per day to inches per day, as shown in the table.

Centimeters per day	Inches per day
40	15.7
80	31.5
120	47.2
160	63.0
200	78.7

Rates of Change

A second common type of linear model is one that involves a known rate of change. In the linear equation

$$y = mx + b$$

you know that m represents the slope of the line. In real-life problems, the slope can often be interpreted as the **rate of change** of y with respect to x. Rates of change should always be listed in appropriate units of measure.

Example 4 Mountain Climbing

A mountain climber is climbing up a 500-foot cliff. At 1 P.M., the climber is 115 feet up the cliff. By 4 P.M., the climber has reached a height of 280 feet, as shown in Figure 2.35.

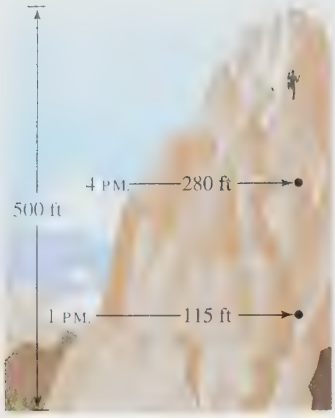


FIGURE 2.35

a. Find the average rate of change of the climber. Use this rate of change to find an equation that relates the height of the climber to the time.

b. Use the equation to estimate the time when the climber reaches the top of the cliff.

SOLUTION

a. Let y represent the climber's height on the cliff (in feet). Let $t = 1$ represent 1 P.M., and let t be measured in hours. Then the two points that represent the climber's two positions are

$$(t_1, y_1) = (1, 115) \quad \text{and} \quad (t_2, y_2) = (4, 280).$$

So, the average rate of change of the climber is

$$\text{Average rate of change} = \frac{y_2 - y_1}{t_2 - t_1} = \frac{280 - 115}{4 - 1} = 55 \text{ feet per hour.}$$

An equation that relates the height of the climber to the time is

$$y - y_1 = m(t - t_1) \qquad \text{Point-slope form}$$
$$y - 115 = 55(t - 1) \qquad \text{Substitute } y_1 = 115,\ t_1 = 1, \text{ and } m = 55.$$
$$y = 55t + 60. \qquad \text{Linear model}$$

If you had chosen to use the point (t_2, y_2) to determine the equation, then you would have obtained a different equation initially: $y - 280 = 55(t - 4)$. However, simplifying this equation yields the same linear model $y = 55t + 60$.

b. To estimate the time when the climber reaches the top of the cliff, let $y = 500$ and solve for t.

$$500 = 55t + 60 \qquad \text{Substitute 500 for } y.$$
$$440 = 55t \qquad \text{Subtract 60 from each side.}$$
$$8 = t \qquad \text{Divide each side by 55.}$$

Because $t = 8$ corresponds to 8 P.M., at the average rate of change, the climber will reach the top at 8 P.M.

Checkpoint 4

At what time does the climber in Example 4 reach a height of 390 feet? ■

Example 5 Population of San Francisco, California

Between 2000 and 2009, the population of San Francisco, California increased at an average rate of approximately 4292 people per year. In 2000, the population was 776,733. Find a mathematical model that gives the population of San Francisco in terms of the year, and use the model to predict the population in 2012. *(Source: U.S. Census Bureau)*

SOLUTION Let y represent the population of San Francisco and let t represent the calendar year, with $t = 0$ corresponding to 2000. It is convenient to let $t = 0$ correspond to 2000 because you are given the population in 2000. Now, using the rate of change of 4292 people per year, you have

Rate of change: 4292 — 2000 population: 776,733

$$y = mt + b$$

$$y = 4292t + 776{,}733.$$

Using this model, you can predict the 2012 population to be

$$\text{2012 population} = 4292(12) + 776{,}733$$
$$= 828{,}237.$$

The graph is shown in Figure 2.36.

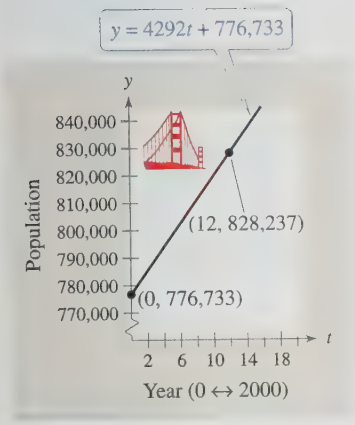


FIGURE 2.36

Checkpoint 5

Use the model in Example 5 to predict the population of San Francisco in 2014.

Example 6 Straight-Line Depreciation

A racing team buys a \$4750 welder that has a useful life of 10 years. The salvage value of the welder at the end of the 10 years is \$400. Write a linear equation that describes the value of the welder throughout its useable life.

SOLUTION Let V represent the value of the welder (in dollars) at the end of year t. Because the value of the welder at the time of purchase is \$4750, you can represent the initial value of the welder by the ordered pair $(t_1, V_1) = (0, 4750)$. You can represent the salvage value by the ordered pair $(t_2, V_2) = (10, 400)$. The slope of the line is

$$m = \frac{400 - 4750}{10 - 0}$$
$$= \frac{-4350}{10}$$
$$= -435$$

which represents the annual depreciation in *dollars per year*. Using the slope-intercept form, you can write the equation of the line.

$$V = -435t + 4750 \qquad \text{Slope-intercept form}$$

The graph of the equation is shown in Figure 2.37.

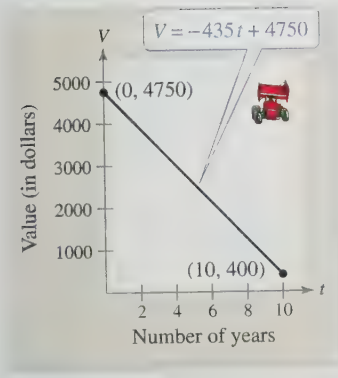


FIGURE 2.37

Checkpoint 6

Write a linear equation to model the value of a new machine that costs \$2300 and is worth \$350 after 10 years.



When you use the *regression* feature of your graphing utility, you may obtain an "*r*-value," which gives a measure of how well the model fits the data (see figure).

```
LinReg
 y=ax+b
 a=1.454545455
 b=125.0545455
 r²=.9465588641
 r=.9729125676
```

The closer the value of $|r|$ is to 1, the better the fit. For the data in Example 7, $r \approx 0.973$, which implies that the model is a good fit. For instructions on how to use the *regression* feature, see Appendix A.

Scatter Plots and Regression Analysis

Another type of linear modeling is a graphical approach that is commonly used in statistics. To find a mathematical model that approximates a set of actual data points, plot the points in a rectangular coordinate system. This collection of points is called a **scatter plot.** You can use the statistical features of a graphing utility to calculate the equation of the best-fitting line for the data in your scatter plot. The statistical method of fitting a line to a collection of points is called **linear regression.** A discussion of linear regression is beyond the scope of this text, but the program in most graphing utilities is easy to use and allows you to analyze linear data that may not be convenient to graph by hand.

Example 7 Individual Tax Returns

The table shows the number of individual income tax returns y (in millions) filed in the United States in the years 2000 to 2009. *(Source: U.S. Internal Revenue Service)*

Year	x	Returns, y
2000	0	125
2001	1	127
2002	2	129
2003	3	130
2004	4	130

Year	x	Returns, y
2005	5	131
2006	6	132
2007	7	135
2008	8	138
2009	9	139

a. Use the *regression* feature of a graphing utility to find a linear model for the data. Let $x = 0$ represent 2000.

b. Use the graphing utility to graph the linear model along with a scatter plot of the data.

c. Use the linear model to predict the number of individual tax returns in 2011.

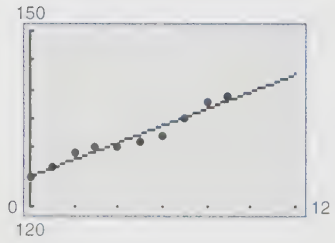


FIGURE 2.38

SOLUTION

a. Enter the data into the graphing utility. Then, using the *regression* feature of the graphing utility, you should obtain a linear model for the data that can be rounded to the following:

$$y = 1.45x + 125.1, \quad 0 \le x \le 9.$$

b. The graph of the equation and the scatter plot are shown in Figure 2.38.

c. Substitute $x = 11$ into the equation found in part (a).

$y = 1.45x + 125.1$ Write equation.

$y = 1.45(11) + 125.1$ Substitute 11 for x.

$y = 141.05$ Simplify.

So, according to the model, about 141,000,000 individual income tax returns will be filed in the United States in 2011.

Checkpoint 7

Repeat Example 7 using only the data for the years 2004–2009. ■

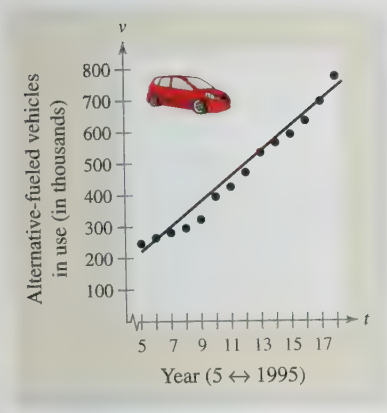


FIGURE 2.39

Example 8 Alternative-Fueled Vehicles

The number of alternative-fueled vehicles v (in thousands) in use in the United States each year from 1995 through 2008 is shown in the table. Construct a scatter plot that represents the data and find a linear model that approximates the data. *(Source: U.S. Energy Information Administration)*

Year	1995	1996	1997	1998	1999	2000	2001
v	246.9	265.0	280.2	295.0	322.3	394.7	425.5

Year	2002	2003	2004	2005	2006	2007	2008
v	471.1	534.0	565.5	592.1	634.6	695.8	775.7

SOLUTION Let $t = 5$ represent 1995. The scatter plot of the data is shown in Figure 2.39. Draw a line on the scatter plot that approximates the data. To find an equation of the line, approximate two points on the line:

$$(t_1, v_1) = (7, 300)$$

and

$$(t_2, v_2) = (17, 700).$$

So, the slope of the line is

$$m \approx \frac{v_2 - v_1}{t_2 - t_1}$$

$$= \frac{700 - 300}{17 - 7}$$

$$= 40.$$

Using the point-slope form, you can determine that an equation of the line is

$$v - 300 = 40(t - 7) \qquad \text{Point-slope form}$$

$$v = 40t + 20. \qquad \text{Slope-intercept form}$$

To check this model, compare the actual v-values with the v-values given by the model.

Checkpoint 8

Repeat Example 8 using only the data for 2003 through 2008.



SUMMARIZE (Section 2.3)

1. State the definition of a mathematical model *(page 183)*. For an example that uses a mathematical model, see Example 1.
2. Describe what it means when two quantities vary directly *(page 184)*. For examples of direct variation, see Examples 2 and 3.
3. Describe the relationship between slope and rate of change *(page 186)*. For examples that involve rates of change, see Examples 4, 5, and 6.
4. State the definition of linear regression *(page 188)*. For examples of linear regression, see Examples 7 and 8.

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SKILLS WARM UP 2.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.1 and 2.2.

In Exercises 1–4, sketch the graph of the line.

1. $y = 2x$

2. $y = \frac{1}{2}x$

3. $y = 2x + 1$

4. $y = \frac{1}{2}x + 1$

In Exercises 5 and 6, find an equation of the line that has the given slope and y-intercept.

5. Slope: 1; y-intercept: $(0, 2)$

6. Slope: $\frac{3}{2}$; y-intercept: $(0, 3)$

In Exercises 7–10, find an equation of the line that passes through the two points.

7. $(1, 3)$ and $(6, 8)$

8. $(0, 4)$ and $(7, 10)$

9. $(1, 5.2)$ and $(5, 4.7)$

10. $(2, 6.5)$ and $(8, 3.6)$

Exercises 2.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Dog Growth The weight of a puppy recorded every two months is shown in the table.

Age (in months)	2	4	6
Weight (in pounds)	21	44	63

Age (in months)	8	10	12
Weight (in pounds)	82	92	101

A linear model that approximates the puppy's weight w (in pounds) in month t is $w = 8.0t + 11$, $2 \le t \le 12$. Plot the actual data with the model. How closely does the model represent the data?

2. Non-Wage Earners The numbers of working-age civilians (in millions) in the United States who were not involved in the labor force from 2000 through 2009 are given by the ordered pairs.

$(2000, 70.0)$ $(2001, 71.4)$ $(2002, 72.7)$
$(2003, 74.7)$ $(2004, 76.0)$ $(2005, 76.8)$
$(2006, 77.4)$ $(2007, 78.7)$ $(2008, 79.5)$
$(2009, 81.7)$

A linear model that approximates the data is

$$y = 1.22t + 70.4, \quad 0 \le t \le 9$$

where y is the number of civilians (in millions) and $t = 0$ represents 2000. Plot the actual data with the model. How closely does the model represent the data? *(Source: U.S. Bureau of Labor Statistics)*

3. Aeropostale Sales The yearly sales y (in millions of dollars) of Aeropostale from 2001 through 2009 are given by the ordered pairs.

$(2001, 304.8)$ $(2002, 550.9)$ $(2003, 734.9)$
$(2004, 964.2)$ $(2005, 1204.3)$ $(2006, 1413.2)$
$(2007, 1590.9)$ $(2008, 1885.5)$ $(2009, 2230.1)$

Use a graphing utility to create a scatter plot of the data. Let $x = 1$ represent 2001. Then use the *regression* feature of the graphing utility to find a best-fitting line for the data. Graph the model and the data together. How closely does the model represent the data? *(Source: Aeropostale, Inc.)*

4. Consumer Price Index For urban consumers of educational and communication materials, the Consumer Price Index y, giving the dollar amount equal to the buying power of $100 in December 1997, is shown for each year from 1998 through 2009 by the ordered pairs.

$(1998, 100.3)$ $(1999, 101.2)$ $(2000, 102.5)$
$(2001, 105.2)$ $(2002, 107.9)$ $(2003, 109.8)$
$(2004, 111.6)$ $(2005, 113.7)$ $(2006, 116.8)$
$(2007, 119.6)$ $(2008, 123.6)$ $(2009, 127.4)$

Use a graphing utility to create a scatter plot of the data. Let $x = 8$ represent 1998. Then use the *regression* feature of the graphing utility to find a best-fitting line for the data. Graph the model and the data together. How closely does the model represent the data? *(Source: U.S. Bureau of Labor Statistics)*

Direct Variation In Exercises 5–10, y is proportional to x. Use the x- and y-values to find a linear model that relates y and x.

5. $x = 8,\ y = 3$ **6.** $x = 5,\ y = 9$

7. $x = 15,\ y = 300$ **8.** $x = 12,\ y = 204$

9. $x = 7,\ y = 3.2$ **10.** $x = 11,\ y = 1.5$

Direct Variation In Exercises 11–14, write a linear model that relates the variables.

11. H varies directly as p; $H = 27$ when $p = 9$

12. s is proportional to t; $s = 32$ when $t = 4$

13. c is proportional to d; $c = 12$ when $d = 20$

14. r varies directly as s; $r = 25$ when $s = 40$

15. Simple Interest The simple interest received from an investment is directly proportional to the amount of the investment. By investing $2000 in a corporate bond, you obtain interest of $120 at the end of 1 year. Find a mathematical model that gives the interest I at the end of 1 year in terms of the amount invested P.

16. Simple Interest The simple interest received from an investment is directly proportional to the amount of the investment. By investing $4000 in a municipal bond, you obtain interest of $280 at the end of 1 year. Find a mathematical model that gives the interest I at the end of 1 year in terms of the amount invested P.

17. Property Tax Your property tax is based on the assessed value of your property. (The assessed value is often lower than the actual value of the property.) A house that has an assessed value of $165,000 has a property tax of $2211.

(a) Find a mathematical model that gives the amount of property tax y in terms of the assessed value x of the property.

(b) Use the model to find the property tax on a house that has an assessed value of $185,000.

18. State Sales Tax An item that sells for $145.99 has a sales tax of $10.22.

(a) Find a mathematical model that gives the amount of sales tax y in terms of the retail price x.

(b) Use the model to find the sales tax on a purchase that has a retail price of $540.50.

19. Miles and Kilometers While driving, your speedometer indicates that your speed is 64 miles per hour, or 103 kilometers per hour.

(a) Use this information to find a mathematical model that relates miles per hour to kilometers per hour.

(b) Use the model to complete the table.

Kilometers per hour	40	60	80	100	120
Miles per hour					

20. Liters and Gallons You are buying gasoline and notice that 14 gallons of gasoline is the same as 53 liters.

(a) Use this information to find a mathematical model that relates gallons to liters.

(b) Use the model to complete the table.

Gallons	5	10	20	25	30
Liters					

Writing a Linear Equation In Exercises 21–26, you are given the 2012 value of a product *and* the rate at which the value is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year. (Let $t = 12$ represent 2012.)

	2012 Value	*Rate*
21.	$2540	$140 increase per year
22.	$156	$4.50 increase per year
23.	$20,400	$2142 decrease per year
24.	$45,000	$2800 decrease per year
25.	$154,000	$10,780 increase per year
26.	$245,000	$5600 increase per year

27. Parachuting After opening the parachute, the descent of a parachutist follows a linear model. At 2:08 P.M., the height of the parachutist is 7000 feet. At 2:10 P.M., the height is 4600 feet.

(a) Write a linear equation that gives the height h of the parachutist in terms of the time t. (Let $t = 0$ represent 2:08 P.M., and let t be measured in seconds.)

(b) Use the equation in part (a) to find the time when the parachutist will reach the ground.

28. Distance Traveled by a Car You are driving at a constant speed. At 4:30 P.M., you drive by a sign that gives the distance to Montgomery, Alabama as 84 miles. At 4:59 P.M., you drive by another sign that gives the distance to Montgomery as 56 miles.

(a) Write a linear equation that gives your distance from Montgomery in terms of time t. (Let $t = 0$ represent 4:30 P.M. and let t be measured in minutes.)

(b) Use the equation in part (a) to find the time when you will reach Montgomery.

29. Straight-Line Depreciation A business purchases a piece of equipment for $875. After 5 years, the equipment will have no value. Write a linear equation giving the value V of the equipment during the 5 years.

30. Straight-Line Depreciation A business purchases a piece of equipment for $25,000. The equipment will be replaced in 10 years, at which time its salvage value is expected to be $2000. Write a linear equation giving the value V of the equipment during the 10 years.

31. Hourly Wages A manufacturer pays its assembly line workers \$11.50 per hour. In addition, workers receive a piecework rate of \$0.75 per unit produced. Write a linear equation for the hourly wages W in terms of the number of units x produced per hour.

32. Sales Commission A salesperson receives a monthly salary of \$2500 plus a commission of 7% of sales. Write a linear equation for the salesperson's monthly wage W in terms of the person's monthly sales S.

33. Hair Growth Rate A beautician cuts a customer's hair to a length of 4 inches. During the next year, the customer's hair grows at a rate of 0.5 inch per month.

(a) Write a linear equation giving the hair length H (in inches) in terms of the number of months t.

(b) The hair keeps growing at this constant rate. Predict the hair length in 1.5 years.

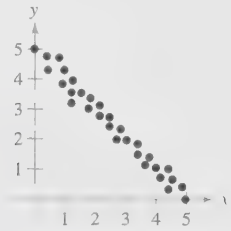
34. Pest Management The cost of implementing an invasive species management system in a forest is related to the area of the forest. It costs \$630 to implement the system in a forest area of 10 acres. It costs \$1070 in a forest area of 18 acres.

(a) Write a linear equation giving the cost of the invasive species management system in terms of the number of acres x of forest.

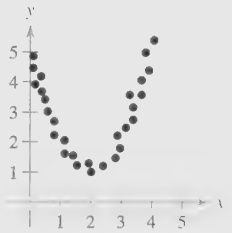
(b) Use the equation in part (a) to find the cost of implementing the system in a forest area of 30 acres.

Finding an Equation of a Line In Exercises 35–40, can the data be approximated by a linear model? If so, sketch the line that best approximates the data. Then find an equation of the line.

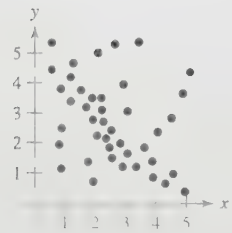
35.



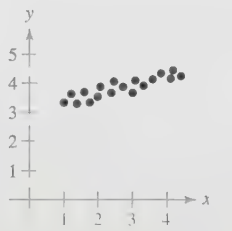
36.



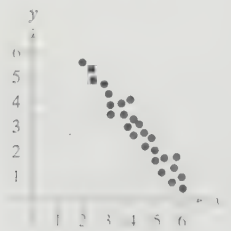
37.



38.



39.



40.



41. Autism The numbers of autistic children C (in thousands) receiving disability services for the years 2001 through 2009 are shown in the table. *(Source: U.S. Department of Education)*

Year	2001	2002	2003	2004	2005
Autistic children, C	98.6	118.8	141.1	166.5	193.8

Year	2006	2007	2008	2009
Autistic children, C	224.6	258.0	292.8	333.0

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 1$ represent 2001. Do the data appear linear?

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) State the slope of the graph of the linear model from part (b) and interpret its meaning in the context of the problem.

(d) Use the linear model to predict the numbers of autistic children receiving disability services in 2011 and 2012. Are your predictions reasonable?

42. Japan The population of Japan is expected to drop by 30% over the next 50 years as the percent of its citizens that are elderly increases. Projections for Japan's population through 2050 are shown in the table. *(Source: National Institute of Population and Social Security Research)*

Year, t	2005	2010	2020
Population, P (in millions)	127.7	127.5	124.1

Year, t	2030	2040	2050
Population, P (in millions)	117.6	109.3	100.6

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 5$ represent 2005. Do the data appear linear?

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Identify the slope of the model from part (b) and interpret its meaning in the context of the problem.

(d) Use the linear model to predict the populations in 2025, 2035, and 2060. Are these predictions reasonable?

43. Yearly Revenue The yearly revenues y (in millions of dollars) for California Pizza Kitchen for the years 2000 through 2009 are given by the ordered pairs. *(Source: California Pizza Kitchen, Inc.)*

(2000, 210.8) (2001, 249.3)
(2002, 306.3) (2003, 359.9)
(2004, 422.5) (2005, 479.6)
(2006, 554.6) (2007, 632.9)
(2008, 677.1) (2009, 664.7)

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 0$ represent 2000.

(b) Use two points on the scatter plot to find an equation of a line that approximates the data.

(c) Use the *regression* feature of the graphing utility to find a linear model for the data. Use this model and the model from part (b) to predict the revenues in 2010 and 2011.

(d) California Pizza Kitchen projected the revenues in 2010 and 2011 to be \$645 million and \$685 million, respectively. How close are these projections to the predictions from the models?

(e) California Pizza Kitchen also expects the yearly revenue to reach \$850 million in 2013, 2014, or 2015. Do the models from parts (b) and (c) support this? Explain your reasoning.

44. Revenue per Share The revenues per share of stock y (in dollars) for California Pizza Kitchen for the years 2000 through 2009 are given by the ordered pairs. *(Source: California Pizza Kitchen, Inc.)*

(2000, 7.86) (2001, 9.02)
(2002, 10.90) (2003, 12.66)
(2004, 14.64) (2005, 16.26)
(2006, 19.16) (2007, 22.32)
(2008, 28.37) (2009, 27.47)

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 0$ represent 2000.

(b) Use two points on the scatter plot to find an equation of a line that approximates the data.

(c) Use the *regression* feature of the graphing utility to find a linear model for the data. Use this model and the model from part (b) to predict the revenues per share in 2010 and 2011.

(d) California Pizza Kitchen projected the revenues per share in 2010 and 2011 to be \$25.80 and \$27.40, respectively. How close are these projections to the predictions from the models?

(e) California Pizza Kitchen also expects the revenue per share to reach \$34.00 in 2013, 2014, or 2015. Do the models from parts (b) and (c) support this? Explain your reasoning.

45. Purchasing Power The value y (in 1982 dollars) of each dollar received by producers in each of the years from 1994 through 2008 in the United States is represented by the ordered pairs. *(Source: U.S. Bureau of Labor Statistics)*

(1994, 0.797) (1995, 0.782)
(1996, 0.762) (1997, 0.759)
(1998, 0.765) (1999, 0.752)
(2000, 0.725) (2001, 0.711)
(2002, 0.720) (2003, 0.698)
(2004, 0.673) (2005, 0.642)
(2006, 0.623) (2007, 0.600)
(2008, 0.565)

(a) Use a spreadsheet software program to generate a scatter plot of the data. Let $t = 4$ represent 1994. Do the data appear linear?

(b) Use the *regression* feature of the spreadsheet software program to find a linear model for the data.

(c) Use the model to predict the value (in 1982 dollars) of 1 dollar received by producers in 2010 and in 2011. Discuss the reliability of your predictions based on your scatter plot and the graph of your linear model for the data.

46. Purchasing Power The value y (in 1982–1984 dollars) of each dollar paid by consumers in each of the years from 1994 through 2008 in the United States is represented by the ordered pairs. *(Source: U.S. Bureau of Labor Statistics)*

(1994, 0.675) (1995, 0.656)
(1996, 0.638) (1997, 0.623)
(1998, 0.613) (1999, 0.600)
(2000, 0.581) (2001, 0.565)
(2002, 0.556) (2003, 0.543)
(2004, 0.529) (2005, 0.512)
(2006, 0.496) (2007, 0.482)
(2008, 0.464)

(a) Use a spreadsheet software program to generate a scatter plot of the data. Let $t = 4$ represent 1994. Do the data appear linear?

(b) Use the *regression* feature of the spreadsheet software program to find a linear model for the data.

(c) Use the model to predict the value (in 1982–1984 dollars) of 1 dollar paid by consumers in 2010 and in 2011. Discuss the reliability of your predictions based on your scatter plot and the graph of your linear model for the data.

47. Health Services The numbers of employees E (in thousands) in the health services industry for the years 2004 through 2009 are shown in the table. *(Source: U.S. Bureau of Labor Statistics)*

Year	2004	2005	2006
Employees, E	14,190	14,536	14,925

Year	2007	2008	2009
Employees, E	15,380	15,798	16,101

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 4$ represent 2004. Do the data appear linear?

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the model to predict the numbers of employees in 2011 and 2013.

(d) Graph the linear model along with the scatter plot of the data. Comparing the data with the model, are the predictions in part (c) most likely high, low, or just about right? Explain your reasoning.

48. Health Care The total yearly health care expenditures E (in billions of dollars) in the United States for the years 2000 through 2009 are shown in the table. *(Source: U.S. Centers for Medicare and Medicaid Services)*

Year	2000	2001	2002	2003	2004
Expenditures, E	1353	1469	1602	1735	1855

Year	2005	2006	2007	2008	2009
Expenditures, E	1983	2113	2240	2339	2486

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 0$ represent 2000. Do the data appear linear?

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the model to predict the health care expenditures in 2011 and 2012.

(d) Graph the linear model along with the scatter plot. The government's projections for 2011 and 2012 are \$2703 billion and \$2850 billion, respectively. Compare these projections with the predictions in part (c). Are the government's projections high, low, or just about right? Explain your reasoning.

49. Think About It Annual data from three years are used to create linear models for the population and the yearly snowfall of Reno, Nevada. Which model is more likely to give better predictions for future years? Discuss the appropriateness of using only three data points in each situation.

50. HOW DO YOU SEE IT? Match the description of the situation with its graph. Find the slope of the line and describe how it is interpreted in the real-life situation. [The graphs are labeled (i), (ii), (iii), and (iv).]

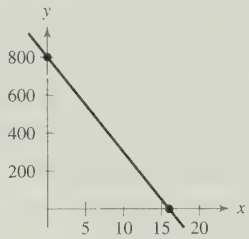
(a) You pay \$0.06 in sales tax for every dollar spent.

(b) A person pays \$40 to reserve a banquet hall, plus \$12 for each hour it is occupied.

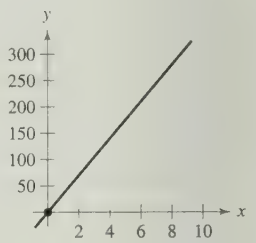
(c) You pay \$800 for a new refrigerator that depreciates \$50 per year.

(d) You pay \$34.99 per day to rent a car.

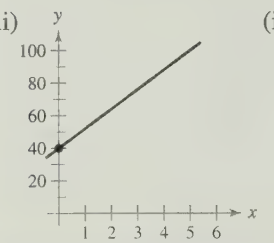
(i)



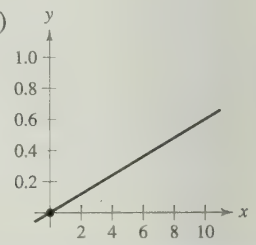
(ii)



(iii)



(iv)



51. Think About It You begin a dance video game with 800 points. You lose 50 points each time you incorrectly complete a dance move. Does this description match graph (i) in Exercise 50? Explain.

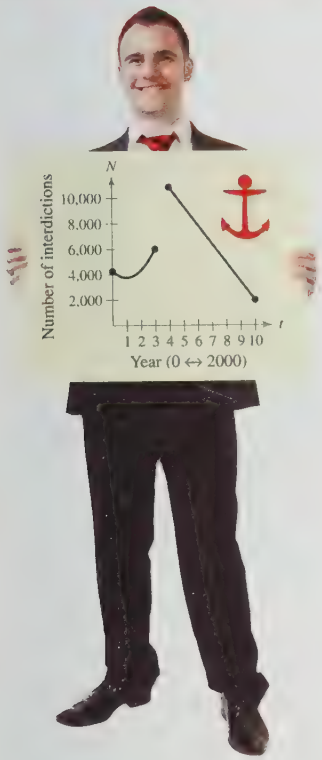
52. Think About It You start with \$12 and save \$40 per week. Does this description match graph (iii) in Exercise 50? Explain.

53. Think About It How can you use the rate of change of y with respect to x to determine whether the data points (1, 10), (2, 20), (3, 30), (4, 40), and (5, 50) are linear?

54. Writing Use your campus library, the Internet, or some other reference source to locate data that you think describes a linear relationship. Use a graphing utility to create a scatter plot of the data and find a linear model that represents the points. Interpret the slope and y-intercept in the context of the data. Write a summary of your findings.

2.4 Functions

- Determine whether a correspondence is a function, and use function notation.
- Find the domain of a function.
- Write a function that relates quantities in an application problem.



In Exercise 86 on page 206, you will use a piecewise-defined function to approximate the numbers of Coast Guard interdictions in various years.

Introduction to Functions and Function Notation

Many everyday phenomena involve two quantities that are related to each other by some rule of correspondence. Here are some examples.

1. The simple interest I earned on \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.
2. The distance d traveled on a bicycle in 2 hours is related to the speed s of the bicycle by the formula $d = 2s$.
3. The area A of a circle is related to its radius r by the formula $A = \pi r^2$.

Not all correspondences between two quantities have simple mathematical formulas. For instance, people commonly match up athletes with jersey numbers and hours of the day with temperatures. In each of these cases, however, there is some rule of correspondence that matches each item from one set with exactly one item from a different set. Such a rule of correspondence is called a **function.**

Definition of a Function

A **function** f from a set A to a set B is a rule of correspondence that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

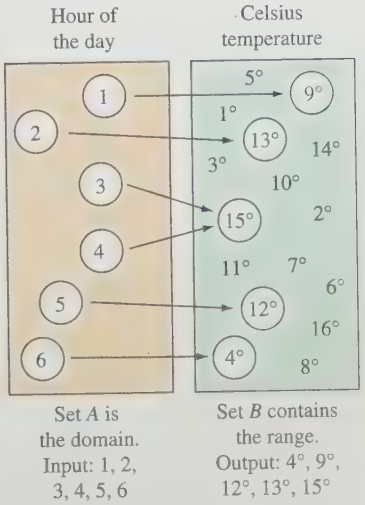
To get a better idea of this definition, look at the function that relates the time of day to the temperature in Figure 2.40. This function can be represented by the following set of ordered pairs.

$$\{(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 4°)\}$$

In each ordered pair, the first coordinate (x-value) is the input and the second coordinate (y-value) is the output. In this example, note the following characteristics of a function.

1. Each element of A (the domain) must be matched with an element of B (the range).
2. Some elements of B may not be matched with any element of A.
3. Two or more elements of A may be matched with the same element of B.
4. An element of A cannot be matched with two different elements of B.

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Function from Set A to Set B

FIGURE 2.40

To determine whether a correspondence is a function, you must decide whether each element of the domain A is matched with exactly one element of the range B. When any element of A is matched with two or more elements of B, the correspondence is not a function. For example, people are not a function of their birthday month because many people are born in any given month.

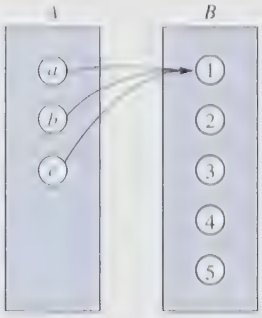
Example 1 **Testing for Functions**

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4, 5\}$. Which of the following sets of ordered pairs or figures represent functions from set A to set B?

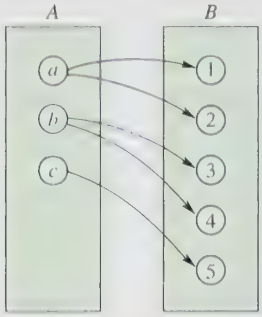
a. $\{(a, 2), (b, 3), (c, 4)\}$

b. $\{(a, 4), (b, 5)\}$

c.



d.



SOLUTION

a. This collection of ordered pairs *does* represent a function from A to B. Each element of A is matched with exactly one element of B.

b. This collection of ordered pairs *does not* represent a function from A to B. Not every element of A is matched with an element of B. The element c of A is not matched with an element of B.

c. This figure *does* represent a function from A to B. It does not matter that each element of A is matched with the same element of B.

d. This figure *does not* represent a function from A to B. The element a of A is matched with *two* elements of B. This is also true of the element b.

Checkpoint 1

Let $A = \{a, b, c, d\}$ and $B = \{1, 3, 5, 7\}$. Decide whether each set of ordered pairs represents a function from set A to set B.

a. $\{(a, 1), (c, 5), (d, 7)\}$

b. $\{(a, 3), (b, 7), (c, 1), (d, 3)\}$

c. $\{(b, 3), (b, 5)\}$

Representing functions by sets of ordered pairs is a common practice in *discrete mathematics*. In algebra, however, it is more common to represent functions by equations or formulas involving two variables. For instance, the equation

$$y = x^2 \qquad y \text{ is a function of } x.$$

represents the variable y as a function of the variable x. In this equation, x is the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Example 2 Testing for Functions Represented by Equations

Which of the equations represent(s) y as a function of x?

a. $x^2 + y = 1$ **b.** $-x + y^2 = 1$

SOLUTION To determine whether y is a function of x, try to solve for y in terms of x.

a. Solving for y yields

$x^2 + y = 1$ Write original equation.

$y = 1 - x^2.$ Solve for y.

To each value of x there corresponds exactly one value of y. So, y *is* a function of x.

b. Solving for y yields

$x + y^2 = 1$ Write original equation.

$y^2 = 1 + x$ Add x to each side.

$y = \pm\sqrt{1 + x}.$ Solve for y.

The $\pm$ indicates that to a given value of x there correspond two values of y. So, y *is not* a function of x.

Checkpoint 2

Determine whether each equation represents y as a function of x.

a. $y - 2 = x^2$ **b.** $x^2 + y^2 = 25$

STUDY TIP

Make sure you understand how function notation is written and how it is read. Frequently, $f(x)$ is misinterpreted as "f times x" rather than "f of x."

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the **function notation** shown below.

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as the **value of f at x** or simply **f of x.** The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *value* of the function at x. For instance, the function given by

$$f(x) = 3 - 2x$$

has *function values* denoted by $f(-1)$, $f(0)$, $f(2)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $f(-1) = 3 - 2(-1)$

$= 3 + 2$

$= 5.$

For $x = 0$, $f(0) = 3 - 2(0)$

$= 3 - 0$

$= 3.$

For $x = 2$, $f(2) = 3 - 2(2)$

$= 3 - 4$

$= 1.$

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable in a function is simply that of a "placeholder." Consequently, the function above could be described by the form

$$f(\quad) = (\quad)^2 - 4(\quad) + 7.$$

Example 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Evaluate $g(x)$ at each specified value and simplify.

a. $g(2)$ **b.** $g(t)$ **c.** $g(x + 2)$

SOLUTION

a. Replace x with 2 in $g(x) = -x^2 + 4x + 1$.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replace x with t.

$$g(t) = -(t)^2 + 4(t) + 1 = -t^2 + 4t + 1$$

c. Replace x with $x + 2$.

$$\begin{aligned} g(x + 2) &= -(x + 2)^2 + 4(x + 2) + 1 \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 \\ &= -x^2 + 5 \end{aligned}$$

STUDY TIP

In Example 3(c), note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

Checkpoint 3

Let $h(x) = 2x^2 + x - 4$. Find $h(-1)$. ■

A function defined by two or more equations over a specified domain is called a **piecewise-defined function.**

Example 4 A Piecewise-Defined Function

Evaluate the function when $x = -1, 0$, and 1.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

SOLUTION For each x-value, only one part of the piecewise function is used. Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 = 2.$$

For $x = 0$, use $f(x) = x - 1$ to obtain $f(0) = (0) - 1 = -1$. For $x = 1$, use $f(x) = x - 1$ to obtain $f(1) = (1) - 1 = 0$. The graph of f is shown in Figure 2.41.

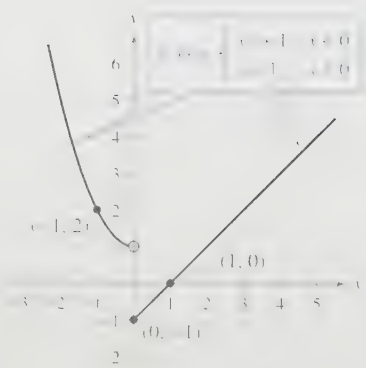


FIGURE 2.41

Checkpoint 4

Evaluate the function in Example 4 when $x = -3$ and 3. ■

Finding the Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function given by

$$f(x) = \frac{1}{x^2 - 4}$$

Domain excludes x-values that result in division by zero.

has an implied domain that consists of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain results from the restrictions needed to avoid even roots of negative numbers. For example, the function given by

$$f(x) = \sqrt{x}$$

Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* result in the even root of a negative number.

Example 5 Finding the Domain of a Function

Find the domain of each function.

a. f: $\{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = \dfrac{1}{x + 5}$

c. Volume of a sphere: $V = \dfrac{4}{3}\pi r^3$

d. $h(x) = \sqrt{4 - x^2}$

e. $r(x) = \sqrt[3]{x + 3}$

SOLUTION

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}$$

b. Excluding x-values that yield zero in the denominator, the domain of g is the set of all real numbers x such that $x \neq -5$.

c. Because this function represents the volume of a sphere, the values of the radius r must be positive. So, the domain is the set of all real numbers r such that $r > 0$.

d. This function is defined only for x-values for which

$$4 - x^2 \geq 0.$$

Using the methods described in Section 1.7, you can conclude that $-2 \leq x \leq 2$. So, the domain of h is the interval $[-2, 2]$.

e. Because the cube root of any real number is defined, the domain of r is the set of all real numbers, or $(-\infty, \infty)$.

✓Checkpoint 5

Find the domain of the function $f(x) = 6 - x^3$.

In Example 5(c), note that the domain of a function may be implied by the physical context. For instance, from the equation $V = \frac{4}{3}\pi r^3$, you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative or zero radius.

Applications

Example 6 The Dimensions of a Container

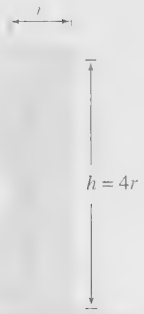


FIGURE 2.42

You are working with a cylindrical beaker in a chemistry lab experiment. The height of the beaker is 4 times the radius, as shown in Figure 2.42.

a. Write the volume of the beaker as a function of the radius r.

b. Write the volume of the beaker as a function of the height h.

SOLUTION

a. $V = \pi r^2 h$ Write volume formula.

$= \pi r^2(4r)$ Substitute $4r$ for h.

$= 4\pi r^3$ V is a function of r.

b. $V = \pi\left(\frac{h}{4}\right)^2 h$ Substitute $\frac{h}{4}$ for r.

$= \frac{\pi h^3}{16}$ V is a function of h.

Checkpoint 6

In Example 6, write the volume of the beaker as a function of the height h when the radius is twice the height.

Example 7 The Path of a Baseball

A baseball is hit at a point 3 feet above home plate at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$y = -0.0032x^2 + x + 3$$

where y and x are measured in feet. Will the baseball clear a 10-foot fence located 300 feet from home plate?

SOLUTION When $x = 300$, the height of the baseball is given by

$$y = -0.0032(300)^2 + 300 + 3 = 15 \text{ feet.}$$

Yes, the ball will clear the fence, as shown in Figure 2.43.

STUDY TIP

Notice that in Figure 2.43, the baseball is not at the point (0, 0) when it is initially hit. This is because the baseball is hit at a point 3 feet above home plate. So, it is hit at the point (0, 3).

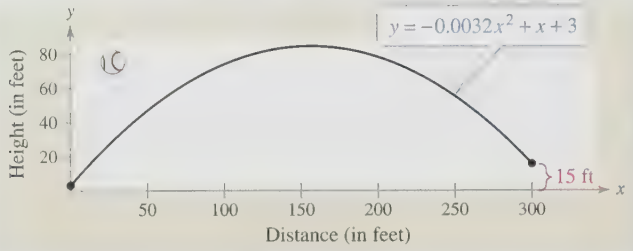


FIGURE 2.43

Checkpoint 7

In Example 7, will the baseball clear a 35-foot fence located 280 feet from home plate?

Example 8 Houses Sold

The number H (in thousands) of new privately owned one-family houses sold in the United States increased in a quadratic pattern from 1998 through 2004. Then, in 2005, the pattern changed from a quadratic to a linear pattern (see Figure 2.44). These two patterns can be approximated by the function

$$H = \begin{cases} 15.036t^2 - 278.68t + 2159.3, & 8 \le t \le 14 \\ -238.20t + 4843.4, & 15 \le t \le 19 \end{cases}$$

with $t = 8$ corresponding to 1998. Use this function to approximate the total number of new privately owned one-family houses sold between 1998 and 2009. *(Source: U.S. Census Bureau and U.S. Department of Housing and Urban Development)*



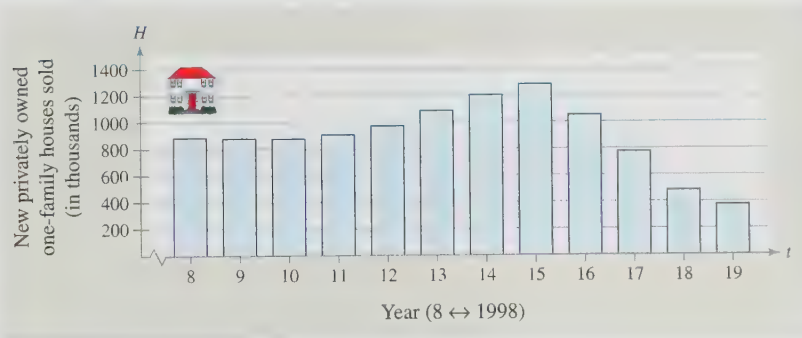


FIGURE 2.44

SOLUTION From 1998 through 2004, use the equation $H = 15.036t^2 - 278.68t + 2159.3$ to approximate the number of houses sold, as shown in the table.

t	8	9	10	11	12	13	14
H	892.2	869.1	876.1	913.2	980.3	1077.5	1204.8

From 2005 through 2009, use the equation $H = -238.20t + 4843.4$ to approximate the number of houses sold, as shown in the table.

t	15	16	17	18	19
H	1270.4	1032.2	794.0	555.8	317.6

To approximate the total number of houses sold from 1998 through 2009, you can add the amounts for the 12 years to obtain the sum 10,783.2. Because H is measured in thousands, you can conclude that the total number of new privately owned one-family houses sold between 1998 and 2009 was approximately 10,783,200.

Checkpoint 8

The median sales price P (in thousands of dollars) of new privately owned one-family houses sold in the United States from 1999 through 2009 can be modeled by the function

$$P = \begin{cases} 8.66t + 82.3, & 9 \le t \le 13 \\ -4.680t^2 + 153.12t - 1004.5, & 14 \le t \le 19 \end{cases}$$

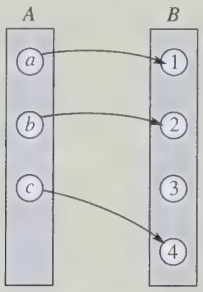
with $t = 9$ corresponding to 1999. Use the function to approximate the median sales price for each year from 1999 through 2009. *(Source: U.S. Census Bureau and U.S. Department of Housing and Urban Development)*

Stephen Coburn/www.shutterstock.com

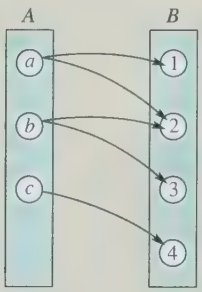
Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

For instance, let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$.



The set of ordered pairs $\{(a, 1), (b, 2), (c, 4)\}$ *is a function.*



The set of ordered pairs $\{(a, 1), (a, 2), (b, 2), (b, 3), (c, 4)\}$ *is not a function.*

Function Notation: $y = f(x)$

f is the **name** of the function.

y is the **dependent variable.**

x is the **independent variable.**

$f(x)$ is the **value of the function at x.**

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, then f is said to be **defined** at x. If x is not in the domain of f, then f is said to be **undefined** at x.

Range: The **range** of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, then the **implied domain** consists of all real numbers for which the expression is defined.



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SUMMARIZE (Section 2.4)

1. State the definition of a function *(page 195)*. For examples of determining functions, see Examples 1 and 2.
2. Describe function notation *(pages 197 and 198)*. For examples of using function notation, see Examples 3 and 4.
3. State the definition of an implied domain of a function *(page 199)*. For an example of finding the domain of a function, see Example 5.
4. Describe a real-life example of how a function can be used to analyze the path of an object *(page 200, Example 7)*.

SKILLS WARM UP 2.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 1.2, and 1.7.

In Exercises 1–4, simplify the expression.

1. $2(-3)^3 + 4(-3) - 7$

2. $4(-1)^2 - 5(-1) + 4$

3. $(x + 1)^2 + 3(x + 1) - 4 - (x^2 + 3x - 4)$

4. $(x - 2)^2 - 4(x - 2) - (x^2 - 4)$

In Exercises 5 and 6, solve for y in terms of x.

5. $2x + 5y - 7 = 0$

6. $y^2 = x^2$

In Exercises 7–10, solve the inequality.

7. $x^2 - 4 \geq 0$

8. $9 - x^2 \geq 0$

9. $x^2 + 2x + 1 \geq 0$

10. $x^2 - 3x + 2 \geq 0$

Exercises 2.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

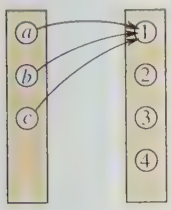
Testing for Functions In Exercises 1–4, decide whether the set of figures represents a function from A to B.

$A = \{a, b, c\}$

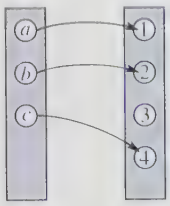
$B = \{1, 2, 3, 4\}$

Give a reason for your answer. *See Example 1.*

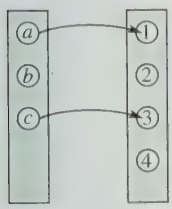
1.



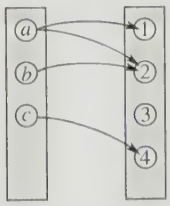
2.



3.



4.



Testing for Functions In Exercises 5–8, decide whether the set of ordered pairs represents a function from A to B.

$A = \{0, 1, 2, 3\}$

$B = \{-2, -1, 0, 1, 2\}$

Give a reason for your answer. *See Example 1.*

5. $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$

6. $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$

7. $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$

8. $\{(0, 2), (3, 0), (1, 1)\}$

Testing for Functions In Exercises 9–14, decide whether the set of ordered pairs represents a function from A to B.

$A = \{a, b, c\}$

$B = \{0, 1, 2, 3\}$

Give a reason for your answer. *See Example 1.*

9. $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$

10. $\{(a, 3), (c, 1)\}$

11. $\{(0, a), (2, c), (3, b)\}$

12. $\{(a, 0), (c, 1), (b, 3)\}$

13. $\{(a, 1), (b, 2), (c, 3)\}$

14. $\{(c, 0), (b, 0), (a, 3)\}$

Testing for Functions Represented by Equations In Exercises 15–24, determine whether the equation represents y as a function of x. *See Example 2.*

15. $x^2 + y^2 = 4$

16. $x = y^2$

17. $x^2 + y = 4$

18. $y = \sqrt{x + 5}$

19. $2x + 3y = 4$

20. $5x - 7 = 3y$

21. $y^2 = x^2 - 1$

22. $x + y^2 = 4$

23. $x^2y - x^2 + 4y = 0$

24. $xy - y - x - 2 = 0$

Evaluating a Function In Exercises 25–28, fill in the blank and simplify. *See Example 3.*

25. $f(x) = 6 - 4x$

(a) $f(3) = 6 - 4(\quad)$

(b) $f(-7) = 6 - 4(\quad)$

(c) $f(t) = 6 - 4(\quad)$

(d) $f(c + 1) = 6 - 4(\quad)$

26. $f(s) = \dfrac{1}{s+1}$

(a) $f(4) = \dfrac{1}{(\quad)+1}$ (b) $f(0) = \dfrac{1}{(\quad)+1}$

(c) $f(4x) = \dfrac{1}{(\quad)+1}$

(d) $f(x+1) = \dfrac{1}{(\quad)+1}$

27. $g(x) = \dfrac{1}{x^2 - 2x}$

(a) $g(1) = \dfrac{1}{(\quad)^2 - 2(\quad)}$

(b) $g(-3) = \dfrac{1}{(\quad)^2 - 2(\quad)}$

(c) $g(t) = \dfrac{1}{(\quad)^2 - 2(\quad)}$

(d) $g(t+1) = \dfrac{1}{(\quad)^2 - 2(\quad)}$

28. $f(t) = \sqrt{25 - t^2}$

(a) $f(3) = \sqrt{25 - (\quad)^2}$

(b) $f(5) = \sqrt{25 - (\quad)^2}$

(c) $f(x+5) = \sqrt{25 - (\quad)^2}$

(d) $f(2x) = \sqrt{25 - (\quad)^2}$

Evaluating a Function In Exercises 29–40, evaluate the function at each specified value of the independent variable and simplify. *See Example 3.*

29. $f(x) = 2x - 3$

(a) $f(1)$

(b) $f(-3)$

(c) $f(x-1)$

(d) $f\left(\frac{1}{4}\right)$

30. $g(y) = 7 - 3y$

(a) $g(0)$

(b) $g\left(\frac{7}{3}\right)$

(c) $g(s)$

(d) $g(s+2)$

31. $h(t) = t^2 - 2t$

(a) $h(2)$

(b) $h(-1)$

(c) $h(x+2)$

(d) $h(1.5)$

32. $k(b) = 2b^2 + 7b + 3$

(a) $k(0)$

(b) $k\left(-\frac{1}{2}\right)$

(c) $k(a)$

(d) $k(x+2)$

33. $V(r) = \dfrac{4}{3}\pi r^3$

(a) $V(3)$

(b) $V(0)$

(c) $V\left(\frac{3}{2}\right)$

(d) $V(2r)$

34. $A(s) = \dfrac{\sqrt{3}s^2}{4}$

(a) $A(1)$

(b) $A(0)$

(c) $A(2x)$

(d) $A(3)$

35. $f(y) = 3 - \sqrt{y}$

(a) $f(4)$

(b) $f(100)$

(c) $f(4x^2)$

(d) $f(0.25)$

36. $f(x) = \sqrt{x+3} - 2$

(a) $f(-3)$

(b) $f(1)$

(c) $f(x-3)$

(d) $f(x+4)$

37. $c(x) = \dfrac{1}{x^2 - 16}$

(a) $c(4)$

(b) $c(0)$

(c) $c(y+2)$

(d) $c(y-2)$

38. $q(t) = \dfrac{2t^2 + 3}{t^2}$

(a) $q(2)$

(b) $q(0)$

(c) $q(x)$

(d) $q(-x)$

39. $f(x) = \dfrac{|x|}{x}$

(a) $f(2)$

(b) $f(-2)$

(c) $f(x^2)$

(d) $f(x-1)$

40. $f(x) = |x| + 4$

(a) $f(2)$

(b) $f(-2)$

(c) $f(x^2)$

(d) $f(x+2)$

A Piecewise-Defined Function In Exercises 41–44, evaluate the function at each specified value of the independent variable and simplify. *See Example 4.*

41. $f(x) = \begin{cases} 3x - 1, & x < 0 \\ 2x + 3, & x \geq 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$ (c) $f(-2)$ (d) $f(2)$

42. $g(x) = \begin{cases} \frac{1}{4}x + 2, & x \leq 4 \\ 8 - x, & x > 4 \end{cases}$

(a) $g(-2)$ (b) $g(0)$ (c) $g(4)$ (d) $g(6)$

43. $h(t) = \begin{cases} t^2 - 2, & t < 2 \\ 3, & t \geq 2 \end{cases}$

(a) $h(-2)$ (b) $h(1)$ (c) $h(2)$ (d) $h(5)$

44. $f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ 2x - 3, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f\left(\frac{3}{2}\right)$ (d) $f(0)$

Values of x for Which the Function Value is Zero In Exercises 45–52, find all real values of x for which $f(x) = 0$.

45. $f(x) = 15 - 3x$

46. $f(x) = \dfrac{2x - 5}{3}$

47. $f(x) = x^2 - 9$

48. $f(x) = 2x^2 - 11x + 5$

49. $f(x) = x^3 - x$

50. $f(x) = x^3 - 3x^2 - 4x + 12$

51. $f(x) = \dfrac{3}{x-1} + \dfrac{4}{x-2}$

52. $f(x) = 3 + \dfrac{2}{x-1}$

Finding the Domain of a Function In Exercises 53–70, find the domain of the function. *See Example 5.*

53. f: {(0, 1), (3, 4), (5, 7), (8, 11), (9, 12)}

54. g: {(10, −4), (20, 1), (30, 6), (40, 9), (50, 13)}

55. h: {(−2, 7.4), (−1, 6.5), (0, 4.8), (1, 4.1), (2, 3.3)}

56. s: {(−5.7, 9.1), (−3.9, 5.1), (−2.9, 3.2), (−1.5, 1.1)}

57. $g(x) = 1 - 2x^2$

58. $f(x) = 5x^2 + 2x - 1$

59. $h(t) = \dfrac{4}{t}$

60. $s(y) = \dfrac{3y}{y + 5}$

61. $g(y) = \sqrt[3]{y - 10}$

62. $f(t) = \sqrt[3]{t + 4}$

63. $f(x) = \sqrt[4]{1 - x^2}$

64. $g(x) = \sqrt{x + 1}$

65. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$

66. $h(x) = \dfrac{10}{x^2 - 2x}$

67. $f(x) = \dfrac{\sqrt{x + 1}}{x - 2}$

68. $f(s) = \dfrac{\sqrt{s - 1}}{s - 4}$

69. $f(x) = \dfrac{x - 4}{\sqrt{x}}$

70. $f(x) = \dfrac{x - 5}{\sqrt{x^2 - 9}}$

Writing a Function as a Set of Ordered Pairs In Exercises 71–74, the domain of f is the set

$A = \{-2, -1, 0, 1, 2\}$.

Write the function as a set of ordered pairs.

71. $f(x) = x^2$

72. $f(x) = \dfrac{2x}{x^2 + 1}$

73. $f(x) = \sqrt{x + 2}$

74. $f(x) = |x + 1|$

75. Think About It Consider

$f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt[3]{x - 2}$.

Why are the domains of f and g different?

76. Think About It A student says that the domain of

$$f(x) = \frac{\sqrt{x + 1}}{x - 3}$$

is all real numbers except $x = 3$. Is the student correct? Explain.

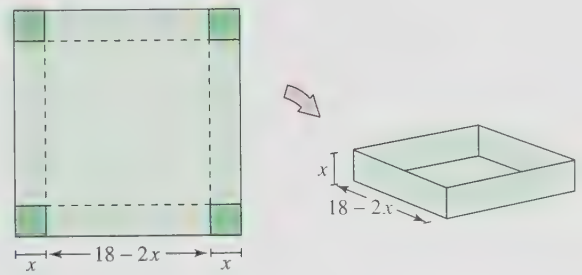
77. Think About It Set A contains n elements and set B contains m elements. How many ordered pairs are there in a set that represents a function f from set A to set B?

78. Discovery Use a graphing utility to graph the functions

$f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x^2 - 4}$.

What is the domain of each function? Do the domains of these two functions overlap? If so, for what values?

79. Volume of a Box An open box is to be made from a square piece of material 18 inches on a side by cutting equal squares from the corners and turning up the sides (see figure).

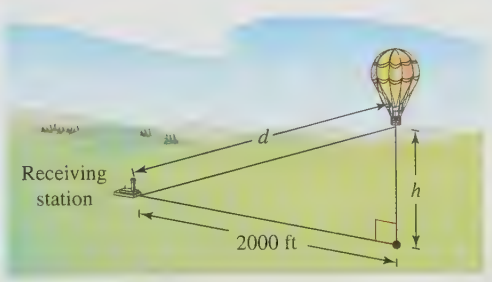


(a) Write the volume V of the box as a function of its height x.

(b) What is the domain of the function?

(c) Determine the volume of a box with a height of 4 inches.

80. Height of a Balloon A balloon carrying a transmitter ascends vertically from a point 2000 feet from the receiving station (see figure). Let d be the distance between the balloon and the receiving station. Write the height h of the balloon as a function of d. What is the domain of this function?



81. Path of a Ball A player throws a baseball toward a teammate. The height y (in feet) of the baseball is given by the function

$$y = -\frac{1}{10}x^2 + 3x + 6$$

where x is the horizontal distance (in feet) from where the ball is thrown. The teammate is 30 feet away and can reach to a maximum height of 7 feet. Can the teammate catch the ball without backing up? Explain.

82. Path of a Salmon Part of the life cycle of a salmon is migration for reproduction. Salmon are anadromous fish. This means that they swim from the ocean to fresh water streams to lay their eggs. During migration, salmon must jump waterfalls to reach their destination. A migrating salmon initiates a jump 4 feet from a waterfall that is 3 feet high. Its path through the air is given by the function

$$h = -0.42x^2 + 2.52x$$

where h is the height (in feet) and x is the horizontal distance (in feet) from where the salmon leaves the water. Will the salmon clear the waterfall?

83. Farmland The number of acres A (in millions) of farmland in the United States each year from 1993 through 2009 can be approximated by the function

$$A = \begin{cases} -3.4t + 979, & 3 \le t \le 11 \\ 0.28t^2 - 11.9t + 1043, & 12 \le t \le 19 \end{cases}$$

where t represents the year, with $t = 3$ corresponding to 1993. Use the function to find the numbers of acres of farmland in 2000 and 2009. *(Source: U.S. Department of Agriculture)*

84. Cost, Revenue, and Profit A company produces a product for which the variable cost is \$11.75 per unit and the fixed costs are \$112,000. The product sells for \$21.95 per unit. Let x be the number of units produced and sold.

(a) Add the variable cost and the fixed costs to write the total cost C as a function of the number of units x.

(b) Write the revenue R as a function of the number of units x.

(c) Use the formula $P = R - C$ to write the profit P as a function of the number of units x.

85. Average Cost A manufacturer determines that the variable cost for a new product is \$2.05 per unit and the fixed costs are \$57,000. All units produced are sold for \$5.89 per unit. Let x be the number of units.

(a) Write the total cost C as a function of the number of units x.

(b) Write the average cost per unit $\overline{C} = C/x$ as a function of x.

(c) Complete the table.

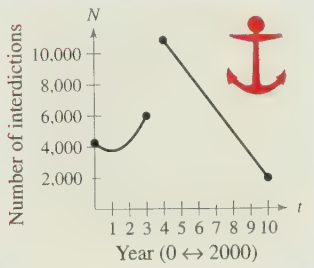
x	100	1000	10,000	100,000
$\overline{C}$				

(d) Write a paragraph analyzing the data in the table. What do you observe about the average cost per unit as x gets larger?

86. Migrant Interdictions The U.S. Coast Guard interdicts undocumented migrants at sea to prevent them from entering the United States, its territories, and its possessions. The number N of migrants interdicted each year from 2000 through 2010 can be approximated by the function

$$N = \begin{cases} 556.50t^2 - 1096.5t + 4280, & 0 \le t \le 3 \\ -1481.9t + 16{,}793, & 4 \le t \le 10 \end{cases}$$

where t represents the year, with $t = 0$ corresponding to 2000 (see figure). *(Source: U.S. Coast Guard)*



(a) Use the function to approximate the numbers of migrants interdicted in 2003, 2004, and 2010.

(b) Use the function to approximate the total number of migrants interdicted from 2000 through 2010.

87. Profits Johnson & Johnson manufactures and sells health care products. The net profits P (in billions of dollars) of Johnson & Johnson for the years 2000 through 2009 are shown in the table. *(Source: Johnson & Johnson)*

Year	2000	2001	2002	2003	2004
Profit, P	4.80	5.89	6.81	8.10	9.30

Year	2005	2006	2007	2008	2009
Profit, P	10.55	11.13	12.09	12.95	12.91

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a linear model and a quadratic model for the data.

(c) Use each model to approximate the net profits for the years 2000 through 2009. Compare the values generated by each model with the actual values shown in the table. Which model is a better fit? Explain.

88. Dividends The dividends D (in dollars) per share declared by Coca-Cola for the years 1995 through 2010 are shown in the table. *(Source: Coca-Cola Company)*

Year	Dividend, D	Year	Dividend, D
1995	0.44	2003	0.88
1996	0.50	2004	1.00
1997	0.56	2005	1.12
1998	0.60	2006	1.24
1999	0.64	2007	1.36
2000	0.68	2008	1.52
2001	0.72	2009	1.64
2002	0.80	2010	1.76

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use the *regression* feature of the graphing utility to find a linear model and a quadratic model for the data.

(c) Use the graphing utility to graph each model from part (b) with the data.

(d) Which model do you think better fits the data? Explain your reasoning.

(e) Use the model you selected in part (d) to predict the dividends per share in 2011 and 2015. Coca-Cola predicts the dividends per share to be about \$1.88 in 2011 and to reach \$2.48 by one of the years from 2013 to 2015. Do your predictions support those of Coca-Cola? Explain.

89. Ripples in a Pond A stone is thrown into the middle of a calm pond, causing ripples to form in concentric circles. The radius r of the outermost ripple increases at the rate of 0.75 foot per second.

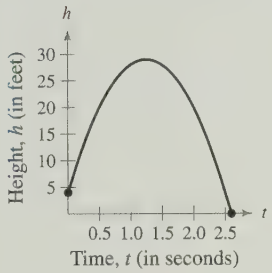
(a) Write a function for the radius r of the circle formed by the outermost ripple in terms of time t.

(b) Write a function for the area A enclosed by the outermost ripple. Complete the table.

Time, t	1	2	3	4	5
Radius, r (in feet)					
Area, A (in square feet)					

(c) Compare the ratios $A(2)/A(1)$ and $A(4)/A(2)$. What do you observe? Based on your observation, predict the area when $t = 8$. Verify by checking $t = 8$ in the area function.

90. HOW DO YOU SEE IT? You pop the cork from a bottle at an outdoor event. The graph represents the height h of the cork after t seconds.



(a) Explain why h is a function of t.

(b) Approximate the height of the cork after 0.5 second and after 1.25 seconds.

(c) Approximate the domain of h.

(d) Is t a function of h? Explain.

Think About It In Exercises 91 and 92, determine whether the statements use the word *function* in ways that are *mathematically* correct. Explain your reasoning.

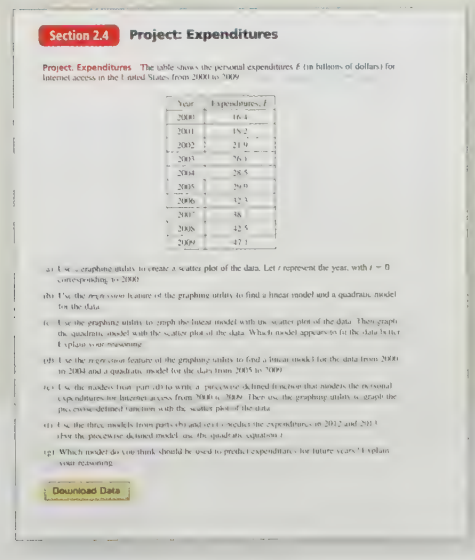
91. (a) The sales tax on a purchased item is a function of the selling price.

(b) Intelligence is a function of education.

92. (a) The market value of a home is a function of its size.

(b) The speed at which a free-falling baseball strikes the ground is a function of its initial height.

93. Project: Expenditures For a project involving the personal expenditures for Internet access in the United States from 2000 to 2009, visit this text's website at *www.cengagebrain.com.* *(Source: Bureau of Economic Analysis)*



QUIZ YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(-3, 2), (4, -5)$ **2.** $(1.3, -4.5), (-3.7, 0.7)$ **3.** $(4, -2), \left(-1, -\frac{5}{2}\right)$

In Exercises 4–6, sketch the graph of the equation. Identify any intercepts and test for symmetry.

4. $y = 9 - x^2$ **5.** $y = x\sqrt{x + 4}$ **6.** $y = |x - 3|$

7. The point $(0, 3)$ lies on a circle whose center is at $(4, 5)$. Write the standard form of the equation of this circle.

8. Write the equation of the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ in standard form. Then sketch the circle.

In Exercises 9–12, find an equation of the line that passes through the point and has the indicated slope. Then sketch the graph of the line.

	Point	*Slope*
9.	$(3, 5)$	$m = \frac{2}{3}$
10.	$(-2, 4)$	$m = 0$
11.	$(2, -3)$	m is undefined.
12.	$(-2, -5)$	$m = -2$

13. A city had a population of 233,134 in 2010 and 244,288 in 2012. Assuming the population followed a linear pattern, predict the population of the city in 2015.

14. Determine whether the equation $x^2 + y = 3$ represents y as a function of x.

In Exercises 15 and 16, let $A = \{6, 8, 10\}$ and $B = \{1, 2, 4, 16\}$. Decide whether the set of ordered pairs represents a function from A to B.

15. $\{(6, 1), (8, 1), (10, 2)\}$ **16.** $\{(6, 2), (10, 4)\}$

In Exercises 17 and 18, evaluate the function at each specified value of the independent variable and simplify.

17. $f(x) = 3(x + 2) - 4$

(a) $f(0)$ (b) $f(-3)$

18. $g(t) = 2t^3 - t^2$

(a) $g(1)$ (b) $g(-2)$

In Exercises 19 and 20, find the domain of the function.

19. $h(x) = \sqrt{x - 4}$ **20.** $f(x) = \dfrac{x}{x + 2}$

In Exercises 21 and 22, use the table, which shows the federal cost C (in millions of dollars) of the National School Lunch Program for each year from 2004 through 2009. *(Source: U.S. Department of Agriculture)*

Year	Cost, C (in millions of dollars)
2004	6663
2005	7055
2006	7389
2007	7707
2008	8265
2009	8873

Table for 21 and 22



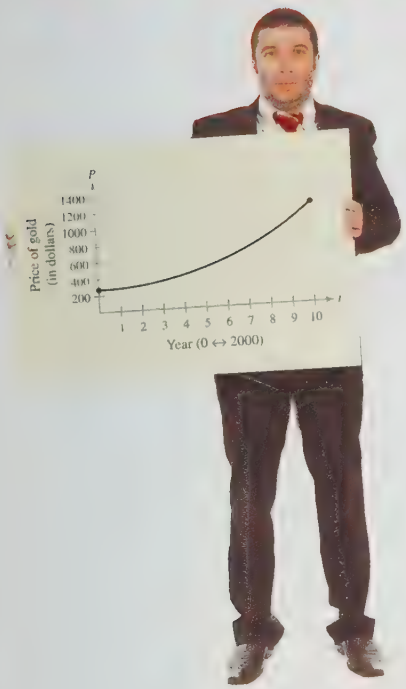
21. Use a graphing utility to create a scatter plot of the data and use the regression feature to find a linear model and a quadratic model for the data. Let $t = 4$ represent 2004.

22. Use each model you found in Exercise 21 to predict the federal costs of the national school lunch program in 2010 and 2011.

23. Write the area A of a circle as a function of its circumference C.

2.5 Graphs of Functions

- Find the domain and range of a function using its graph.
- Identify the graph of a function using the Vertical Line Test.
- Describe the increasing, decreasing, and constant behavior of a function.
- Find the relative minima and relative maxima of the graph of a function.
- Sketch the graph of a step function.
- Classify a function as even or odd.
- Identify six common graphs and use them to sketch the graphs of functions.



In Exercise 79 on page 218, you will use a graph to determine in which years the price of an ounce of gold was increasing or decreasing.

The Graph of a Function

In Section 2.4, you studied functions from an algebraic point of view. In this section, you will study functions from a graphical perspective.

The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember that x is the directed distance from the y-axis and $f(x)$ is the directed distance from the x-axis, as shown in Figure 2.45. If the graph of a function has an x-intercept at $(a, 0)$, then a is a **zero** of the function. In other words, the zeros of a function are the values of x for which $f(x) = 0$. For instance, the function given by $f(x) = x^2 - 4$ has two zeros: -2 and 2.

The **range** of a function (the set of values assumed by the dependent variable) is often easier to determine graphically than algebraically.

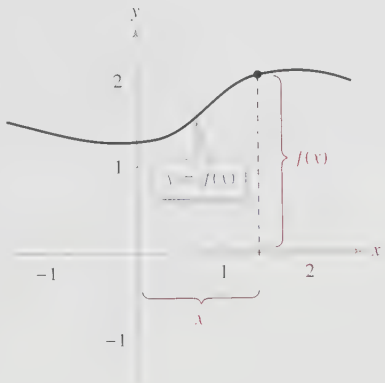


FIGURE 2.45

Example 1 Finding the Domain and Range of a Function

Use the graph of the function f, shown in Figure 2.46, to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

SOLUTION

a. Because the graph does not extend beyond $x = -1$ (on the left) and $x = 4$ (on the right), the domain of f is all x in the interval $[-1, 4]$.

b. Because $(-1, -5)$ is a point on the graph of f, it follows that

$$f(-1) = -5.$$

Similarly, because $(2, 4)$ is a point on the graph of f, it follows that

$$f(2) = 4.$$

c. Because the graph does not extend below $f(-1) = -5$ or above $f(2) = 4$, the range of f is the interval $[-5, 4]$.

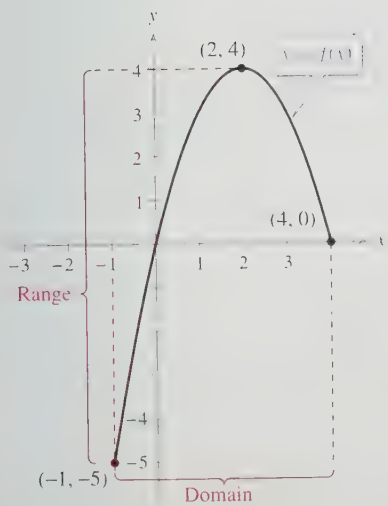


FIGURE 2.46

Checkpoint 1

Use the graph of $f(x) = x^2 - 3$ to find the domain and range of f.

Blaj Gabriel/www.shutterstock.com

Vertical Line Test

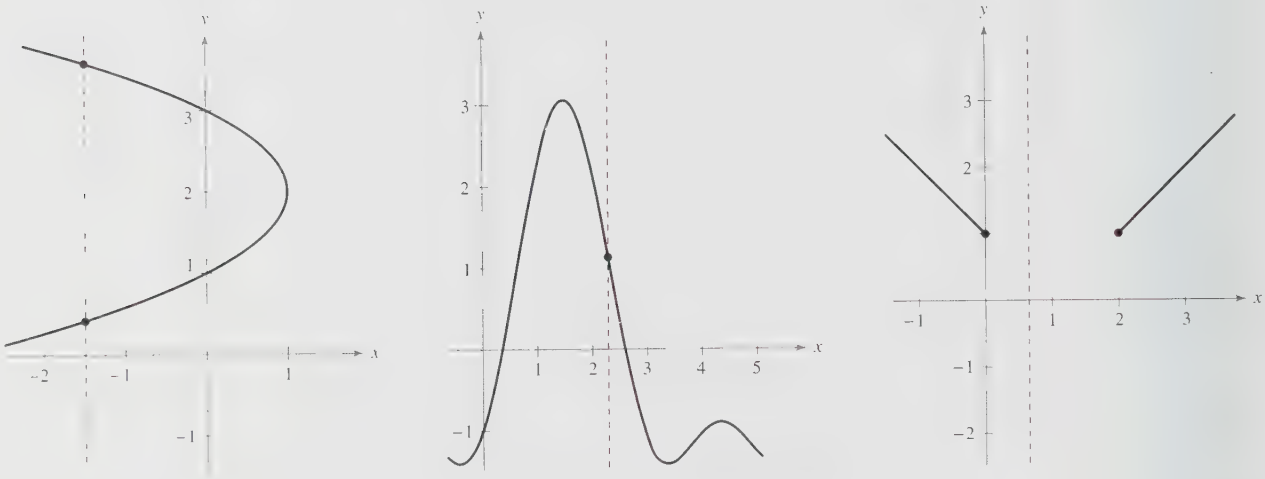
By the definition of a function, at most one y-value corresponds to a given x-value. This means that the graph of a function cannot have two or more different points with the same x-coordinate, and no two points on the graph of a function can be vertically above or below each other. It follows, then, that a vertical line can intersect the graph of a function at most once. This observation provides a convenient visual test called the **Vertical Line Test** for functions.

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no vertical line intersects the graph at more than one point.

Example 2 **Using the Vertical Line Test**

Use the Vertical Line Test to decide whether the graphs in Figure 2.47 represent y as a function of x.



(a) (b) (c)

FIGURE 2.47

SOLUTION

a. This *is not* a graph of y as a function of x because you can find a vertical line that intersects the graph twice. That is, for a particular input x, there is more than one output y.

b. This *is* a graph of y as a function of x because every vertical line intersects the graph at most once. That is, for a particular input x, there is at most one output y.

c. This *is* a graph of y as a function of x. That is, for a particular input x, there is at most one output y. Note that if a vertical line does not intersect the graph, it simply means that the function is undefined for that particular value of x.

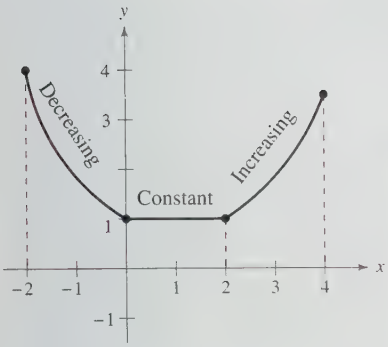
Checkpoint 2

Use the Vertical Line Test to decide whether the graph of each equation represents y as a function of x.

a. $x^2 + y = 2$ **b.** $y = 2x^3$

Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. For example, consider the graph that is shown in Figure 2.48. As you move from *left to right*, this graph decreases, then is constant, and then increases.



As x increases from left to right, notice how y decreases, then remains constant, and then increases.

FIGURE 2.48

Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies

$$f(x_1) < f(x_2).$$

A function f is **decreasing** on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies

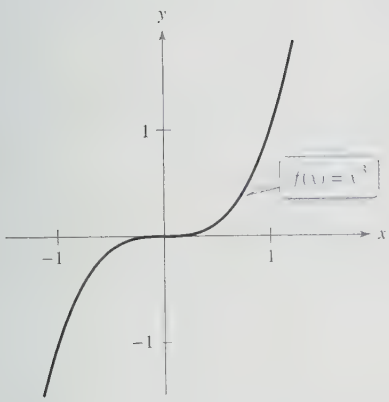
$$f(x_1) > f(x_2).$$

A function f is **constant** on an interval if, for any x_1 and x_2 in the interval,

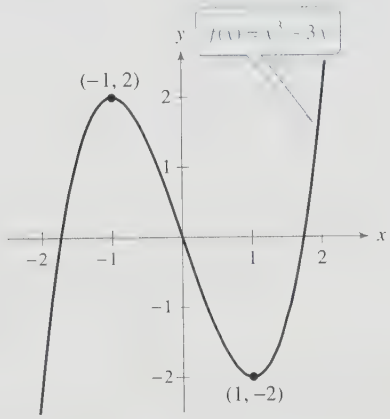
$$f(x_1) = f(x_2).$$

Example 3 Describing Function Behavior

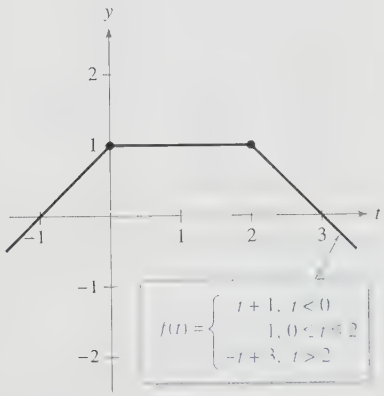
Describe the increasing, decreasing, and constant behavior of each function shown in Figure 2.49.



(a)



(b)



(c)

FIGURE 2.49

SOLUTION

a. This function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.

Checkpoint 3

Describe the increasing, decreasing, and constant behavior of the function $f(x) = x^2 + 3x$.

Make sure you understand that the behavior of functions (increasing, decreasing, or constant) is determined by the y-values, but that the intervals are written in terms of x.

Relative Minima and Relative Maxima

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the **relative minimum** or **relative maximum** values of the function.

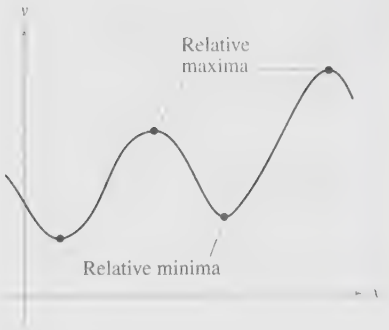


FIGURE 2.50

Definition of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \leq f(x).$$

A function value $f(a)$ is called a **relative maximum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \geq f(x).$$

Figure 2.50 shows several examples of relative minima and relative maxima. In Section 3.1, you will study a technique for finding the *exact point* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

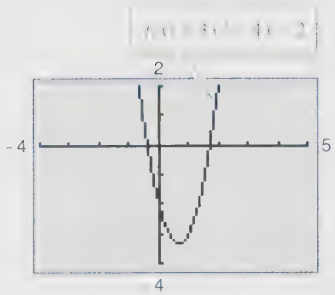


FIGURE 2.51

Example 4 Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function given by $f(x) = 3x^2 - 4x - 2$.

SOLUTION The graph of f is shown in Figure 2.51. By using the *zoom* and *trace* features of a graphing utility, you can estimate that the function has a relative minimum at the point

$$(0.67, -3.33). \qquad \text{Relative minimum}$$

Later, in Section 3.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

Checkpoint 4

Use a graphing utility to approximate the relative maximum of the function given by $f(x) = -x^2 + 4x - 2$. ■

TECH TUTOR

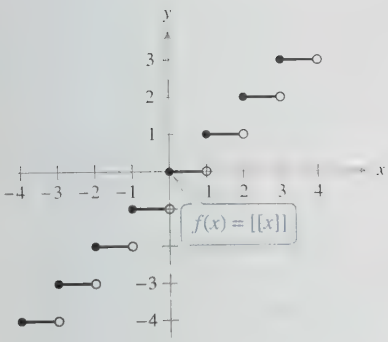
For instructions on how to use the *table* feature and the *minimum* feature, see Appendix A.

You can also use the *table* feature of a graphing utility to approximate numerically the relative minimum of the function in Example 4. Using a table that begins at 0.6 and increments the value of x by 0.01, you can approximate the minimum of $f(x) = 3x^2 - 4x - 2$ to be -3.33, which occurs at $(0.67, -3.33)$. A third way to find the relative minimum is to use the *minimum* feature of a graphing utility.

TECH TUTOR

When using a graphing utility to estimate the x- and y-values of a relative minimum or relative maximum, the *zoom* feature will often produce graphs that are nearly flat. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will stretch vertically if the values of Ymin and Ymax are closer together.

Step Functions



Greatest Integer Function

FIGURE 2.52

The **greatest integer function** is denoted by $[\![x]\!]$ and is defined as

$$f(x) = [\![x]\!] = \text{the greatest integer less than or equal to } x.$$

The graph of this function is shown in Figure 2.52. Note that the graph of the greatest integer function jumps vertically one unit at each integer and is constant (a horizontal line segment) between each pair of consecutive integers. Because of the jumps in its graph, the greatest integer function is an example of a type of function called a **step function.** Some values of the greatest integer function are as follows.

$$[\![-1]\!] = -1 \qquad [\![-0.5]\!] = -1$$
$$[\![0]\!] = 0 \qquad [\![0.5]\!] = 0$$
$$[\![1]\!] = 1 \qquad [\![1.5]\!] = 1$$

The range of the greatest integer function is the set of all integers.

When using a graphing utility to graph a step function, set the utility to *dot* mode rather than *connected* mode.

Example 5 The Cost of a Machinery Rental

The cost of renting a warehouse forklift is \$200 for up to, but not including, the first day and \$125 for each additional day (or portion of a day). The greatest integer function

$$C = 200 + 125[\![t]\!], \quad t > 0$$

can be used to model the cost of this rental, where C is the total cost of the rental (in dollars) and t is the length of the rental (in days).

a. Sketch the graph of this function.

b. How long can a company rent the forklift without spending more than \$1000?

SOLUTION

a. For rentals up to, but not including, 1 day, the cost is \$200. For rentals between 1 and 2 days, the cost is \$325, for rentals between 2 and 3 days, the cost is \$450, and so on, as shown in the table. Using these and other values, you can sketch the graph shown in Figure 2.53.

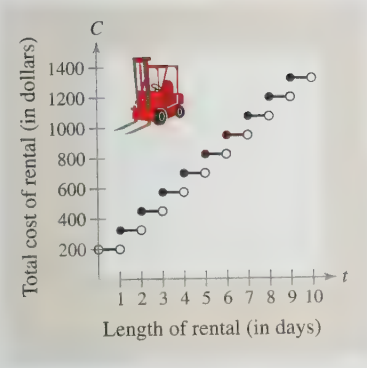


FIGURE 2.53

Length of rental, t	Cost of rental, C
$0 < t < 1$	\$200
$1 \le t < 2$	\$325
$2 \le t < 3$	\$450
⋮	⋮
$9 \le t < 10$	\$1325

b. From the graph, you can see that the company must limit the rental period to 7 days to avoid spending more than \$1000.

Checkpoint 5

Repeat Example 5 when the cost of each additional day (or portion of a day) is \$100.

Even and Odd Functions

In Section 2.1, you studied different types of symmetry of a graph. A function is said to be **even** if its graph is symmetric with respect to the y-axis and **odd** if its graph is symmetric with respect to the origin. The symmetry tests in Section 2.1 yield the following tests for even and odd functions. Even though symmetry with respect to the x-axis is introduced in Section 2.1, it will not be discussed here because a graph that is symmetric with respect to the x-axis is not a function.

Tests for Even and Odd Functions

A function given by $y = f(x)$ is even if, for each x in the domain of f,

$$f(-x) = f(x).$$

A function given by $y = f(x)$ is odd if, for each x in the domain of f,

$$f(-x) = -f(x).$$

Example 6 Identifying Even and Odd Functions

Decide whether each function is even, odd, or neither.

a. $g(x) = x^3 - x$ **b.** $h(x) = x^2 + 1$

SOLUTION

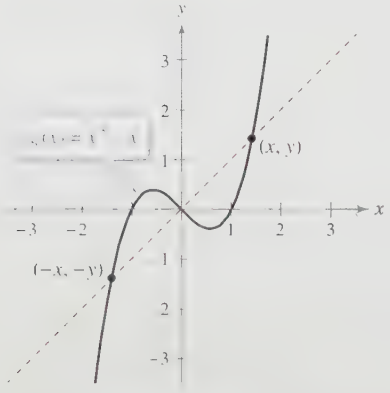
a. The function given by $g(x) = x^3 - x$ is odd because

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) = -g(x). \end{aligned}$$

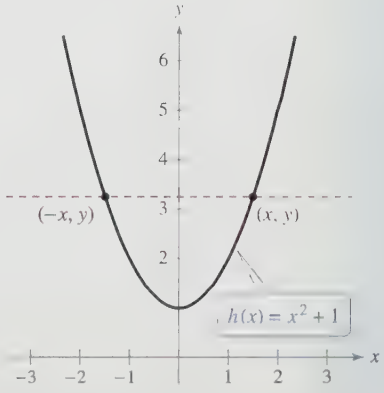
b. The function given by $h(x) = x^2 + 1$ is even because

$$\begin{aligned} h(-x) &= (-x)^2 + 1 \\ &= x^2 + 1 = h(x). \end{aligned}$$

The graphs of the two functions are shown in Figure 2.54.



(a) Odd function (symmetric about origin)



(b) Even function (symmetric about y-axis)

FIGURE 2.54

Checkpoint 6

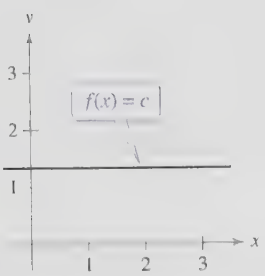
Decide whether the function $f(x) = -2x^2 + x - 1$ is even, odd, or neither. ■

Common Graphs

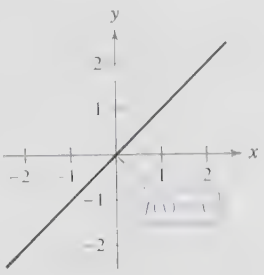
Figure 2.55 shows the graphs of six common functions. You need to be familiar with these graphs. They can be used as an aid when sketching other graphs. For instance, the graph of the absolute value function given by

$$f(x) = |x - 2|$$

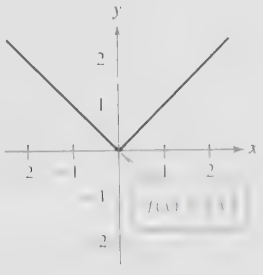
is V-shaped.



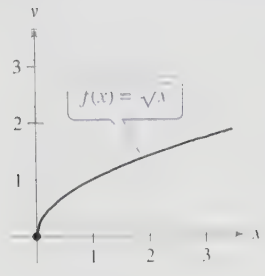
(a) Constant function



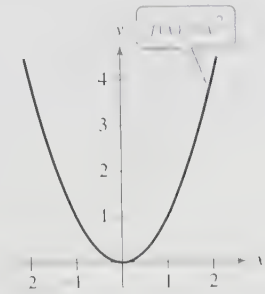
(b) Identity function



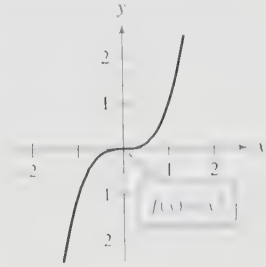
(c) Absolute value function



(d) Square root function



(e) Squaring function



(f) Cubing function

FIGURE 2.55



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SUMMARIZE (Section 2.5)

1. State the definition of the graph of a function *(page 209)*. For an example of finding the domain and range of a function using its graph, see Example 1.
2. Describe the Vertical Line Test *(page 210)*. For an example that uses the Vertical Line Test, see Example 2.
3. Describe when a function is increasing, decreasing, or constant on an interval *(page 211)*. For an example of describing the increasing, decreasing, or constant behavior of a function, see Example 3.
4. State the definitions of relative minimum and relative maximum *(page 212)*. For an example of approximating a relative minimum using a graphing utility, see Example 4.
5. Describe the greatest integer function *(page 213)*. For an example that uses a greatest integer function, see Example 5.
6. State the definitions of an even function and an odd function *(page 214)*. For an example of identifying an even or odd function, see Example 6.

SKILLS WARM UP 2.5 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.4, 1.5, and 2.4.

1. Find $f(2)$ for $f(x) = -x^3 + 5x$.
2. Find $f(6)$ for $f(x) = x^2 - 6x$.
3. Find $f(-x)$ for $f(x) = 3/x$.
4. Find $f(-x)$ for $f(x) = x^2 + 3$.

In Exercises 5 and 6, solve the equation.

5. $x^3 - 16x = 0$
6. $2x^2 - 3x + 1 = 0$

In Exercises 7–10, find the domain of the function.

7. $g(x) = 4(x - 4)^{-1}$
8. $f(x) = 2x/(x^2 - 9x + 20)$
9. $h(t) = \sqrt[4]{5 - 3t}$
10. $f(t) = t^3 + 3t - 5$

Exercises 2.5

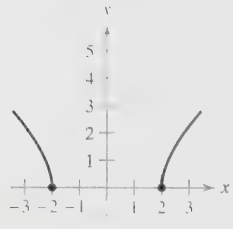
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding the Domain and Range of a Function In Exercises 1–8, find the domain and range of the function. Then evaluate f at the given x-value. *See Example 1.*

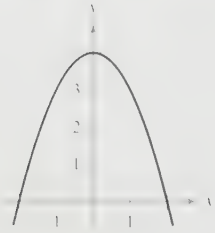
1. $f(x) = \sqrt{x - 1}$, $x = 1$



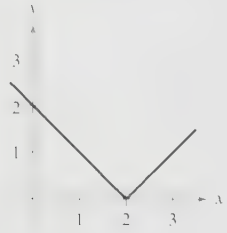
2. $f(x) = \sqrt{x^2 - 4}$, $x = -2$



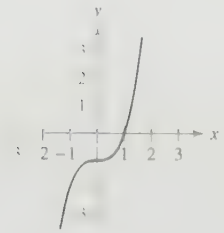
3. $f(x) = 4 - x^2$, $x = 0$



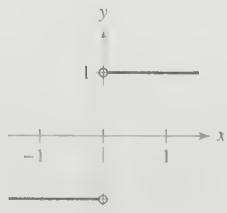
4. $f(x) = |x - 2|$, $x = 2$



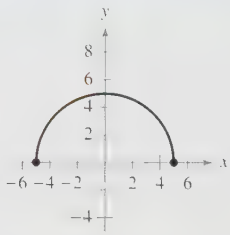
5. $f(x) = x^3 - 1$, $x = 0$



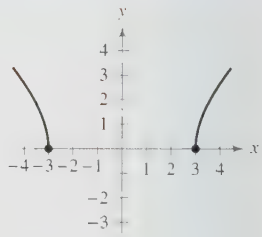
6. $f(x) = \dfrac{|x|}{x}$, $x = 5$



7. $f(x) = \sqrt{25 - x^2}$, $x = 0$

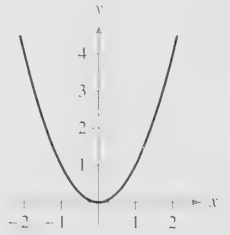


8. $f(x) = \sqrt{x^2 - 9}$, $x = 3$

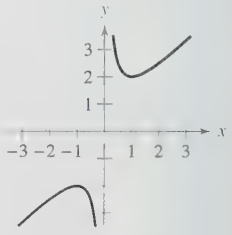


Using the Vertical Line Test In Exercises 9–16, use the Vertical Line Test to decide whether y is a function of x. *See Example 2.*

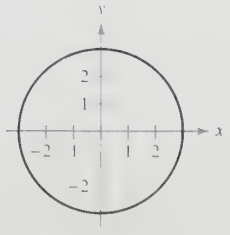
9. $y = x^2$



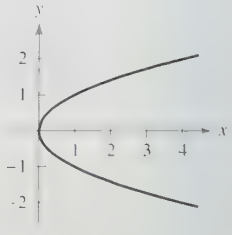
10. $x^2 = xy - 1$



11. $x^2 + y^2 = 9$



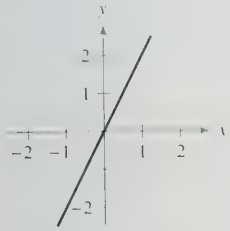
12. $x - y^2 = 0$



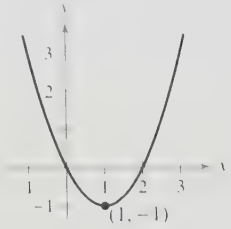
13. $y = \frac{1}{2}x^2$
14. $y = \frac{1}{4}x^3$
15. $x^2 + y^2 = 25$
16. $x = |y + 2|$

Describing Function Behavior In Exercises 17–28, describe the increasing, decreasing, and constant behavior of the function. Find the point or points where the behavior of the function changes. *See Example 3.*

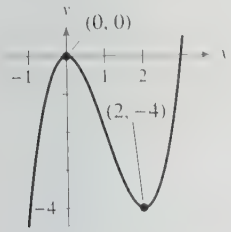
17. $f(x) = 2x$



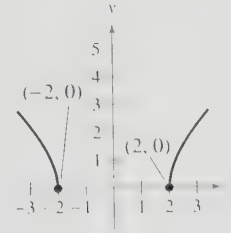
18. $f(x) = x^2 - 2x$



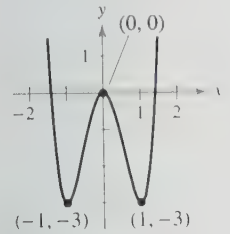
19. $f(x) = x^3 - 3x^2$



20. $f(x) = \sqrt{x^2 - 4}$



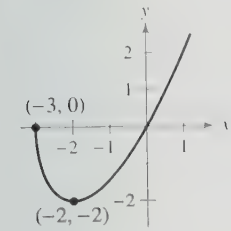
21. $f(x) = 3x^4 - 6x^2$



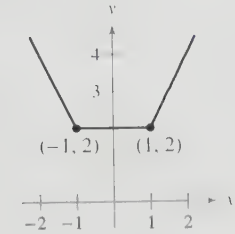
22. $f(x) = x^{2/3}$



23. $y = x\sqrt{x + 3}$



24. $y = |x + 1| + |x - 1|$



25. $f(x) = x^2 - 4x$

26. $f(x) = \frac{3}{2}x$

27. $f(x) = x^3 - 3x^2 + 2$

28. $f(x) = \sqrt{x^2 - 1}$



Approximating a Relative Minimum and Maximum In Exercises 29–34, use a graphing utility to graph the function, approximate the relative minimum or maximum of the function, and estimate the open intervals on which the function is increasing or decreasing. *See Example 4.*

29. $f(x) = x^2 - 4x + 1$

30. $f(x) = -x^2 + 6x + 3$

31. $f(x) = x^3 - 3x^2$

32. $f(x) = -x^3 + 3x + 1$

33. $f(x) = \frac{1}{4}(-4x^4 - 5x^3 + 10x^2 + 8x + 6)$

34. $f(x) = \frac{1}{4}(x^4 + x^3 - 10x^2 + 2x - 15)$

Using the Greatest Integer Function In Exercises 35–38, evaluate the function at each specified value of the independent variable.

35. $f(x) = [\![x]\!]$

(a) $f(2)$ (b) $f(2.5)$

(c) $f(-2.5)$ (d) $f(-4)$

36. $f(x) = [\![-x]\!]$

(a) $f(3)$ (b) $f(6.1)$

(c) $f(-5.9)$ (d) $f(-9)$

37. $f(x) = [\![x - 1.8]\!]$

(a) $f(4)$ (b) $f(3.7)$

(c) $f(-5.8)$ (d) $f(-6.3)$

38. $f(x) = [\![x + 0.3]\!]$

(a) $f(2.9)$ (b) $f(4.6)$

(c) $f(-2.3)$ (d) $f(-4.2)$

Identifying Even and Odd Functions In Exercises 39–44, decide whether the function is even, odd, or neither. *See Example 6.*

39. $f(x) = x^6 - 2x^2 + 3$

40. $f(t) = t^2 + 3t - 10$

41. $g(x) = x^3 - 5x$

42. $h(x) = x^3 + 3$

43. $f(x) = x\sqrt{4 - x^2}$

44. $g(s) = 4s^{2/3}$

Sketching Graphs of Functions In Exercises 45–58, sketch the graph of the function and determine whether the function is even, odd, or neither.

45. $f(x) = 3$

46. $g(x) = x$

47. $f(x) = 5 - 3x$

48. $h(x) = x^2 - 4$

49. $g(s) = \frac{s^3}{4}$

50. $f(t) = -t^4$

51. $f(x) = \sqrt{1 - x}$

52. $g(t) = \sqrt[3]{t - 1}$

53. $f(x) = x^{3/2}$

54. $f(x) = |x + 2|$

55. $f(x) = \begin{cases} x^2 + 1, & x \le 1 \\ 3x - 1, & x > 1 \end{cases}$

56. $f(x) = \begin{cases} 2x - 1, & x \le -1 \\ x^2 - 1, & x > -1 \end{cases}$

57. $f(x) = \begin{cases} x + 1, & x \le 0 \\ 4, & 0 < x \le 2 \\ 3x - 1, & x > 2 \end{cases}$

58. $f(x) = \begin{cases} 2x - 1, & x \le 1 \\ 3, & 1 < x \le 3 \\ 2x + 1, & x > 3 \end{cases}$

Graphing Functions In Exercises 59–64, use a graphing utility to graph the function and determine whether the function is even, odd, or neither.

59. $f(x) = x^2 - x^4$

60. $g(x) = 2x^3 + 1$

61. $h(x) = x^5 - 2x^3 + x$

62. $f(x) = x^5 - 2x^4 + x - 2$

63. $g(x) = 2 - x^6 - x^8$

64. $h(x) = x^9 + 3x^5 - x^3 + x$

Sketching Graphs of Functions In Exercises 65–78, sketch the graph of the function.

65. $f(x) = 4 - x$

66. $f(x) = 4x + 2$

67. $f(x) = x^2 - 9$

68. $f(x) = x^2 - 4x$

69. $f(x) = 1 - x^4$

70. $f(x) = x^4 - 4x^2$

71. $f(x) = \frac{1}{3}(3 + |x|)$

72. $f(x) = -1(1 + |x|)$

73. $f(x) = \sqrt{x + 3}$

74. $f(x) = \sqrt{x - 1}$

75. $f(x) = -[\![x]\!]$

76. $f(x) = 2[\![x]\!]$

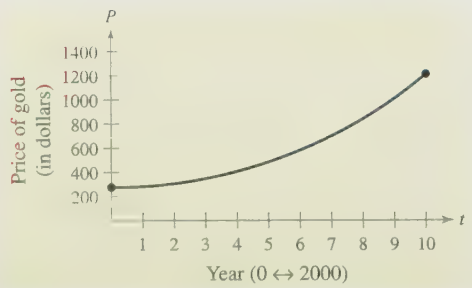
77. $f(x) = [\![x - 1]\!]$

78. $f(x) = [\![x + 1]\!]$

79. Price of Gold The price P (in dollars) of an ounce of gold from 2000 through 2010 can be approximated by the model

$$P = 0.038214t^4 - 0.48703t^3 + 11.0120t^2 - 6.018t + 276.49, \quad 0 \le t \le 10$$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: World Gold Council)*



(a) Use the graph of P to find the maximum price of gold between 2000 and 2010.

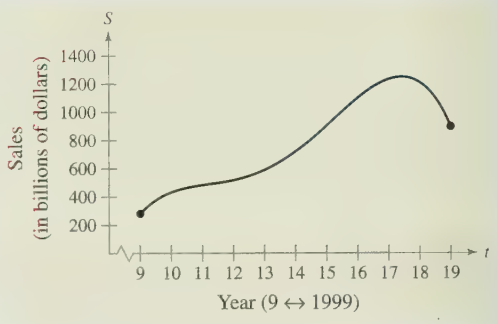
(b) During which year(s) was the price decreasing? During which year(s) was the price increasing?

(c) Is it realistic to assume that the price of gold will continue to follow this model?

80. Sales The sales S (in billions of dollars) of petroleum and coal products from 1999 through 2009 can be approximated by the model

$$S = -1.28418t^4 + 67.9435t^3 - 1316.592t^2 + 11{,}169.74t - 34{,}707.3, \quad 9 \le t \le 19$$

where t represents the year, with $t = 9$ corresponding to 1999. *(Source: U.S. Census Bureau)*



(a) Use the graph of S to find the maximum sales of these products between 1999 and 2009.

(b) During which year(s) were sales decreasing? During which year(s) were sales increasing?

(c) Is it realistic to assume that sales will continue to follow this model?

81. Lung Volume The change in volume V (in milliliters) of the lungs as they expand and contract during a breath can be approximated by the model

$$V = (-6.549s^2 + 26.20s - 3.8)^2, \quad 0 \le s \le 4$$

where s represents the number of seconds.

(a) Graph the volume function with a graphing utility and use the *trace* feature to estimate the numbers of seconds in which the volume is increasing and in which the volume is decreasing.

(b) Find the maximum change in volume between 0 and 4 seconds.

82. Maximum Profit The marketing department of a company estimates that the demand for a product is given by $p = 100 - 0.0001x$, where p is the price per unit and x is the number of units. The cost C of producing x units is given by

$$C = 350{,}000 + 30x$$

and the profit P for producing and selling x units is given by

$$P = R - C = xp - C.$$

Sketch the graph of the profit function and estimate the number of units that would produce a maximum profit. Verify your estimate using a graphing utility.

83. Maximum Profit The marketing department of a company estimates that the demand for a product is given by $p = 125 - 0.0002x$, where p is the price per unit and x is the number of units. The cost C of producing x units is given by

$$C = 225{,}000 + 80x$$

and the profit P for producing and selling x units is given by

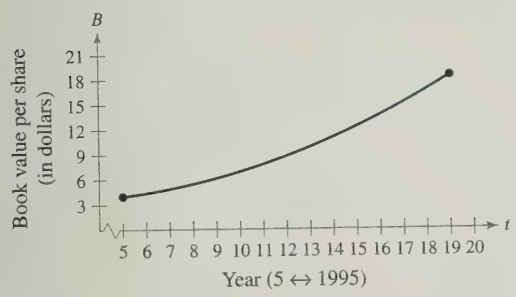
$$P = R - C = xp - C.$$

Sketch the graph of the profit function and estimate the number of units that would produce a maximum profit. Verify your estimate using a graphing utility.

84. HOW DO YOU SEE IT? For the years 1995 through 2009, the book value B (in dollars) of a share of Wells Fargo stock can be approximated by the model

$$B = 0.0497t^2 - 0.163t + 3.53, \quad 5 \le t \le 19$$

where t represents the year, with $t = 5$ corresponding to 1995 (see figure). *(Source: Wells Fargo)*



(a) Estimate the maximum book value per share from 1995 through 2009.

(b) Estimate the minimum book value per share from 1995 through 2009.

(c) Assume the book value B continues to follow the model through 2011. In which year is B at a maximum?

85. Cost of Overnight Delivery The cost of sending an overnight package from New York to Atlanta is \$26.10 for up to, but not including, the first pound and \$4.25 for each additional pound (or portion of a pound). A model for the total cost C of sending the package is $C = 26.1 + 4.25[\![x]\!], x > 0$, where x is the weight of the package (in pounds). Sketch the graph of this function.

86. Cost of Overnight Delivery The cost of sending an overnight package from Los Angeles to Miami is \$31.60 for up to, but not including, the first pound and \$5.50 for each additional pound (or portion of a pound). A model for the total cost C of sending the package is $C = 31.6 + 5.5[\![x]\!], x > 0$, where x is the weight of the package (in pounds). Sketch the graph of this function.

87. Strategic Reserve The total volume V (in millions of barrels) of the Strategic Oil Reserve in the United States from 1998 through 2009 can be approximated by

$$V = \begin{cases} -2.2292t^4 + 94.319t^3 - 1472.15t^2 + \\ \quad 10{,}050.1t - 24{,}773, & 8 \le t \le 13 \\ 0.796t^3 - 38.13t^2 + 611.8t - 2601, \\ \quad 14 \le t \le 19 \end{cases}$$

where t represents the year, with $t = 8$ corresponding to 1998. Sketch the graph of this function. *(Source: U.S. Energy Information Administration)*

88. Grade Level Salaries The 2010 salary S (in dollars) for federal employees at the Step 1 level can be approximated by the model

$$S = \begin{cases} 3109.2x + 13{,}012, & x = 1, 2, \ldots, 10 \\ 12{,}310.5x - 86{,}725, & x = 11, \ldots, 15 \end{cases}$$

where x represents the "GS" grade. Sketch a *bar graph* that represents this function. *(Source: U.S. Office of Personnel Management)*

89. Revenues The revenues of Symantec Corporation (in millions of dollars) from 2000 through 2009 are given by the ordered pairs. *(Source: Symantec Corporation)*

(2000, 853.6), (2001, 1071.4), (2002, 1406.9), (2003, 1870.1), (2004, 2582.8), (2005, 4143.4), (2006, 5199.4), (2007, 5874.4), (2008, 6149.9), (2009, 5985.0)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data from 2000 through 2004. Let t represent the year, with $t = 0$ corresponding to 2000. Then determine the domain and range of the function.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data from 2005 through 2009. Let t represent the year, with $t = 5$ corresponding to 2005. Then determine the domain and range of the function.

(c) Use your results from parts (a) and (b) to construct a piecewise model for all of the data. Use the graphing utility to graph the function.

(d) During which years did the revenues increase? During which year did the revenue decrease?

90. Think About It If f is an even function, determine whether g is even, odd, or neither. Explain.

(a) $g(x) = -f(x)$ (b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$ (d) $g(x) = f(x - 2)$

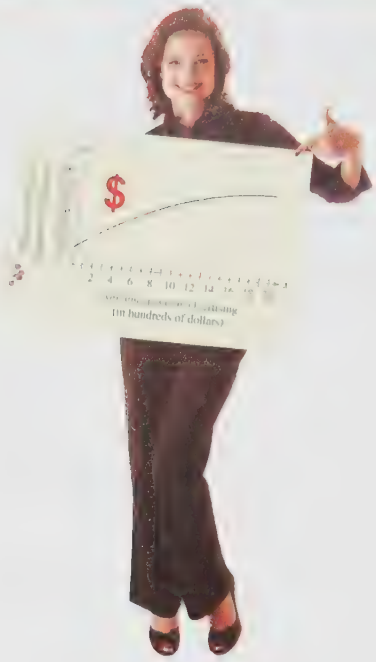
Think About It In Exercises 91–94, find the coordinates of a second point on the graph of a function f when the given point is on the graph and the function is (a) even and (b) odd.

91. $\left(-\frac{3}{2}, 4\right)$ **92.** $\left(-\frac{5}{3}, -7\right)$

93. $(4, 9)$ **94.** $(5, -1)$

2.6 Transformations of Functions

- Use vertical and horizontal shifts to sketch graphs of functions.
- Use reflections to sketch graphs of functions.
- Use nonrigid transformations to sketch graphs of functions.



In Exercise 65 on page 229, you will see how a transformation affects the graph of a company's profit function.

Vertical and Horizontal Shifts

Many functions have graphs that are simple transformations of the common graphs that are summarized on page 215. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ *upward* two units, as shown in Figure 2.56. In function notation, h and f are related as shown below.

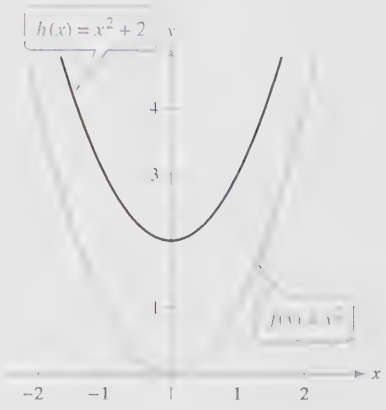
$$h(x) = x^2 + 2 = f(x) + 2 \qquad \text{Upward shift of two units}$$

Similarly, you can obtain the graph of

$$g(x) = (x - 2)^2$$

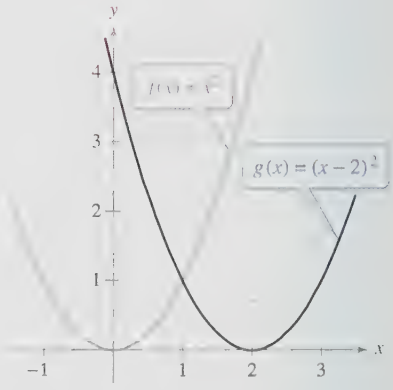
by shifting the graph of $f(x) = x^2$ to the *right* two units, as shown in Figure 2.57. In this case, the functions g and f have the following relationship.

$$g(x) = (x - 2)^2 = f(x - 2) \qquad \text{Right shift of two units}$$



Vertical Shift Upward

FIGURE 2.56



Horizontal Shift to the Right

FIGURE 2.57

The following list summarizes this discussion about vertical and horizontal shifts.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical** and **horizontal shifts** of the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units **upward:** $h(x) = f(x) + c$
2. Vertical shift c units **downward:** $h(x) = f(x) - c$
3. Horizontal shift c units to the **right:** $h(x) = f(x - c)$
4. Horizontal shift c units to the **left:** $h(x) = f(x + c)$

STUDY TIP

In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift.

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Some graphs can be obtained from a combination of vertical and horizontal shifts, as demonstrated in Example 1(b). Vertical and horizontal shifts generate a *family of functions*, each with the same shape but at different locations in the plane.

Example 1 Shifts in the Graph of a Function

Use the graph of $f(x) = x^3$ to sketch the graph of each function.

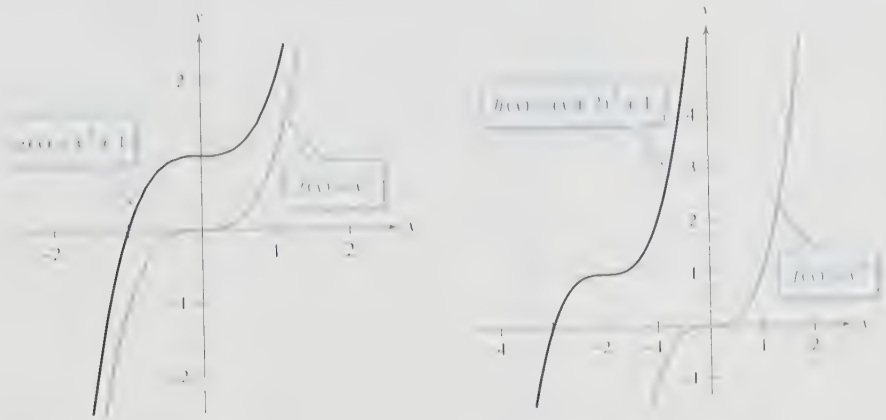
a. $g(x) = x^3 + 1$

b. $h(x) = (x + 2)^3 + 1$

SOLUTION

a. Relative to the graph of $f(x) = x^3$, the graph of $g(x) = x^3 + 1$ is an upward shift of one unit, as shown in Figure 2.58(a).

b. Relative to the graph of $f(x) = x^3$, the graph of $h(x) = (x + 2)^3 + 1$ involves a left shift of two units *and* an upward shift of one unit, as shown in Figure 2.58(b).



(a) Vertical shift: one unit upward

(b) Horizontal shift: two units left; Vertical shift: one unit upward

FIGURE 2.58

Note that the functions f, g, and h belong to the family of cubic functions.

✓ Checkpoint 1

Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of

$$g(x) = \sqrt{x - 3} - 4.$$



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TECH TUTOR

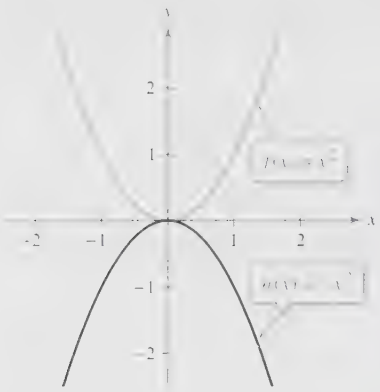


Graphing utilities are ideal tools for exploring transformations of functions. Try to predict how the graphs of g and h relate to the graph of f. Graph f, g, and h in the same viewing window to confirm your prediction.

a. $f(x) = x^2$, $g(x) = (x - 4)^2$, $h(x) = (x - 4)^2 + 3$

b. $f(x) = x^2$, $g(x) = (x + 1)^2$, $h(x) = (x + 1)^2 - 2$

c. $f(x) = x^2$, $g(x) = (x + 4)^2$, $h(x) = (x + 4)^2 + 2$



Reflection

FIGURE 2.59

Reflections

Another common type of transformation is a **reflection.** For instance, if you consider the x-axis to be a mirror, the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$, as shown in Figure 2.59.

Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. **Reflection in the x-axis:** $g(x) = -f(x)$
2. **Reflection in the y-axis:** $h(x) = f(-x)$

Example 2 **Reflections of the Graph of a Function**

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$

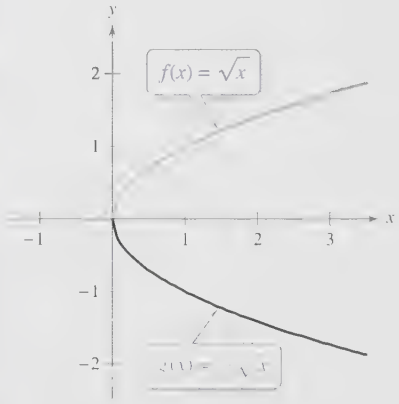
SOLUTION

a. The graph of g is a reflection of the graph of f in the x-axis because

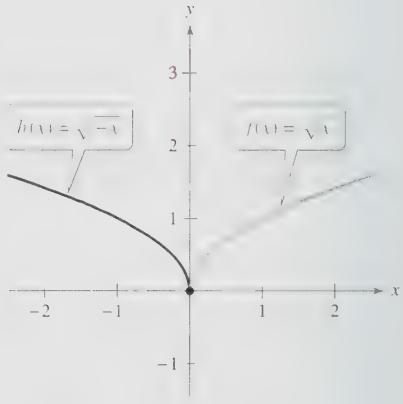
$$g(x) = -\sqrt{x} = -f(x).$$ See Figure 2.60(a).

b. The graph of h is a reflection of the graph of f in the y-axis because

$$h(x) = \sqrt{-x} = f(-x).$$ See Figure 2.60(b).



(a) Reflection in x-axis



(b) Reflection in y-axis

FIGURE 2.60

TECH TUTOR

Several models of graphing utilities have programs that will give you practice working with reflections, horizontal shifts, and vertical shifts. These programs will graph the function

$$y = R(x + H)^2 + V$$

where $R = \pm 1$, H is an integer between -6 and 6, and V is an integer between -3 and 3. Each time you run the program, different values of R, H, and V are possible. From the graph, you should be able to determine the values of R, H, and V.

Checkpoint 2

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $g(x) = -|x|$ **b.** $h(x) = |-x|$

When sketching the graph of a function involving square roots, remember that the domain must be restricted to exclude numbers that make the radicand negative. For instance, here are the domains of the functions in Example 2.

Domain of $g(x) = -\sqrt{x}$: $x \geq 0$

Domain of $h(x) = \sqrt{-x}$: $x \leq 0$

Example 3 **Reflections and Shifts**

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

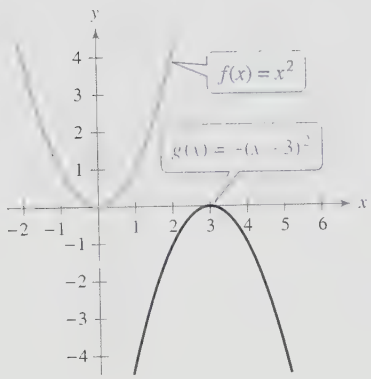
a. $g(x) = -(x - 3)^2$

b. $h(x) = -x^2 + 2$

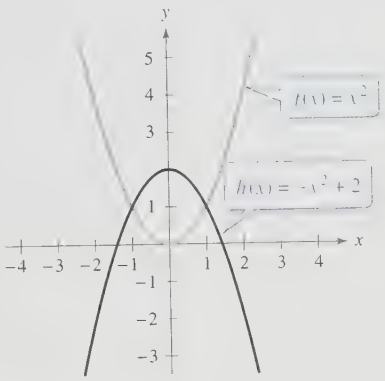
SOLUTION

a. To sketch the graph of $g(x) = -(x - 3)^2$, first shift the graph of $f(x) = x^2$ to the right three units. Then reflect the result in the x-axis. See Figure 2.61(a).

b. To sketch the graph of $h(x) = -x^2 + 2$, first reflect the graph of $f(x) = x^2$ in the x-axis. Then shift the result upward two units. See Figure 2.61(b).



(a) Shift and then reflect in x-axis.



(b) Reflect in x-axis and then shift.

FIGURE 2.61

Checkpoint 3

Use the graph of $f(x) = x^3$ to sketch the graph of each function.

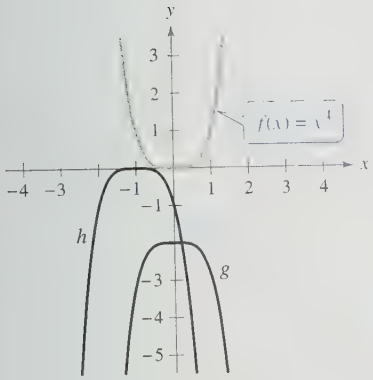
a. $g(x) = -(x + 2)^3$

b. $h(x) = -x^3 - 3$

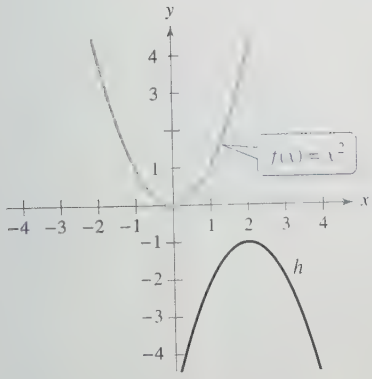
Example 4 **Writing Equations from Graphs**

The graphs labeled g and h in Figure 2.62(a) are transformations of the graph of $f(x) = x^4$. Write an equation for each function.

SOLUTION The graph of g is a reflection in the x-axis *followed* by a downward shift of two units of the graph of $f(x) = x^4$. So, the equation for g is $g(x) = -x^4 - 2$. The graph of h is a horizontal shift of one unit to the left *followed* by a reflection in the x-axis of the graph of $f(x) = x^4$. So, the equation for h is $h(x) = -(x + 1)^4$.



(a)



(b)

FIGURE 2.62

Checkpoint 4

The graph labeled h in Figure 2.62(b) is a transformation of the graph of $f(x) = x^2$. Write an equation for the function h.

Can you think of another way to find an equation for g in Example 4? When you shift the graph of f upward two units and then reflect the graph in the x-axis, you get $g(x) = -(x^4 + 2)$. The Distributive Property yields $g(x) = -x^4 - 2$, which is the same equation obtained in Example 4.

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are **rigid** transformations because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the xy-plane. A **nonrigid** transformation is one that causes a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** if $c > 1$ and a **vertical shrink** if $0 < c < 1$.

Example 5 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$

b. $g(x) = \frac{1}{3}|x|$

SOLUTION

a. Relative to the graph of $f(x) = |x|$, the graph of

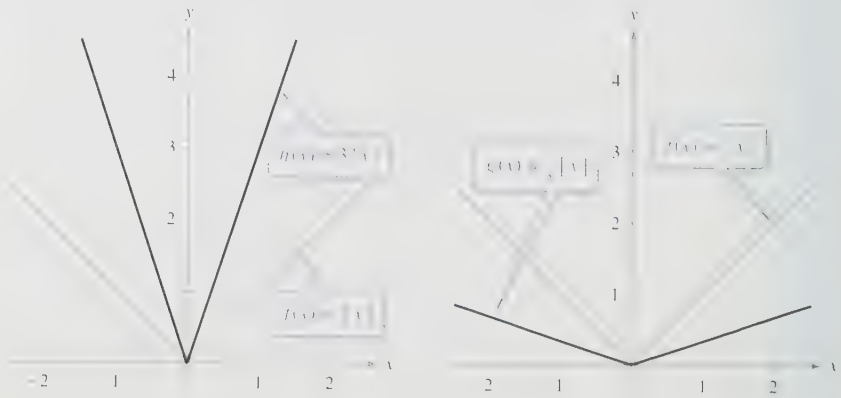
$$h(x) = 3|x| = 3f(x)$$

is a vertical stretch (each y-value is multiplied by 3) of the graph of f. See Figure 2.63(a).

b. Relative to the graph of $f(x) = |x|$, the graph of

$$g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$$

is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f. See Figure 2.63(b).



(a) (b)

FIGURE 2.63

Checkpoint 5

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = 4\sqrt{x}$

b. $h(x) = \frac{1}{4}\sqrt{x}$

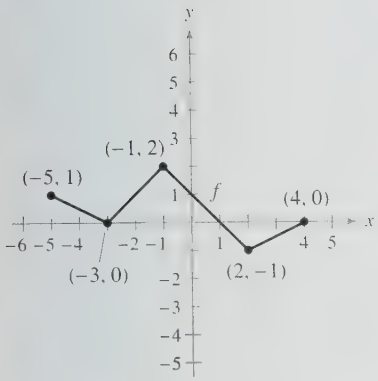


FIGURE 2.64

Example 6 **Rigid and Nonrigid Transformations**

Use the graph of f shown in Figure 2.64 to sketch each graph.

a. $g(x) = f(x - 2) + 1$

b. $h(x) = \frac{1}{2}f(x)$

SOLUTION

a. The graph of g is a horizontal shift to the right two units and a vertical shift upward one unit of the graph of f. The graph of g is shown in Figure 2.65(a).

b. The graph of h is a vertical shrink of the graph of f.

For $x = -5$, $h(-5) = \frac{1}{2}f(-5) = \frac{1}{2}(1) = \frac{1}{2}$.

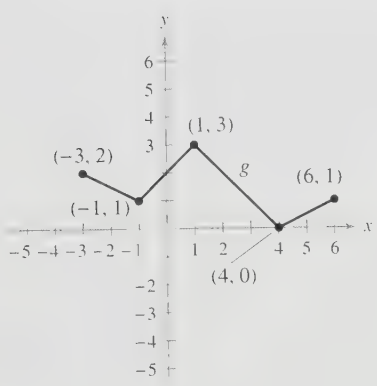
For $x = -3$, $h(-3) = \frac{1}{2}f(-3) = \frac{1}{2}(0) = 0$.

For $x = -1$, $h(-1) = \frac{1}{2}f(-1) = \frac{1}{2}(2) = 1$.

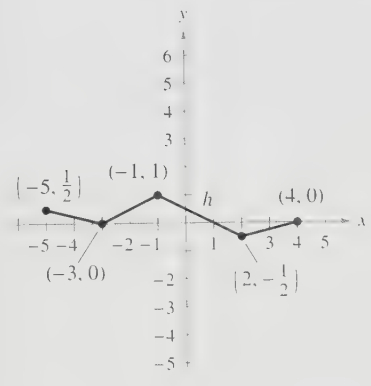
For $x = 2$, $h(2) = \frac{1}{2}f(2) = \frac{1}{2}(-1) = -\frac{1}{2}$.

For $x = 4$, $h(4) = \frac{1}{2}f(4) = \frac{1}{2}(0) = 0$.

The graph of h is shown in Figure 2.65(b).



(a)



(b)

FIGURE 2.65

Checkpoint 6

Use the graph of g shown in Figure 2.65(a) to sketch the graph of

$$p(x) = 2g(x) - 1.$$



SUMMARIZE (Section 2.6)

1. Describe how to shift the graph of $y = f(x)$ vertically and horizontally *(page 220)*. For an example of shifting the graph of a function, see Example 1.
2. Describe how to reflect the graph of $y = f(x)$ in the x-axis and in the y-axis *(page 222)*. For examples of reflecting and shifting the graph of a function, see Examples 2, 3, and 4.
3. State the definitions of a rigid transformation, a nonrigid transformation, a vertical stretch, and a vertical shrink *(page 224)*. For examples of rigid and nonrigid transformations, see Examples 5 and 6.

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SKILLS WARM UP 2.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.4, 1.5, 2.4, and 2.5.

In Exercises 1 and 2, evaluate the function at the indicated value.

1. Find $f(3)$ for $f(x) = x^2 - 4x + 15$.
2. Find $f(-x)$ for $f(x) = 2x/(x - 3)$.

In Exercises 3 and 4, solve the equation.

3. $-x^3 + 10x = 0$
4. $3x^2 + 2x - 8 = 0$

In Exercises 5–10, sketch the graph of the function.

5. $f(x) = -2$
6. $f(x) = -x$
7. $f(x) = x + 5$
8. $f(x) = 2 - x$
9. $f(x) = 3x - 4$
10. $f(x) = 9x + 10$

Exercises 2.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Describing Transformations In Exercises 1–8, describe the sequence of transformations from $f(x) = x^2$ to g. Then sketch the graph of g by hand. Verify with a graphing utility. *See Examples 1, 2, and 3.*

1. $g(x) = x^2 - 4$
2. $g(x) = x^2 + 1$
3. $g(x) = (x + 2)^2$
4. $g(x) = (x - 3)^2$
5. $g(x) = (x - 4)^2 + 2$
6. $g(x) = (x + 1)^2 - 3$
7. $g(x) = -x^2 + 1$
8. $g(x) = -(x - 2)^2$

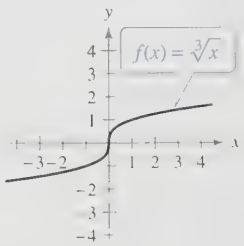
Describing Transformations In Exercises 9–16, describe the sequence of transformations from $f(x) = |x|$ to g. Then sketch the graph of g by hand. Verify with a graphing utility. *See Examples 1, 2, and 3.*

9. $g(x) = |x| + 2$
10. $g(x) = |x| - 3$
11. $g(x) = |x - 1|$
12. $g(x) = |x + 4|$
13. $g(x) = -|x| + 3$
14. $g(x) = 5 - |x - 1|$
15. $g(x) = |x + 1| - 4$
16. $g(x) = |x - 2| - 1$

Describing Transformations In Exercises 17–24, describe the sequence of transformations from $f(x) = \sqrt{x}$ to g. Then sketch the graph of g by hand. Verify with a graphing utility. *See Examples 1, 2, 3, and 5.*

17. $g(x) = \sqrt{x - 3}$
18. $g(x) = \sqrt{x + 4}$
19. $g(x) = \sqrt{x - 3} + 1$
20. $g(x) = \sqrt{x + 5} - 2$
21. $g(x) = \sqrt{2x}$
22. $g(x) = \sqrt{2x} - 5$
23. $g(x) = 2 + \sqrt{-x - 4}$
24. $g(x) = \sqrt{-x} + 1$

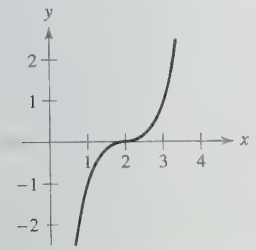
Describing Transformations In Exercises 25–34, describe the sequence of transformations from $f(x) = \sqrt[3]{x}$ to y. Then sketch the graph of y by hand. Verify with a graphing utility. *See Examples 1, 2, 3, and 5.*



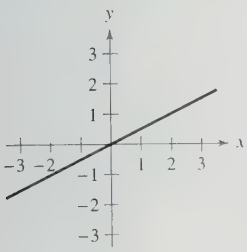
25. $y = \sqrt[3]{-x}$
26. $y = -\sqrt[3]{x}$
27. $y = \sqrt[3]{x} - 2$
28. $y = \sqrt[3]{x + 3}$
29. $y = 2 - \sqrt[3]{x + 1}$
30. $y = -\sqrt[3]{x - 1} - 4$
31. $y = 2\sqrt[3]{x + 1} - 1$
32. $y = 2\sqrt[3]{x - 2} + 1$
33. $y = \frac{1}{2}\sqrt[3]{x}$
34. $y = \frac{1}{2}\sqrt[3]{x} - 3$

Writing Equations from Graphs In Exercises 35–42, identify the transformation shown in the graph and identify the associated common function. Write the equation of the graphed function. *See Example 4.*

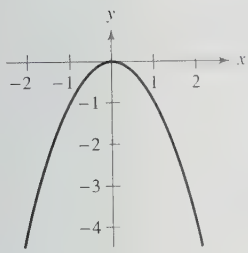
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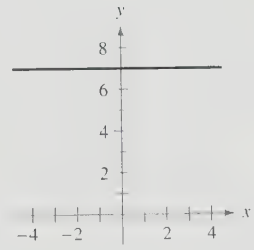
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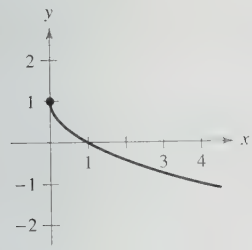
37.



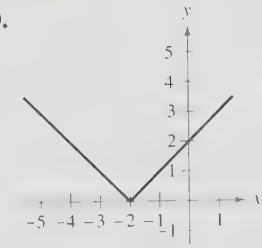
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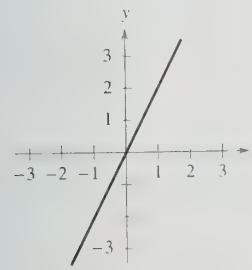
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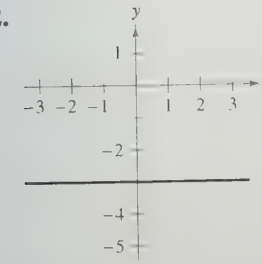
40.



41.



42.



43. Comparing Graphs of Functions Use a graphing utility to graph f for $c = -2$, 0, and 2 in the same viewing window.

(a) $f(x) = \frac{1}{2}x + c$ (b) $f(x) = \frac{1}{2}(x - c)$

(c) $f(x) = \frac{1}{2}(cx)$

In each case, compare the graph with the graph of $y = \frac{1}{2}x$.

44. Comparing Graphs of Functions Use a graphing utility to graph f for $c = -2$, 0, and 2 in the same viewing window.

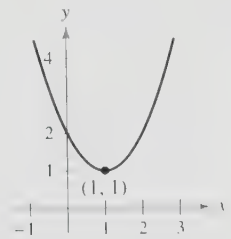
(a) $f(x) = x^3 + c$ (b) $f(x) = (x - c)^3$

(c) $f(x) = (x - 2)^3 + c$

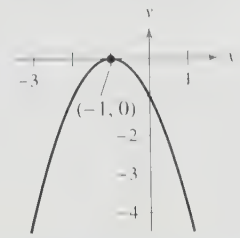
In each case, compare the graph with the graph of $y = x^3$.

45. Writing Equations from Graphs Use the graph of $f(x) = x^2$ to write equations for the functions whose graphs are shown. *See Example 4.*

(a)

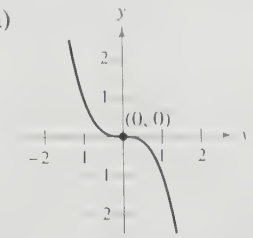


(b)

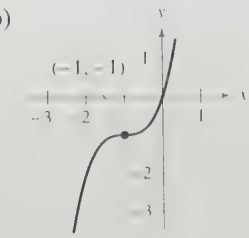


46. Writing Equations from Graphs Use the graph of $f(x) = x^3$ to write equations for the functions whose graphs are shown. *See Example 4.*

(a)



(b)



47. Sketching Transformations Use the graph of f (see figure) to sketch each graph. *See Example 6.*

(a) $y = f(x) + 2$

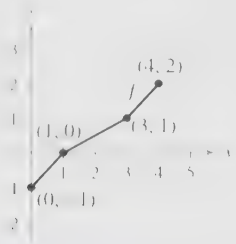
(b) $y = -f(x)$

(c) $y = f(x - 2)$

(d) $y = f(x + 3)$

(e) $y = 2f(x)$

(f) $y = f(-x)$



48. Sketching Transformations Use the graph of f (see figure) to sketch each graph. *See Example 6.*

(a) $y = f(x) - 1$

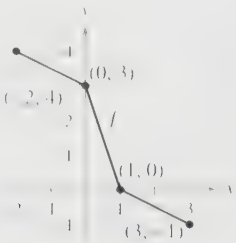
(b) $y = f(x + 1)$

(c) $y = f(x - 1)$

(d) $y = -f(x - 2)$

(e) $y = f(-x)$

(f) $y = \frac{1}{2}f(x)$



49. Sketching Transformations Use the graph of f (see figure) to sketch each graph. *See Example 6.*

(a) $y = f(x - 5)$

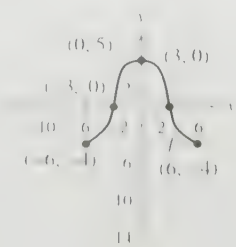
(b) $y = -f(x) + 3$

(c) $y = \frac{1}{3}f(x)$

(d) $y = f(x + 1)$

(e) $y = f(-x)$

(f) $y = f(x) - 5$



50. HOW DO YOU SEE IT? Consider the graph of $f(x) = |x|$. Match the description of the transformation with its graph.

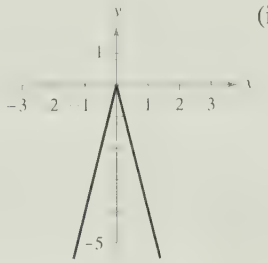
(a) The graph of f is shifted three units to the right and two units upward.

(b) The graph of f is reflected in the x-axis, shifted two units to the left, and shifted three units upward.

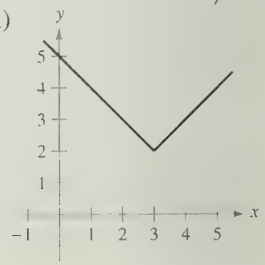
(c) The graph of f is vertically stretched by a factor of 4 and reflected in the x-axis.

(d) The graph of f is vertically shrunk by a factor of $\frac{1}{3}$ and shifted two units to the left.

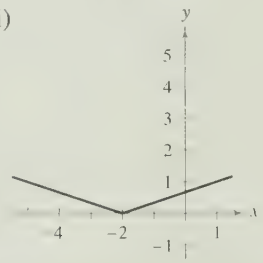
(i)



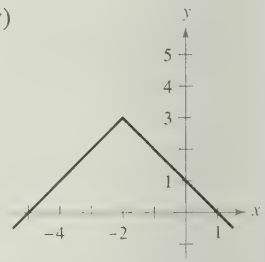
(ii)



(iii)



(iv)



Writing an Equation from a Description In Exercises 51–54, consider the graph of $f(x) = x^3$. Use your knowledge of rigid and nonrigid transformations to write an equation for the description. Verify with a graphing utility.

51. The graph of f is shifted two units downward.

52. The graph of f is shifted three units to the left.

53. The graph of f is vertically stretched by a factor of 4.

54. The graph of f is vertically shrunk by a factor of $\frac{1}{3}$.

Writing an Equation from a Description In Exercises 55–58, consider the graph of $g(x) = \sqrt{x}$. Use your knowledge of rigid and nonrigid transformations to write an equation for the description. Verify with a graphing utility.

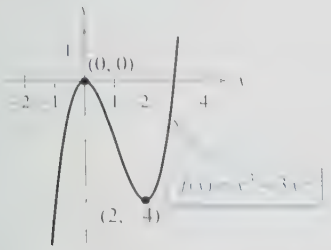
55. The graph of g is shifted four units to the right and three units downward.

56. The graph of g is reflected in the x-axis, shifted two units to the left, and shifted one unit upward.

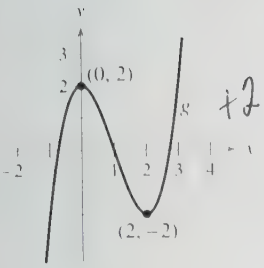
57. The graph of g is vertically shrunk by a factor of $\frac{1}{2}$ and shifted three units to the right.

58. The graph of g is vertically stretched by a factor of 2, reflected in the x-axis, and shifted three units upward.

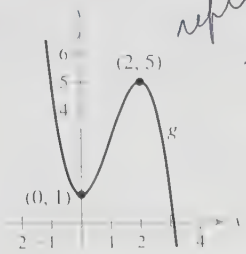
Writing an Equation from a Graph In Exercises 59–62, use the graph of $f(x) = x^3 - 3x^2$ to write an equation for the function g.



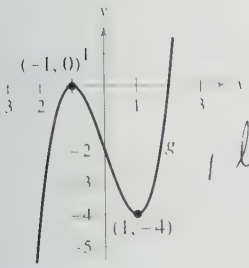
59.



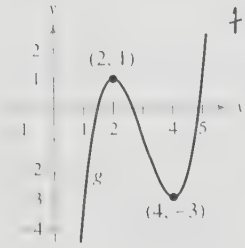
60.



61.



62.



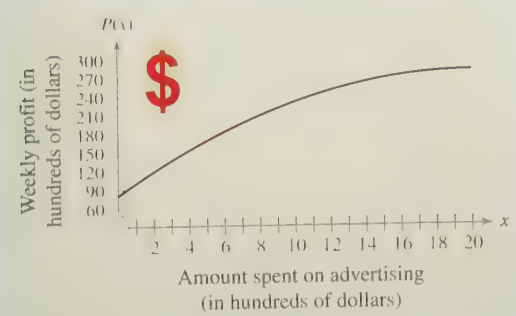
63. Exploration The point $(3, 9)$ on the graph of $f(x) = x^2$ has been shifted to the point $(4, 7)$ after a rigid transformation. Identify the shift and write the new function g in terms of f.

64. Exploration The point $(8, 2)$ on the graph of $f(x) = \sqrt[3]{x}$ has been shifted to the point $(5, 0)$ after a rigid transformation. Identify the shift and write the new function h in terms of f.

65. Profit A company's weekly profit P (in hundreds of dollars) from a product is given by the model

$$P(x) = 80 + 20x - 0.5x^2, \quad 0 \le x \le 20$$

where x is the amount (in hundreds of dollars) spent on advertising.



(a) The company estimates that taxes and operating costs will increase by an average of \$2500 per week during the next year. Rewrite the profit equation to reflect this expected decrease in profits. Identify the type of transformation applied to the graph of the equation.

(b) Use a graphing utility to graph the profit equation from part (a).

(c) Rewrite the original profit equation so that x measures advertising expenditures in dollars. [Find $P(x/100)$.] Identify the type of transformation applied to the graph of the equation.

66. Automobile Aerodynamics The number of horsepower H required to overcome wind drag on an automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \quad 10 \le x \le 100$$

where x is the speed of the car (in miles per hour).

(a) Use a graphing utility to graph the function.

(b) Rewrite the horsepower function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.] Identify the type of transformation applied to the graph of the horsepower function.

67. Finance The amount M (in billions of dollars) of home mortgage debt outstanding in the United States from 1995 through 2009 can be approximated by the model

$$M(t) = 34.0t^2 + 3705, \quad 5 \le t \le 19$$

where t represents the year, with $t = 5$ corresponding to 1995. *(Source: Board of Governors of the Federal Reserve System)*

(a) Describe the sequence of transformations from $f(t) = t^2$ to M.

(b) Use a graphing utility to graph the function M.

(c) According to the model, when did the amount of debt exceed 13 trillion dollars?

(d) Rewrite the function so that $t = 5$ represents 2005. Identify the type of transformation applied to the graph of the home mortgage debt function M.

68. Exploration Use a graphing utility to graph the six functions below in the same viewing window. Describe any similarities and differences you observe among the graphs.

(a) $y = x$ (b) $y = x^2$

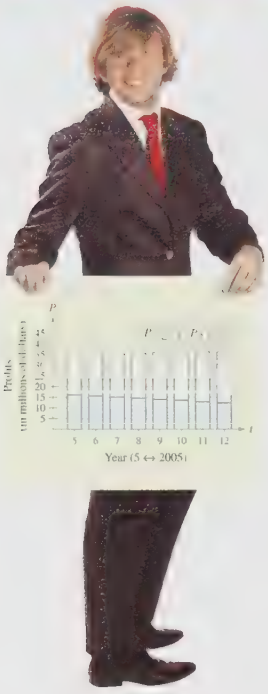
(c) $y = x^3$ (d) $y = x^4$

(e) $y = x^5$ (f) $y = x^6$

69. Reasoning Use the results of Exercise 68 to make a conjecture about the shapes of the graphs of $y = x^7$ and $y = x^8$. Use a graphing utility to verify your conjecture.

2.7 The Algebra of Functions

- Find the sum, difference, product, and quotient of two functions.
- Form the composition of two functions, find the domain of a composite function, and identify a function as the composition of two functions.
- Use combinations and compositions of functions to solve application problems.



In Exercise 56 on page 237, you will use a composite function to determine the trend of the profits for a company that owns two manufacturing plants.

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can be combined to create new functions. For example, the functions given by $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined as shown.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4 \quad \text{Sum}$$

$$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2 \quad \text{Difference}$$

$$f(x)g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3 \quad \text{Product}$$

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1, \quad g(x) \neq 0 \quad \text{Quotient}$$

The domain of an arithmetic combination of the functions f and g consists of all real numbers that are common to the domains of f and g.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the **sum, difference, product,** and **quotient** of f and g are defined as follows.

1. *Sum:* $(f + g)(x) = f(x) + g(x)$
2. *Difference:* $(f - g)(x) = f(x) - g(x)$
3. *Product:* $(fg)(x) = f(x) \cdot g(x)$
4. *Quotient:* $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$.

SOLUTION The sum of the functions f and g is given by

$$(f + g)(x) = f(x) + g(x) \quad \text{Definition of sum of two functions}$$

$$= (2x + 1) + (x^2 + 2x - 1) \quad \text{Substitute for } f(x) \text{ and } g(x).$$

$$= x^2 + 4x. \quad \text{Simplify.}$$

Checkpoint 1

Given $f(x) = x^2 - 4$ and $h(x) = x^2 + x + 3$, find $(f + h)(x)$.

Ana-Maria Tanasescu /Shutterstock.com

Example 2 **Finding the Difference of Two Functions**

Given the functions

$$f(x) = 2x + 1 \quad \text{and} \quad g(x) = x^2 + 2x - 1$$

find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

SOLUTION The difference of the functions f and g is given by

$$\begin{aligned}(f - g)(x) &= f(x) - g(x) && \text{Definition of difference of two functions}\\ &= (2x + 1) - (x^2 + 2x - 1) && \text{Substitute for } f(x) \text{ and } g(x).\\ &= -x^2 + 2. && \text{Simplify.}\end{aligned}$$

When $x = 2$, the value of this difference is

$$(f - g)(2) = -(2)^2 + 2 = -2.$$

STUDY TIP

Note that in Example 2, $(f - g)(2)$ can also be evaluated as follows.

$$\begin{aligned}&(f - g)(2)\\ &= f(2) - g(2)\\ &= [2(2) + 1]\\ &\quad - [2^2 + 2(2) - 1]\\ &= 5 - 7\\ &= -2\end{aligned}$$

Checkpoint 2

Given $f(x) = x^2 - 4$ and $h(x) = x^2 + x + 3$, find $(f - h)(x)$. Then evaluate the difference when $x = 3$.

In Examples 1 and 2, both f and g have domains that consist of all real numbers. So, the domains of $(f + g)$ and $(f - g)$ are also the set of all real numbers. Remember that any restrictions on the domains of f and g must be considered when forming the sum, difference, product, or quotient of f and g.

Example 3 **The Quotient of Two Functions**

Find the domains of $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{g}{f}\right)(x)$ for the functions

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{4 - x^2}.$$

SOLUTION The quotient of f and g is given by

$$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} && \text{Definition of quotient of two functions}\\ &= \frac{\sqrt{x}}{\sqrt{4 - x^2}} && \text{Substitute for } f(x) \text{ and } g(x).\end{aligned}$$

and the quotient of g and f is given by

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} && \text{Definition of quotient of two functions}\\ &= \frac{\sqrt{4 - x^2}}{\sqrt{x}}. && \text{Substitute for } g(x) \text{ and } f(x).\end{aligned}$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these two domains is $[0, 2]$, which implies that the domains of f/g and g/f are as follows. Notice that the domains differ slightly.

$$\text{Domain of } \frac{f}{g}\text{: } [0, 2) \qquad \text{Domain of } \frac{g}{f}\text{: } (0, 2]$$

Checkpoint 3

Find the domains of $\left(\frac{f}{h}\right)(x)$ and $\left(\frac{h}{f}\right)(x)$ for $f(x) = x - 1$ and $h(x) = x - 3$.

Composition of Functions

Another way to combine two functions is to form the **composition** of one with the other. To form a composition, the output of one function becomes the input for another function. For instance, when $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is given by

$$\begin{aligned} f(g(x)) &= f(x + 1) \\ &= (x + 1)^2. \end{aligned}$$

This composition is denoted as $f \circ g$ and is read as "f composed with g."

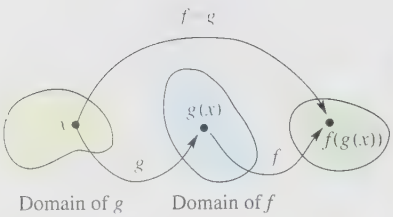


FIGURE 2.66

Definition of the Composition of Two Functions

The **composition** of the functions f and g is given by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 2.66.)

From the definition above, it follows that the domain of $f \circ g$ is always a subset of the domain of g, and the range of $f \circ g$ is always a subset of the range of f.

Example 4 Composition of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, find each composition.

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

SOLUTION

a. The composition of f with g is

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(4 - x^2) && \text{Definition of } g(x) \\ &= (4 - x^2) + 2 && \text{Definition of } f(x) \\ &= -x^2 + 6. && \text{Simplify.} \end{aligned}$$

b. The composition of g with f is

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\ &= g(x + 2) && \text{Definition of } f(x) \\ &= 4 - (x + 2)^2 && \text{Definition of } g(x) \\ &= 4 - (x^2 + 4x + 4) && \text{Expand.} \\ &= -x^2 - 4x. && \text{Simplify.} \end{aligned}$$

Note that, in this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

✓Checkpoint 4

Given $f(x) = x^2 - 2$ and $g(x) = x + 1$, find each composition.

a. $(f \circ g)(x)$

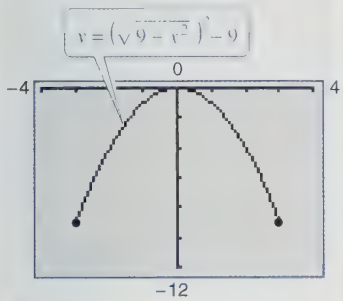
b. $(g \circ f)(x)$

TECH TUTOR

In Example 5, the domain of the composite function is $[-3, 3]$. To convince yourself of this, use a graphing utility to graph

$$y = \left(\sqrt{9 - x^2}\right)^2 - 9$$

as shown in the figure below. Notice that the graphing utility does not extend the graph to the left of $x = -3$ or to the right of $x = 3$.



Example 5 Finding the Domain of a Composite Function

Find the composition $(f \circ g)(x)$ for the functions given by

$$f(x) = x^2 - 9 \quad \text{and} \quad g(x) = \sqrt{9 - x^2}.$$

Then find the domain of $f \circ g$.

SOLUTION The composition of the functions is

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{9 - x^2}\right) \\ &= \left(\sqrt{9 - x^2}\right)^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2.\end{aligned}$$

From this result, it might appear that the domain of the composition is the set of all real numbers. However, because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $f \circ g$ is $[-3, 3]$.

Checkpoint 5

Find the composition $(f \circ g)(x)$ for the functions given by $f(x) = \sqrt{x}$ and $g(x) = 3 - x^2$. Then find the domain of $f \circ g$.

In Examples 4 and 5, you formed the composition of two functions. To "decompose" a composite function, look for an "inner" function and an "outer" function. For instance, the function h given by

$$h(x) = (3x - 5)^3$$

is the composition of f with g, where $f(x) = x^3$ and $g(x) = 3x - 5$. That is,

$$h(x) = (3x - 5)^3 = [g(x)]^3 = f(g(x)).$$

In the function h, $g(x) = 3x - 5$ is the *inner* function and $f(x) = x^3$ is the *outer* function.

STUDY TIP

When writing a function as a composition of two functions, there can be many correct answers. For instance,

$$h(x) = (3x - 5)^3$$

can be written as the composition of f with g, where $f(x) = (x - 5)^3$ and $g(x) = 3x$.

Example 6 Identifying a Composite Function

Write the function given by $h(x) = \dfrac{1}{(x - 2)^2}$ as a composition of two functions.

SOLUTION One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x - 2)^2} = (x - 2)^{-2} = f(x - 2) = f(g(x)).$$

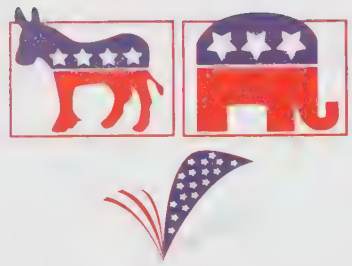
Checkpoint 6

Write the function given by

$$h(x) = (x - 1)^2 + 2$$

as a composition of two functions.

Applications



In recent years, no party has had a strong majority in the Senate, which can make it difficult to pass legislation.

Example 7 Political Makeup of the U.S. Senate

Consider three functions R, D, and I that represent the numbers of Republicans, Democrats, and Independents, respectively, in the U.S. Senate from 1977 to 2011. Sketch the graphs of R, D, and I and the sum of R, D, and I in the same coordinate plane. The numbers of senators from the three political parties immediately following the elections are shown below. *(Source: United States Senate)*

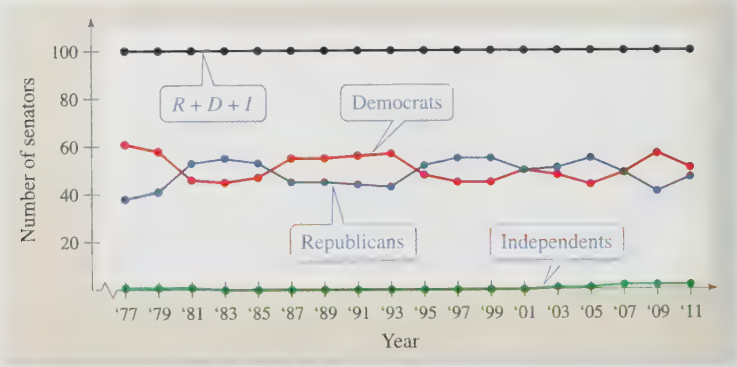
Year	R	D	I
1977	38	61	1
1979	41	58	1
1981	53	46	1
1983	55	45	0
1985	53	47	0
1987	45	55	0
1989	45	55	0
1991	44	56	0
1993	43	57	0

Year	R	D	I
1995	52	48	0
1997	55	45	0
1999	55	45	0
2001	50	50	0
2003	51	48	1
2005	55	44	1
2007	49	49	2
2009	41	57	2
2011	47	51	2

SOLUTION The graphs of R, D, and I are shown in Figure 2.67. Note that the sum of R, D, and I is the constant function

$$R + D + I = 100.$$

This follows from the fact that the number of senators in the United States Senate is 100 (two from each state).



Numbers of U.S. Senators by Political Party

FIGURE 2.67

Checkpoint 7

In Example 7, consider the function f given by

$$f = 100 - (R + D).$$

What does f represent in the context of the real-life situation?

Arsen/www.shutterstock.com

Example 8 Bacteria Count

The number N of bacteria in a certain food is given by

$$N(T) = 20T^2 - 80T + 500, \quad 2 \le T \le 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \quad 0 \le t \le 3$$

where t is the time in hours. Find

a. the composition $N(T(t))$,

b. the number of bacteria in the food when $t = 2$ hours, and

c. how long the food can remain unrefrigerated before the bacteria count reaches 2000.

SOLUTION

a. $N(T(t)) = 20(4t + 2)^2 - 80(4t + 2) + 500$

$= 20(16t^2 + 16t + 4) - 320t - 160 + 500$

$= 320t^2 + 320t + 80 - 320t - 160 + 500$

$= 320t^2 + 420$

b. When $t = 2$, the number of bacteria is

$N(T(2)) = 320(2)^2 + 420$

$= 1280 + 420$

$= 1700.$

c. The bacteria count will reach $N = 2000$ when $320t^2 + 420 = 2000$. By solving this equation, you can determine that the bacteria count will reach 2000 when $t \approx 2.2$ hours. So, the food can remain unrefrigerated for about 2 hours and 12 minutes.



✓Checkpoint 8

In Example 8, how long can the food remain unrefrigerated before the bacteria count reaches 1000?

SUMMARIZE (Section 2.7)

1. Describe how to find the sum, difference, product, and quotient of two functions with overlapping domains *(page 230)*. For examples of these arithmetic combinations of two functions, see Examples 1, 2, and 3.
2. State the definition of the composition of two functions *(page 232)*. For examples that use the composition of two functions, see Examples 4, 5, and 6.
3. Describe real-life examples that use a combination of two functions *(pages 234 and 235, Examples 7 and 8)*.

Sean Nel /Shutterstock.com

SKILLS WARM UP 2.7

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.7.

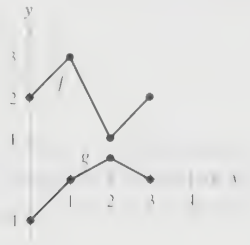
In Exercises 1–10, perform the indicated operations and simplify the result.

1. $\frac{1}{x} + \frac{1}{1-x}$
2. $\frac{2}{x+3} - \frac{2}{x-3}$
3. $\frac{3}{x-2} - \frac{2}{x(x-2)}$
4. $\frac{x}{x-5} + \frac{1}{3}$
5. $(x-1)\left(\frac{1}{\sqrt{x}-1}\right)$
6. $\left(\frac{x}{x^2-4}\right)\left(\frac{x^2-x-2}{x^2}\right)$
7. $(x^2-4) \div \left(\frac{x+2}{5}\right)$
8. $\left(\frac{x}{x^2+3x-10}\right) \div \left(\frac{x^2+3x}{x^2+6x+5}\right)$
9. $\frac{(1/x)+5}{3-(1/x)}$
10. $\frac{(x/4)-(4/x)}{x-4}$

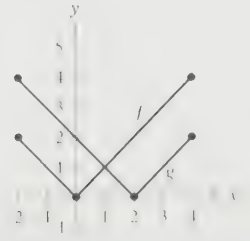
Exercises 2.7

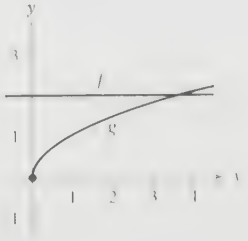
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Graphing the Sum of Two Functions In Exercises 1–4, use the graphs of f and g to graph $h(x) = (f+g)(x)$.

1. 

2. 

3. 

4. 

Finding Arithmetic Combinations of Functions In Exercises 5–12, find (a) $(f+g)(x)$, (b) $(f-g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g? *See Examples 1, 2, and 3.*

5. $f(x) = x + 1, \quad g(x) = x - 1$
6. $f(x) = 2x - 3, \quad g(x) = 1 - x$
7. $f(x) = 3x - 1, \quad g(x) = 1 - x^2$
8. $f(x) = 2x + 3, \quad g(x) = x^2 - 1$
9. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1-x}$
10. $f(x) = x^2 - 4, \quad g(x) = \sqrt{x-3}$
11. $f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x^2}$
12. $f(x) = \frac{1}{x}, \quad g(x) = \frac{2}{x^3}$

Evaluating Arithmetic Combinations of Functions In Exercises 13–24, evaluate the function for

$f(x) = 2x + 1$ and $g(x) = x^2 - 2$.

13. $(f+g)(3)$
14. $(f-g)(-2)$
15. $(f-g)(2t)$
16. $(f+g)(3t)$
17. $(fg)(-2)$
18. $(fg)(-6)$
19. $\left(\frac{f}{g}\right)(5)$
20. $\left(\frac{f}{g}\right)(0)$
21. $(f-g)(0)$
22. $(f+g)(1)$
23. $(4f)(2) - (2g)(4)$
24. $(2f)(5) + (3g)(-4)$

Composition of Functions In Exercises 25–28, find (a) $f \circ g$, (b) $g \circ f$, and (c) $f \circ f$. *See Example 4.*

25. $f(x) = 3x, \quad g(x) = 2x + 5$
26. $f(x) = 2x - 1, \quad g(x) = 7 - x$
27. $f(x) = x^2, \quad g(x) = 3x + 1$
28. $f(x) = x^3, \quad g(x) = \frac{1}{x}$

Composition of Functions In Exercises 29–36, find (a) $f \circ g$ and (b) $g \circ f$. *See Example 4.*

29. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 1$

30. $f(x) = \frac{1}{2}x + 1, \quad g(x) = 2x + 3$

31. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2 - 4$

32. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$

33. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$

34. $f(x) = x^2, \quad g(x) = x^2$

35. $f(x) = |x|, \quad g(x) = x + 6$

36. $f(x) = x - 4, \quad g(x) = |x|$

Finding the Domain of a Composite Function In Exercises 37–40, determine the domain of (a) f, (b) g, and (c) $f \circ g$. *See Example 5.*

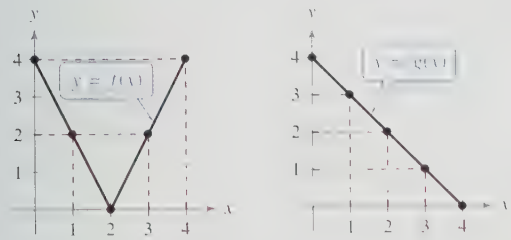
37. $f(x) = x^2 + 3, \quad g(x) = \sqrt{x}$

38. $f(x) = \sqrt[3]{x + 1}, \quad g(x) = x^3$

39. $f(x) = \dfrac{1}{x^2}, \quad g(x) = x - 2$

40. $f(x) = \dfrac{5}{x^2 - 4}, \quad g(x) = x + 3$

Evaluating Combinations of Functions In Exercises 41–44, use the graphs of f and g to evaluate the functions.



41. (a) $(f + g)(3)$ (b) $\left(\dfrac{f}{g}\right)(2)$

42. (a) $(f - g)(1)$ (b) $(fg)(4)$

43. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

44. (a) $(f \circ g)(0)$ (b) $(g \circ f)(3)$

Identifying a Composite Function In Exercises 45–52, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.) *See Example 6.*

45. $h(x) = (2x + 1)^2$

46. $h(x) = (1 - x)^3$

47. $h(x) = \sqrt[3]{x^2 - 4}$

48. $h(x) = \sqrt{9 - x}$

49. $h(x) = \dfrac{1}{x + 2}$

50. $h(x) = \dfrac{4}{(5x + 2)^2}$

51. $h(x) = (x + 4)^2 + 2(x + 4)$

52. $h(x) = (x + 3)^{3/2}$

53. Stopping Distance While driving at x miles per hour, you are required to stop quickly to avoid an accident. The distance the car travels (in feet) during your reaction time is given by $R(x) = \frac{3}{4}x$. The distance the car travels (in feet) while you are braking is given by $B(x) = \frac{1}{15}x^2$. Find the function that represents the total stopping distance T. (*Hint:* $T = R + B$.) Graph the functions R, B, and T on the same set of coordinate axes for $0 \le x \le 60$.

54. Cost The weekly cost C of producing x units in a manufacturing process is given by the function

$$C(x) = 70x + 800.$$

The number of units x produced in t hours is given by

$$x(t) = 40t.$$

Find and interpret $(C \circ x)(t)$.

55. Cost The weekly cost C of producing x units in a manufacturing process is given by the function

$$C(x) = 50x + 495.$$

The number of units x produced in t hours is given by

$$x(t) = 30t.$$

Find and interpret $(C \circ x)(t)$.

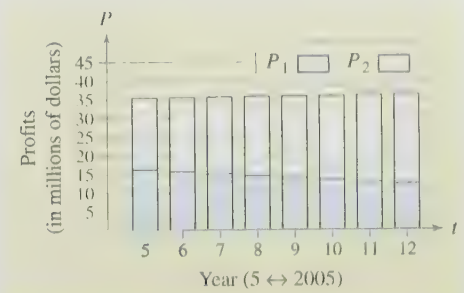
56. Profits A manufacturing company has two plants, one in New Jersey and another in California. From 2005 through 2012, the profits for the plant in New Jersey decreased according to the function

$$P_1 = 18.97 - 0.55t, \quad t = 5, 6, 7, 8, 9, 10, 11, 12$$

and the profits for the plant in California increased according to the function

$$P_2 = 15.85 + 0.67t, \quad t = 5, 6, 7, 8, 9, 10, 11, 12$$

where P_1 and P_2 represent the yearly profits (in millions of dollars) and t represents the year, with $t = 5$ corresponding to 2005 (see figure).



(a) Use the stacked bar graph to determine whether the company's total yearly profits for both plants decreased or increased from 2005 through 2012.

(b) Write a function that represents the total yearly profits for both plants from 2005 through 2012. Explain how the function supports your answer in part (a).

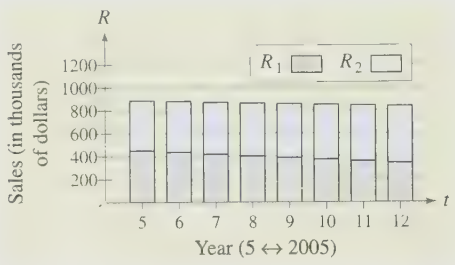
57. Sales You own two fast-food restaurants. From 2005 through 2012, the sales of one restaurant decreased according to the function

$$R_1 = 525 - 15.2t, \quad t = 5, 6, 7, 8, 9, 10, 11, 12$$

and the sales of the other restaurant increased according to the function

$$R_2 = 392 + 8.5t, \quad t = 5, 6, 7, 8, 9, 10, 11, 12$$

where R_1 and R_2 represent the yearly sales (in thousands of dollars) and t represents the year, with $t = 5$ corresponding to 2005 (see figure).



(a) Use the stacked bar graph to determine whether your total yearly sales of both restaurants decreased or increased from 2005 through 2012.

(b) Write a function that represents your total yearly sales of both restaurants from 2005 through 2012. Explain how the function supports your answer in part (a).

58. Cost, Revenue, and Profit The table shows the revenues y_1 (in thousands of dollars) and total costs y_2 (in thousands of dollars) for a sports memorabilia store for the years 2003 through 2012.

Year	y_1	y_2
2003	46.3	32.9
2004	51.3	36.5
2005	55.9	39.9
2006	60.8	43.8
2007	65.9	46.9
2008	71.0	51.1
2009	75.7	53.7
2010	80.8	57.6
2011	85.6	62.1
2012	90.7	68.7

(a) Use the *regression* feature of a graphing utility to find linear models for y_1 and y_2. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the graphing utility to graph the models for

$$y_1, \quad y_2, \quad \text{and} \quad y_3 = y_1 - y_2$$

in the same viewing window. What does y_3 represent in the context of the problem? Predict the value of y_3 in 2014.

(c) Create a stacked bar graph for y_2 and y_3. What do the heights of the bars represent?

59. Employment The table shows the numbers of employees in the offices of health care practitioners from 2000 through 2009. The numbers of employees (in thousands) in the offices of physicians, dentists, and other health care practitioners are represented by y_1, y_2, and y_3, respectively. *(Source: U.S. Bureau of Labor Statistics)*

Year	y_1	y_2	y_3
2000	1840	688	438
2001	1911	705	457
2002	1968	725	486
2003	2003	744	503
2004	2048	760	527
2005	2094	774	549
2006	2148	786	573
2007	2202	808	600
2008	2253	818	627
2009	2280	818	651

(a) Create a stacked bar graph for the data.

(b) Use the *regression* feature of a graphing utility to find linear models for y_1, y_2, and y_3. Let t represent the year, with $t = 0$ corresponding to 2000.

(c) Use the graphing utility to graph the models for

$$y_1, \quad y_2, \quad y_3, \quad \text{and} \quad y_4 = y_1 + y_2 + y_3$$

in the same viewing window. Use y_4 to predict the total number of employees in the offices of health care practitioners in 2011 and in 2013.

60. Bacteria Count A package of food is removed from a refrigerator. The number N of bacteria in the food is given by

$$N(T) = 10T^2 - 20T + 600, \quad 1 \le T \le 16$$

where T is the temperature (in degrees Celsius) of the food. The temperature of the food increases according to the function

$$T(t) = 3t + 1, \quad 0 \le t \le 5$$

where t is the time in hours.

(a) Find the composite function $N(T(t))$.

(b) Find the number of bacteria in the food when $t = 2$ hours.

(c) Find the time t when the bacteria count reaches 1500.

61. Ripples in Water A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outermost ripple is given by

$r(t) = 0.6t$

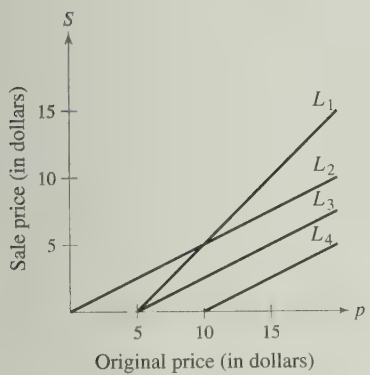
where t is time in seconds after the pebble strikes the water. The area of the outermost circle is given by the function

$A(r) = \pi r^2$.

Find and interpret

$(A \circ r)(t)$.

62. HOW DO YOU SEE IT? The graphs labeled L_1, L_2, L_3, and L_4 represent four different pricing discounts, where p is the original price (in dollars) and S is the sale price (in dollars). Match each function with its graph. Describe the situations in parts (c) and (d).



(a) $f(p)$: A 50% discount is applied.

(b) $g(p)$: A \$5 discount is applied.

(c) $(g \circ f)(p)$

(d) $(f \circ g)(p)$

Price-Earnings Ratio The average annual price-earnings ratio for a corporation's stock is defined as the average price of the stock divided by the earnings per share. In Exercises 63 and 64, the average price of a corporation's stock is given as the function P, and the earnings per share is given as the function E. Find the price-earnings ratios, P/E, for the years 2005 through 2009.

63. *DISH Network*

Year	2005	2006	2007	2008	2009
P	\$29.06	\$31.92	\$43.51	\$25.15	\$16.05
E	\$1.74	\$1.52	\$1.68	\$1.98	\$1.42

(Source: DISH Network)

64. *Walt Disney*

Year	2005	2006	2007	2008	2009
P	\$26.72	\$27.53	\$34.18	\$32.09	\$22.75
E	\$1.31	\$1.61	\$1.92	\$2.26	\$1.82

(Source: Walt Disney)

65. Think About It Find the domains of $(f/g)(x)$ and $(g/f)(x)$ for the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{9 - x^2}$. Why do the two domains differ?

True or False? In Exercises 66 and 67, determine whether the statement is true or false. Justify your answer.

66. If $f(x) = x + 1$ and $g(x) = 6x$, then

$(f \circ g)(x) = (g \circ f)(x)$.

67. Given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.



AP Photo/Jeff Chiu

Business Capsule

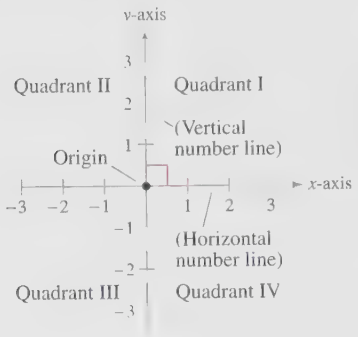
SunPower Corporation develops and manufactures solar-electric power products. SunPower's high-efficiency solar cells generate up to 50% more power than other solar technologies. SunPower's technology was developed by Dr. Richard Swanson and his students while he was Professor of Engineering at Stanford University. From 2005 to 2010, SunPower's revenues increased by almost 3000%.

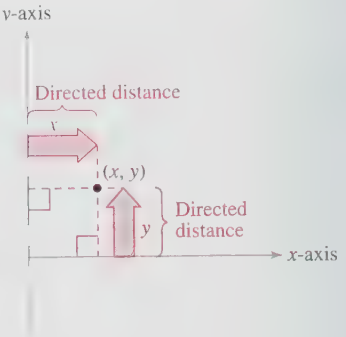
68. Research Project Use your campus library, the Internet, or some other reference source to find information about an alternative-energy business experiencing strong growth similar to the example above. Write a brief report about the business.

ALGEBRA TUTOR

Identifying and Plotting Points in the Cartesian Plane

To be successful at graphing equations, you must be good at plotting points in the Cartesian plane. The Cartesian plane is formed by the intersection of a horizontal real number line called the x-axis and a vertical real number line called the y-axis. These axes intersect at a point called the origin and divide the plane into four quadrants, as shown below.





STUDY TIP

Remember that a point on an axis is not considered to be in any of the four quadrants.

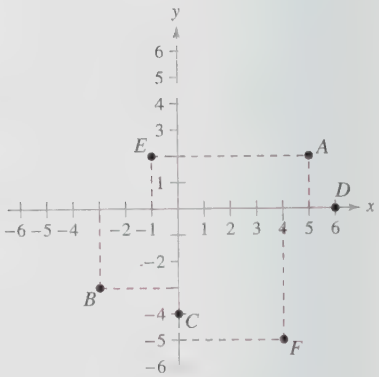
Each point in the plane corresponds to an ordered pair (x, y) of real numbers x and y, called the coordinates of the point. The x-coordinate represents the directed distance from the y-axis to the point, and the y-coordinate represents the directed distance from the x-axis to the point, as shown above.

Example 1 Identifying Points in the Cartesian Plane

Give the coordinates of each point.

a. A **b.** B **c.** C

d. D **e.** E **f.** F



SOLUTION

a. Point A is 5 units to the right of the origin and 2 units up. So, the x-coordinate is 5 and the y-coordinate is 2. The coordinates are $(5, 2)$.

b. Point B is 3 units to the left of the origin and 3 units down. So, the x-coordinate is -3 and the y-coordinate is -3. The coordinates are $(-3, -3)$.

c. Point C is 4 units down from the origin. So, the x-coordinate is 0 and the y-coordinate is -4. The coordinates are $(0, -4)$.

d. Point D is 6 units to the right of the origin. So, the x-coordinate is 6 and the y-coordinate is 0. The coordinates are $(6, 0)$.

e. Point E is 1 unit to the left of the origin and 2 units up. So, the x-coordinate is -1 and the y-coordinate is 2. The coordinates are $(-1, 2)$.

f. Point F is 4 units to the right of the origin and 5 units down. So, the x-coordinate is 4 and the y-coordinate is -5. The coordinates are $(4, -5)$.

Example 2 Plotting Points in the Cartesian Plane

Plot each point in the Cartesian plane.

a. $P(7, -2)$ **b.** $Q(-1, -6)$

c. $R(0, 4)$ **d.** $S(-2, 0)$

e. $T(8, 3)$ **f.** $U(-3, 5)$

g. $V\left(-\frac{3}{2}, \frac{7}{2}\right)$ **h.** $W\left(4, \frac{1}{2}\right)$

i. $X(2.5, -6.5)$

SOLUTION

a. Start at the origin. The x-coordinate is 7, so move right 7 units. The y-coordinate is -2, so move down 2 units. Plot the point at $(7, -2)$ and label it P.

b. Start at the origin. The x-coordinate is -1, so move left 1 unit. The y-coordinate is -6, so move down 6 units. Plot the point at $(-1, -6)$ and label it Q.

c. Start at the origin. The y-coordinate is 4, so move up 4 units. Plot the point at $(0, 4)$ and label it R.

d. Start at the origin. The x-coordinate is -2, so move left 2 units. Plot the point at $(-2, 0)$ and label it S.

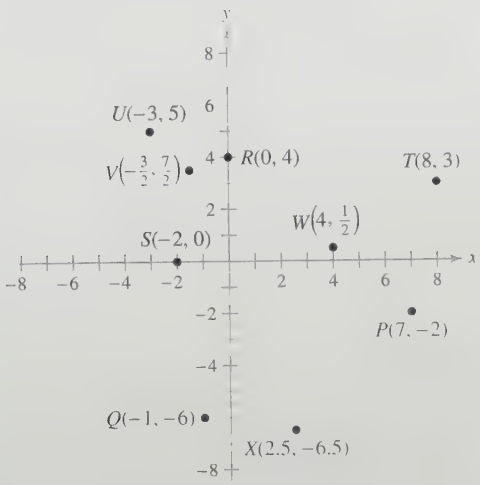
e. Start at the origin. The x-coordinate is 8, so move right 8 units. The y-coordinate is 3, so move up 3 units. Plot the point at $(8, 3)$ and label it T.

f. Start at the origin. The x-coordinate is -3, so move left 3 units. The y-coordinate is 5, so move up 5 units. Plot the point at $(-3, 5)$ and label it U.

g. Start at the origin. The x-coordinate is $-\frac{3}{2}$, so move left $\frac{3}{2}$ units. The y-coordinate is $\frac{7}{2}$, so move up $\frac{7}{2}$ units. Plot the point at $\left(-\frac{3}{2}, \frac{7}{2}\right)$ and label it V.

h. Start at the origin. The x-coordinate is 4, so move right 4 units. The y-coordinate is $\frac{1}{2}$, so move up $\frac{1}{2}$ unit. Plot the point at $\left(4, \frac{1}{2}\right)$ and label it W.

i. Start at the origin. The x-coordinate is 2.5, so move right 2.5 units. The y-coordinate is -6.5, so move down 6.5 units. Plot the point at $(2.5, -6.5)$ and label it X.



SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 244. Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 2.1

Review Exercises

- Plot points in the Cartesian plane, find the distance between two points, and find the midpoint of a line segment joining two points. — *1–6*

  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

  $\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

- Determine whether a point is a solution of an equation. — *7, 8*
- Sketch the graph of an equation. — *9, 10*
- Find the x- and y-intercepts, and determine the symmetry, of the graph of an equation. — *11–16*
- Write the equation of a circle in standard form. — *17–20*

  $(x - h)^2 + (y - k)^2 = r^2$

Section 2.2

- Find the slope of a line passing through two points. — *21–24*

  $m = \frac{y_2 - y_1}{x_2 - x_1}$

- Use the point-slope form to find the equation of a line. — *25–28*

  $y - y_1 = m(x - x_1)$

- Use the slope-intercept form to sketch the graph of a line. — *29–32*

  $y = mx + b$

- Use slope to determine whether lines are parallel or perpendicular, and write the equation of a line parallel or perpendicular to a given line. — *33–42*

  Parallel lines: $m_1 = m_2$

  Perpendicular lines: $m_1 = -\frac{1}{m_2}$

Section 2.3

- Construct and use a linear model to relate quantities that vary directly. — *43–52*

  Direct variation: $y = mx$

- Construct and use a linear model with slope as the rate of change. — *53–55*
- Use a scatter plot to find a linear model that fits a set of data. — *56*

* A wide range of valuable study aids are available to help you master the material in this chapter. The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *www.cengagebrain.com* offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

Section 2.4

	Review Exercises
■ Determine whether a correspondence is a function.	*57–62*
■ Use function notation, evaluate a function, and find the domain of a function.	*63–71*
■ Write a function that relates quantities in an application problem.	*72–74*

Section 2.5

■ Find the domain and range of a function using its graph.	*75–78*
■ Identify the graph of a function using the Vertical Line Test.	*79–82*
■ Describe the increasing and decreasing behavior of a function.	*75–78*
■ Find the relative minima and relative maxima of the graph of a function.	*75–78*
■ Sketch the graph of a step function.	*85, 86*
■ Classify a function as even or odd.	*75–78*
In an even function, $f(-x) = f(x)$.	
In an odd function, $f(-x) = -f(x)$.	
■ Identify six common graphs and use them to sketch the graphs of functions.	*83–90*

Section 2.6

■ Use vertical and horizontal shifts, reflections, and nonrigid transformations to sketch graphs of functions.	*91–98*
Vertical shifts: $h(x) = f(x) + c$ or $h(x) = f(x) - c$	
Horizontal shifts: $h(x) = f(x - c)$ or $h(x) = f(x + c)$	
Reflection in the x-axis: $h(x) = -f(x)$	
Reflection in the y-axis: $h(x) = f(-x)$	
Nonrigid transformation: $h(x) = cf(x)$	

Section 2.7

■ Find the sum, difference, product, and quotient of two functions.	*99–104*
■ Form the composition of two functions and determine its domain.	*105–108*
■ Identify a function as the composition of two functions.	*109–112*
■ Use combinations and compositions of functions to solve application problems.	*113–115*

Study Strategies

■ **To Memorize or Not To Memorize?** When studying mathematics, you often need to memorize formulas, rules, and properties. The formulas that you use most often can become committed to memory through practice. Some formulas, however, are used infrequently or may be easily forgotten. When you are unsure of a formula, you may be able to *derive* it using other information that you know. For instance, if you forget the standard form of the equation of a circle, you can use the Distance Formula and properties of a circle to derive it, as shown on page 166. If you also forget the Distance Formula, you can depict the distance between two generic points graphically and use the Pythagorean Theorem to derive the formula, as shown on page 159.

■ **Choose Convenient Values for Yearly Data** When you work with data involving years, you may want to reassign simpler values to represent the years. For instance, you might represent the years 1993 through 2010 by the x-values 3 through 10. If you sketch a graph of these data, be sure to account for this in the x-axis title: Year ($3 \leftrightarrow 1993$).

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Using the Distance and Midpoint Formulas In Exercises 1–4, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(3, 2), (-3, -5)$
2. $(-9, 3), (5, 7)$
3. $(3.45, 6.55), (-1.06, -3.87)$
4. $(-6.7, -3.9), (5.1, 8.2)$

Using the Distance Formula In Exercises 5 and 6, find the values of x for which the distance between the points is 25.

5. $(10, 10), (x, -5)$
6. $(x, -5), (-15, 10)$

Solution of an Equation In Exercises 7 and 8, determine whether each point is a solution of the equation.

7. $y = 2x^2 - 7x - 15$
   (a) $(5, 0)$ (b) $(-2, 7)$
8. $y = \sqrt{16 - x^2}$
   (a) $(1, 5)$ (b) $(4, 0)$

Sketching the Graph of an Equation In Exercises 9 and 10, complete the table. Use the resulting solution points to sketch the graph of the equation.

9. $y = -\frac{1}{2}x + 2$

x	-2	0	2	3	4
y					

10. $y = x^2 - 3x$

x	-1	0	1	2	3
y					

Sketching the Graph of an Equation In Exercises 11–16, sketch the graph of the equation. Identify any intercepts and test for symmetry.

11. $y = x^2 + 3$
12. $y^2 = x$
13. $y = 3x - 4$
14. $y = \sqrt{9 - x}$
15. $y = x^3 + 1$
16. $y = |x - 3|$

Finding the Equation of a Circle In Exercises 17 and 18, find the standard form of the equation of the circle.

17. Center: $(-1, 2)$; radius: 6
18. Endpoints of a diameter: $(-2, -3), (4, 5)$

Completing the Square to Sketch a Circle In Exercises 19 and 20, write the equation of the circle in standard form. Then sketch the circle.

19. $x^2 + y^2 - 4x + 6y - 12 = 0$
20. $4x^2 + 4y^2 - 4x - 8y - 11 = 0$

Finding the Slope of a Line Through Two Points In Exercises 21–24, plot the points and find the slope of the line passing through the points.

21. $(3, 7), (2, -1)$
22. $(3, -2), (-1, -2)$
23. $(3, 4), (3, -2)$
24. $(-1, 5), (2, -3)$

The Point-Slope Form of the Equation of a Line In Exercises 25–28, find an equation of the line that passes through the point and has the indicated slope. Then sketch the graph of the line.

	Point	*Slope*
25.	$(0, -5)$	$m = \frac{3}{2}$
26.	$(3, 0)$	$m = -\frac{1}{3}$
27.	$(-2, 6)$	$m = 0$
28.	$(5, 4)$	m is undefined.

Sketching the Graph of a Linear Equation In Exercises 29–32, find the slope and y-intercept (if possible) of the line specified by the equation. Then sketch the graph of the line.

29. $8x - 4y + 16 = 0$
30. $3y - 2 = 0$
31. $17 - 5x = 10$
32. $16x + 12y - 24 = 0$

Parallel and Perpendicular Lines In Exercises 33–36, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

33. L_1: $(0, 3), (-2, 1)$; L_2: $(-8, -3), (4, 9)$
34. L_1: $(-1, 0), (5, 5)$; L_2: $(2, 1), (8, 6)$
35. L_1: $(3, 6), (-1, -5)$; L_2: $(-2, 3), (4, 7)$
36. L_1: $(-1, 2), (-1, 4)$; L_2: $(7, 3), (4, 7)$

Equations of Parallel and Perpendicular Lines In Exercises 37–42, write equations of the lines passing through the given point that are (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*
37.	$(4, 3)$	$y = \frac{1}{2}x - 1$
38.	$(1, -4)$	$y = -2x + 3$
39.	$(3, -2)$	$5x - 4y = 8$

	Point	*Line*
40.	$(-8, 3)$	$2x + 3y = 5$
41.	$(-1, -2)$	$y = 2$
42.	$(0, 5)$	$x = -3$

Direct Variation In Exercises 43–46, y is proportional to x. Use the x- and y-values to find a linear model that relates y and x.

43. $x = 3, y = 7$

44. $x = 5, y = 7.5$

45. $x = 10, y = 3480$

46. $x = 14, y = 1.95$

Direct Variation In Exercises 47–50, write a linear model that relates the variables.

47. A varies directly as r; $A = 30$ when $r = 6$.

48. y varies directly as z; $y = 7$ when $z = 14$.

49. a is proportional to b; $a = 15$ when $b = 20$.

50. m varies directly as n; $m = 12$ when $n = 36$.

51. Property Tax The property tax in a city is based on the assessed value of the property. A house that has an assessed value of \$150,000 has a property tax of \$1260. Find a mathematical model that gives the amount of property tax y in terms of the assessed value of the property x. Use the model to find the property tax on a house that has an assessed value of \$175,000.

52. Feet and Meters You are driving and you notice a billboard that indicates it is 1000 feet or 305 meters to the next restaurant of a national fast-food chain. Use this information to find a linear model that relates feet to meters. Use the model to complete the table.

Feet	20	50	100	120
Meters				

53. Fourth-Quarter Sales During the second and third quarters of the year, a business had sales of \$2,950,000 and \$3,150,000, respectively. Assume the growth of the sales follows a linear pattern. What will sales be during the fourth quarter?

54. Dollar Value The dollar value of a product in 2011 is \$75 and the item is expected to increase in value at a rate of \$4.75 per year. Write a linear equation that gives the dollar value of the product in terms of the year. Use this model to predict the dollar value of the product in 2013. (Let $t = 11$ represent 2011.)

55. Straight-Line Depreciation A small business purchases a piece of equipment for \$135,000. After 10 years, the equipment will have to be replaced. Its salvage value at that time is expected to be \$5500. Write a linear equation giving the value V of the equipment during the 10 years it will be used.

56. Sales The sales S (in millions of dollars) for Intuit Corporation for the years 2005 through 2010 are shown in the table. *(Source: Intuit Corporation)*

Year	Sales S (in millions of dollars)
2005	2079.9
2006	2342.3
2007	2672.9
2008	3071.0
2009	3182.5
2010	3455.0

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 5$ corresponding to 2005. Do the data appear linear?

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the linear model from part (b) to predict sales in 2011.

(d) Intuit Corporation predicts sales of \$3820 million for 2011. Does your prediction from part (c) agree with that of Intuit Corporation? Which prediction do you think is more reasonable? Explain.

Testing for Functions Represented by Equations In Exercises 57–60, decide whether the equation represents y as a function of x.

57. $3x + y = 12$

58. $y^2 = x^2 - 9$

59. $y^2 = x + 3$

60. $x^2 + y^2 - 6x + 8y = 0$

Testing for Functions In Exercises 61 and 62, decide whether the set of ordered pairs represents a function from A to B.

$A = \{1, 2, 3\}$ $B = \{-3, -4, -7\}$

Give a reason for your answer.

61. $\{(1, -3), (2, -7), (3, -3)\}$

62. $\{(1, -4), (2, -3), (3, -9)\}$

Evaluating a Function In Exercises 63 and 64, evaluate the function at each specified value of the independent variable and simplify.

63. $f(x) = \sqrt{x + 4} - 5$

(a) $f(5)$ (b) $f(0)$ (c) $f(-4)$ (d) $f(x + 3)$

64. $f(x) = \begin{cases} 2x - 1, & x \leq 1 \\ x^2 + 2, & x > 1 \end{cases}$

(a) $f(0)$ (b) $f(1)$ (c) $f(3)$ (d) $f(-4)$

Finding the Domain of a Function In Exercises 65–70, find the domain of the function.

65. $f(x) = 2x^2 + 7x + 3$

66. $g(t) = 3t^2 + 1$

67. $h(x) = \sqrt{x + 5}$

68. $f(t) = \sqrt[3]{t - 3}$

69. $g(t) = \dfrac{\sqrt{t - 1}}{t - 4}$

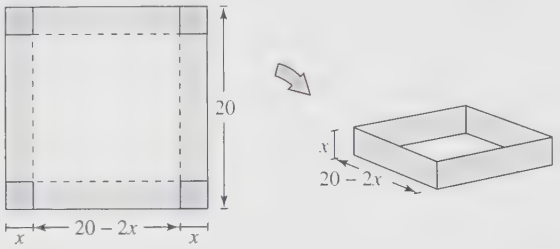
70. $h(x) = \dfrac{2}{x^2 - 4}$

71. Reasoning A student has difficulty understanding why the domains of

$$h(x) = \frac{x^2 - 4}{x} \quad \text{and} \quad k(x) = \frac{x}{x^2 - 4}$$

are different. How would you explain their respective domains algebraically? How could you use a graphing utility to explain their domains?

72. Volume of a Box An open box is to be made from a square piece of material 20 inches on a side by cutting equal squares from the corners and turning up the sides (see figure).



(a) Write the volume V of the box as a function of its height x.

(b) What is the domain of the function?

(c) Use a graphing utility to graph the function.

(d) Determine the volume of the box with a height of 3 inches.

73. Balance in an Account A person deposits $6500 in an account that pays 5.75% interest compounded quarterly.

(a) Write the balance of the account in terms of the time t that the principal is left in the account.

(b) What is the domain of this function?

74. Vertical Motion The velocity v (in feet per second) of a ball thrown vertically upward from ground level is given by

$$v(t) = -32t + 80$$

where t is the time (in seconds).

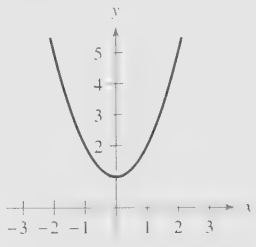
(a) Find the velocity when $t = 1$.

(b) Find the time when the ball reaches its maximum height. [*Hint:* Find the time when $v(t) = 0$.]

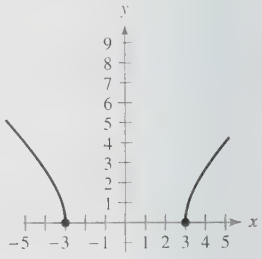
(c) Find the velocity when $t = 3$.

Describing Function Behavior In Exercises 75–78, (a) determine the domain and range of the function, (b) determine the intervals over which the function is increasing, decreasing, or constant, (c) determine whether the function is even, odd, or neither, and (d) approximate any relative minimum or relative maximum values of the function.

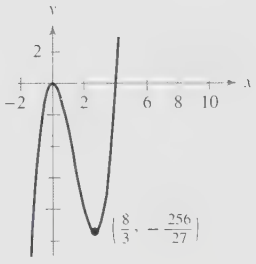
75. $f(x) = x^2 + 1$



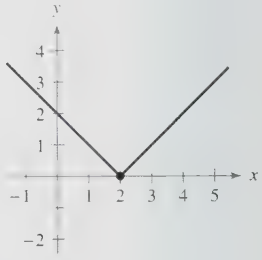
76. $f(x) = \sqrt{x^2 - 9}$



77. $f(x) = x^3 - 4x^2$

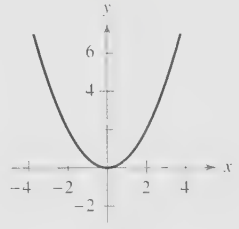


78. $f(x) = |x - 2|$

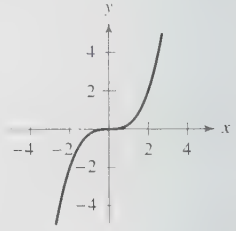


Using the Vertical Line Test In Exercises 79–82, use the Vertical Line Test to decide whether y is a function of x.

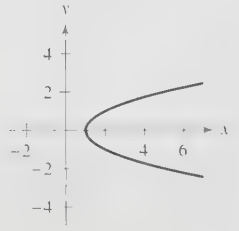
79. $y = \frac{1}{2}x^2$



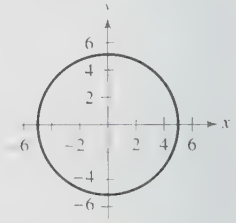
80. $y = \frac{1}{4}x^3$



81. $x - y^2 = 1$



82. $x^2 + y^2 = 25$



Sketching Graphs In Exercises 83–90, sketch the graph of the function.

83. $f(x) = |x + 3|$

84. $g(x) = \sqrt{x^2 - 16}$

85. $h(x) = -[\![x]\!] + 1$

86. $f(x) = 2[\![x]\!] + 2$

87. $g(x) = \begin{cases} x + 2, & x < 0 \\ 2, & x = 0 \\ x^2 + 2, & x > 0 \end{cases}$

88. $g(x) = \begin{cases} 3x + 1, & x < -1 \\ x^2 - 3, & x \geq -1 \end{cases}$

89. $h(x) = x^2 - 3x$

90. $f(x) = \sqrt{9 - x^2}$

Describing Transformations In Exercises 91 and 92, describe the sequence of transformations from $f(x) = x^2$ to g. Then sketch the graph of g by hand. Verify with a graphing utility.

91. $g(x) = -(x - 1)^2 - 2$

92. $g(x) = -x^2 + 3$

Describing Transformations In Exercises 93 and 94, describe the sequence of transformations from $f(x) = \sqrt{x}$ to g. Then sketch the graph of g by hand. Verify with a graphing utility.

93. $g(x) = \sqrt{x - 2}$

94. $g(x) = \sqrt{x} + 2$

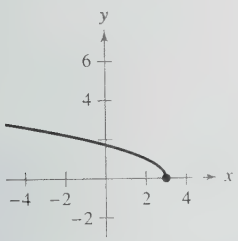
Describing Transformations In Exercises 95 and 96, describe the sequence of transformations from $f(x) = \sqrt[3]{x}$ to g. Then sketch the graph of g by hand. Verify with a graphing utility.

95. $g(x) = \sqrt[3]{x + 2}$

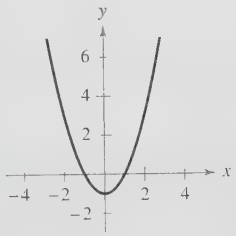
96. $g(x) = 2\sqrt[3]{x} + 1$

Identifying Transformations In Exercises 97 and 98, identify the transformation shown in the graph and identify the associated common function. Write the equation of the graphed function.

97.



98.



Finding Arithmetic Combinations of Functions In Exercises 99 and 100, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

99. $f(x) = 3x - 1, \quad g(x) = x^2 + 2x$

100. $f(x) = 3x, \quad g(x) = \sqrt{x^2 + 1}$

Evaluating Arithmetic Combinations of Functions In Exercises 101–104, evaluate the function for $f(x) = x^2 + 3x$ and $g(x) = 2x - 5$.

101. $(f + g)(2)$

102. $(f - g)(-1)$

103. $(fg)(3)$

104. $\left(\frac{f}{g}\right)(0)$

Composition of Functions In Exercises 105–108, find and determine the domains of (a) $f \circ g$ and (b) $g \circ f$.

105. $f(x) = x^2, \quad g(x) = x + 3$

106. $f(x) = 2x - 5, \quad g(x) = x^2 + 2$

107. $f(x) = \frac{1}{x}, \quad g(x) = 3x + x^2$

108. $f(x) = \frac{1}{x^2}, \quad g(x) = x^3$

Identifying a Composite Function In Exercises 109–112, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

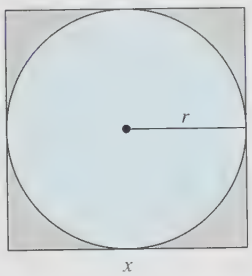
109. $h(x) = (6x - 5)^2$

110. $h(x) = \sqrt[3]{x + 2}$

111. $h(x) = \frac{1}{(x - 1)^2}$

112. $h(x) = (x - 3)^3 + 2(x - 3)$

113. Area A square concrete foundation is prepared as a base for a large cylindrical aquatic tank that is to be used in ecology experiments (see figure).



(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

114. Salary You are a sales representative for an automobile manufacturer. You are paid an annual salary plus a bonus of 4% of your sales over \$500,000. Consider the two functions given by

$f(x) = x - 500{,}000$

and

$g(x) = 0.04x.$

If x is greater than \$500,000, does $f(g(x))$ or $g(f(x))$ represent your bonus? Explain.

115. Bacteria The number N of bacteria in a petri dish is given by $N(T) = 8T^2 - 14T + 200, \ 2 \leq T \leq 32$, where T is the temperature (in degrees Fahrenheit). The temperature is $T(t) = 2t + 2, \ 0 \leq t \leq 15$, where t is the time in hours. Find and interpret $(N \circ T)(t)$.

TEST YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, find the distance between the points and the midpoint of the line segment joining the points.

1. $(-3, 2), (5, -2)$
2. $(3.25, 7.05), (-2.37, 1.62)$

3. Find the x- and y-intercepts of the graph of $y = (x + 5)(x - 3)$.
4. Describe the symmetry of the graph of

$$y = \frac{x}{x^2 - 4}.$$

5. Write the equation of the circle in standard form and sketch its graph.

$$x^2 + y^2 - 6x + 4y - 3 = 0$$

6. Find an equation of the line that passes through $(-3, 5)$ and has a slope of $\frac{2}{3}$.

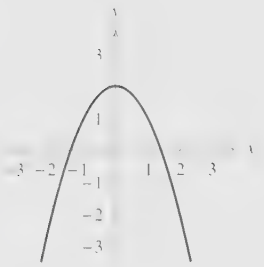


Figure for 9

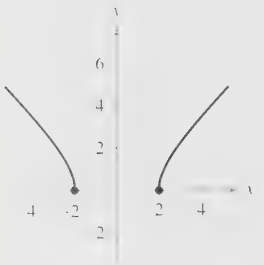


Figure for 10

In Exercises 7 and 8, decide whether the statement is true or false. Explain.

7. The equation $2x - 3y = 5$ represents y as a function of x.
8. If $A = \{3, 4, 5\}$ and $B = \{-1, -2, -3\}$, then the set $\{(3, -9), (4, -2), (5, -3)\}$ represents a function from A to B.

In Exercises 9 and 10, (a) find the domain and range of the function, (b) determine the intervals over which the function is increasing, decreasing, or constant, (c) determine whether the function is even or odd, and (d) approximate any relative minimum or relative maximum values of the function.

9. $f(x) = 2 - x^2$ (See figure.)
10. $g(x) = \sqrt{x^2 - 4}$ (See figure.)

In Exercises 11 and 12, describe the sequence of transformations from $f(x) = x^2$ to g. Then sketch the graph of g by hand.

11. $g(x) = -(x + 2)^2$
12. $g(x) = (x - 3)^2 + 4$

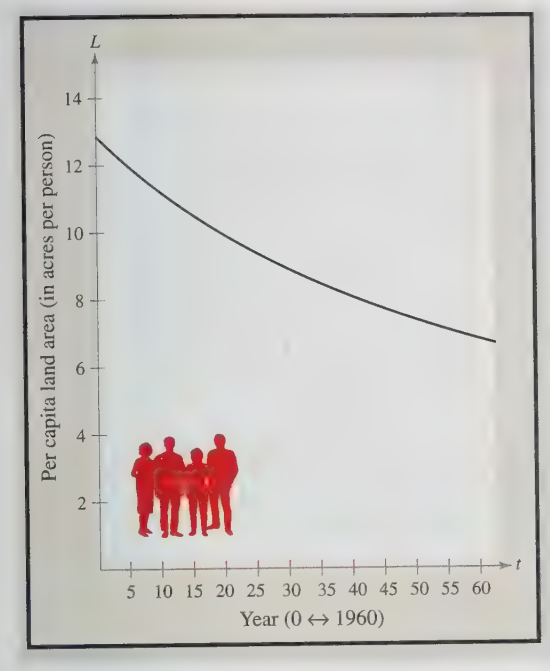
In Exercises 13–16, use $f(x) = x^2 + 1$ and $g(x) = 2x - 2$ to find the combination or composition.

13. $(f - g)(x)$
14. $(fg)(x)$
15. $(f \circ g)(x)$
16. $(g \circ f)(x)$

17. A business purchases a piece of equipment for \$30,000. After 5 years, the equipment will be worth only \$3500. Write a linear equation that gives the value V of the equipment during the 5 years.
18. The projected populations P (in millions) of children under the age of 5 in the United States for selected years from 2015 through 2050 are shown in the table. Use a graphing utility to create a scatter plot of the data and find a linear model for the data. Let t represent the year, with $t = 15$ corresponding to 2015. *(Source: U.S. Census Bureau)*

Year	Population, P (in millions)
2015	22.1
2020	22.8
2025	23.5
2030	24.2
2035	25.1
2040	26.1
2045	27.2
2050	28.1

Table for 18



Example 8 on page 321 shows how a rational function can be used to model the per capita land area of the United States.

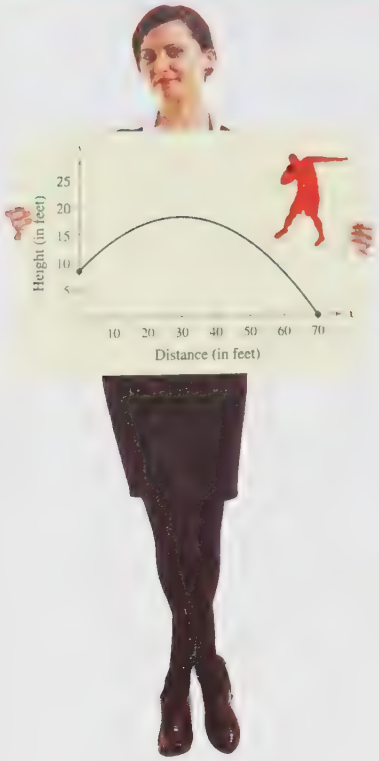
3 Polynomial and Rational Functions

3.1 Quadratic Functions and Models
3.2 Polynomial Functions of Higher Degree
3.3 Polynomial Division
3.4 Real Zeros of Polynomial Functions
3.5 Complex Numbers
3.6 The Fundamental Theorem of Algebra
3.7 Rational Functions



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3.1 Quadratic Functions and Models



In Exercise 56 on page 260, you will use a quadratic function to determine the distance traveled by a shot in a shot put event.

- Sketch the graph of a quadratic function.
- Use the standard form of a quadratic function to sketch the graph of a parabola and find the equation of a parabola given its vertex and a point on its graph.
- Construct and use a quadratic model to solve an application problem.

The Graph of a Quadratic Function

In this and the next section, you will study the graphs of polynomial functions.

Definition of a Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x with degree n.**

For instance, each of the following functions is a polynomial function.

$$f(x) = 3x + 5 \qquad g(x) = \frac{1}{4}x^2 \qquad h(x) = 2x^3 + 7x^2 - 5x + 11$$

Polynomial functions are classified by degree. Recall that the degree of a polynomial is the highest degree of its terms. For instance, the polynomial function given by

$$f(x) = a, \quad a \neq 0 \qquad \text{Constant function}$$

has degree 0 and is called a **constant function.** In Chapter 2, you learned that the graph of this type of function is a horizontal line. The polynomial function given by

$$f(x) = ax + b, \quad a \neq 0 \qquad \text{Linear function}$$

has degree 1 and is called a **linear function.** In Chapter 2, you learned that the graph of the linear function given by $f(x) = ax + b$ is a line whose slope is a and whose y-intercept is $(0, b)$. In this section, you will study second-degree polynomial functions, which are called **quadratic functions.**

For instance, each of the following functions is a quadratic function.

$$f(x) = x^2 + 6x + 2 \qquad g(x) = 2(x + 1)^2 - 3 \qquad h(x) = (x - 2)(x + 1)$$

Definition of a Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function of x given by

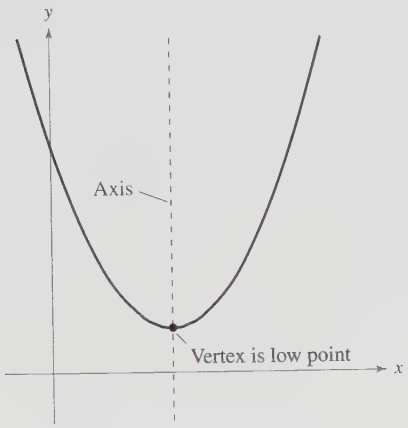
$$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$

is called a quadratic function.

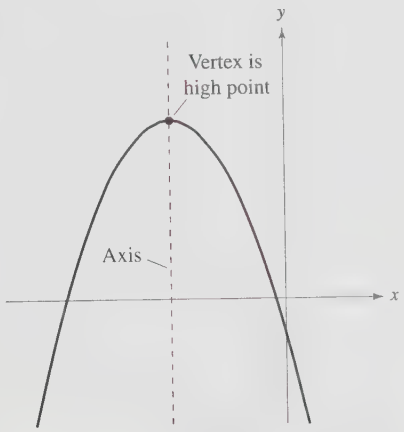
The graph of a quadratic function is called a **parabola.** It is "∪"-shaped and can open upward or downward. Parabolas occur in many real-life applications, such as modeling the path of a baseball or the revenue and production costs of a company. You will learn about such applications later in this section.

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All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point at which the axis intersects the parabola is the **vertex** of the parabola, as shown in Figure 3.1. When the leading coefficient is positive, the graph of $f(x) = ax^2 + bx + c$ is a parabola that opens upward. When the leading coefficient is negative, the graph of $f(x) = ax^2 + bx + c$ is a parabola that opens downward. Later in this section, you will learn ways to find the coordinates of the vertex of a parabola.



$a > 0$: Parabola opens upward.



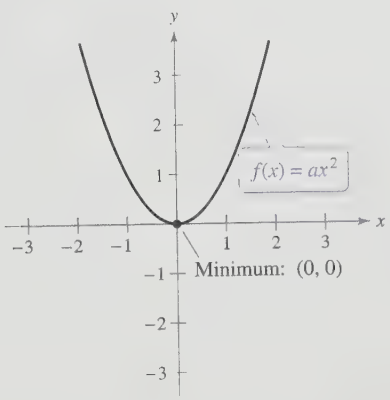
$a < 0$: Parabola opens downward.

FIGURE 3.1

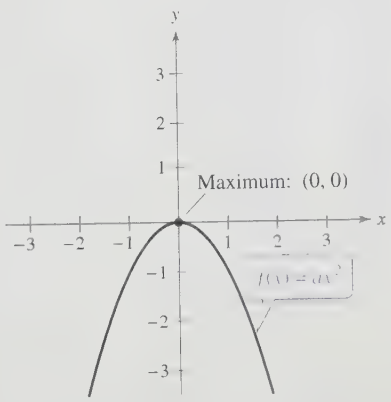
The simplest type of quadratic function is

$f(x) = ax^2$.

Its graph is a parabola whose vertex is $(0, 0)$. When $a > 0$, the vertex is the point with the *minimum* y-value on the graph, and when $a < 0$, the vertex is the point with the *maximum* y-value on the graph, as shown in Figure 3.2.



$a > 0$: Parabola opens upward.



$a < 0$: Parabola opens downward.

FIGURE 3.2

When sketching the graph of

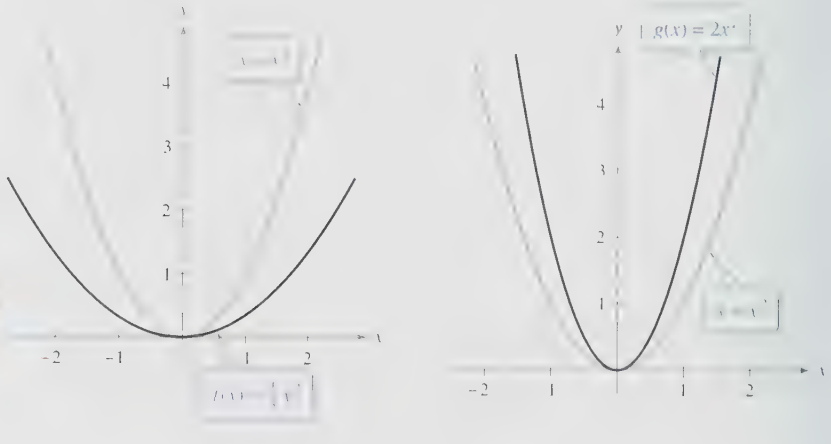
$f(x) = ax^2$

it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 2.6. There you learned that when $a > 1$, the graph of $y = af(x)$ is a vertical stretch of the graph of $y = f(x)$. When $0 < a < 1$, the graph of $y = af(x)$ is a vertical shrink of the graph of $y = f(x)$. This is demonstrated again in Example 1.

Example 1 Sketching the Graph of a Quadratic Function

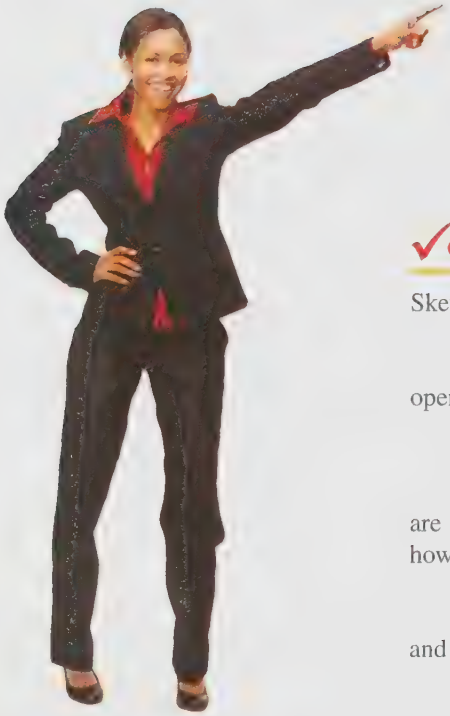
a. Compared with the graph of $y = x^2$, each output of $f(x) = \frac{1}{3}x^2$ vertically "shrinks" the graph by a factor of $\frac{1}{3}$, creating the wider parabola shown in Figure 3.3(a).

b. Compared with the graph of $y = x^2$, each output of $g(x) = 2x^2$ vertically "stretches" the graph by a factor of 2, creating the narrower parabola shown in Figure 3.3(b).



(a) (b)

FIGURE 3.3



✓Checkpoint 1

Sketch the graph of $f(x) = 4x^2$. Then compare this graph with the graph of $y = x^2$.

Note that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens. When $|a|$ is small, the parabola opens more widely than when $|a|$ is large.

Recall from Section 2.6 that the graphs of

$$y = f(x \pm c), \quad y = f(x) \pm c, \quad y = -f(x), \quad \text{and} \quad y = f(-x)$$

are rigid transformations of the graph of $y = f(x)$. For instance, in Figure 3.4, notice how the graph of $y = x^2$ can be transformed to produce the graphs of

$$f(x) = -x^2 + 1$$

and

$$g(x) = (x + 2)^2 - 3.$$

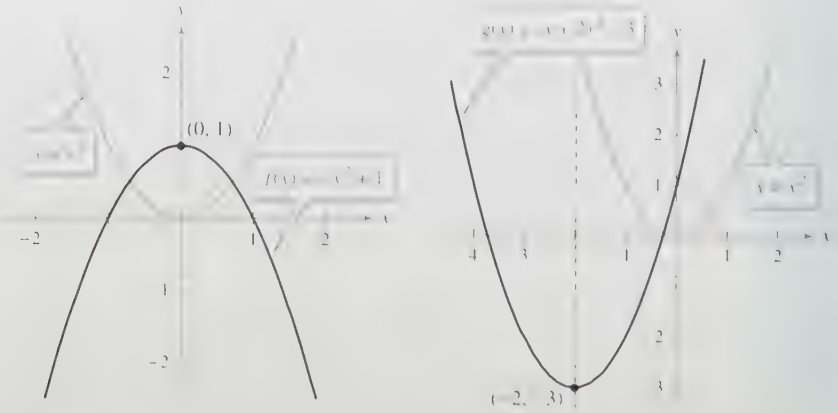


FIGURE 3.4

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The Standard Form of a Quadratic Function

The **standard form** of a quadratic function is

$$f(x) = a(x - h)^2 + k.$$

This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola.

ALGEBRA TUTOR

For help in factoring polynomial expressions, see the *Chapter 3 Algebra Tutor* on page 326.

Standard Form of a Quadratic Function

The quadratic function given by

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is said to be in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k).

When $a > 0$, the parabola opens upward.

When $a < 0$, the parabola opens downward.

To write a quadratic function in standard form, you can use the process of *completing the square*, as illustrated in Example 2.

Example 2 Using Standard Form to Graph a Parabola

Sketch the graph of $f(x) = 2x^2 + 8x + 7$. Identify the vertex.

SOLUTION Begin by writing the quadratic function in standard form. The first step in completing the square is to factor out any coefficient of x^2 that is not 1.

$$\begin{aligned} f(x) &= 2x^2 + 8x + 7 && \text{Write original function.} \\ &= (2x^2 + 8x) + 7 && \text{Group } x\text{-terms.} \\ &= 2(x^2 + 4x) + 7 && \text{Factor 2 out of } x\text{-terms.} \\ &= 2(x^2 + 4x + 4 - 4) + 7 && \text{Add and subtract 4 within parentheses to complete the square.} \\ & \qquad\quad \underbrace{\qquad}_{(\frac{1}{2})^2} \end{aligned}$$

After adding and subtracting 4 within the parentheses, regroup the terms to form a perfect square trinomial. The -4 can be removed from inside the parentheses. But, because of the 2 outside the parentheses, you must multiply -4 by 2 as shown below.

$$\begin{aligned} f(x) &= 2(x^2 + 4x + 4) - 2(4) + 7 && \text{Regroup terms.} \\ &= 2(x^2 + 4x + 4) - 8 + 7 && \text{Simplify.} \\ &= 2(x + 2)^2 - 1 && \text{Standard form} \end{aligned}$$

From this form, you can see that the graph of f is a parabola that opens upward with vertex $(-2, -1)$. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of $y = 2x^2$, as shown in Figure 3.5.

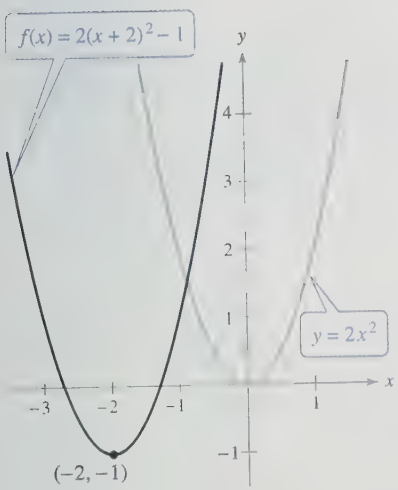


FIGURE 3.5

Checkpoint 2

Sketch the graph of each quadratic function. Identify the vertex.

a. $f(x) = x^2 + 2x + 4$

b. $f(x) = 2x^2 - 12x + 20$

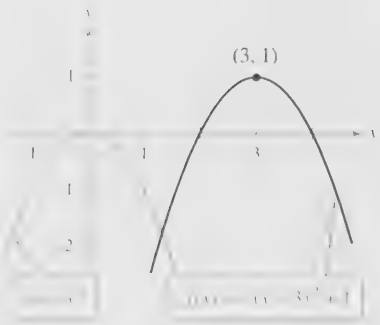


FIGURE 3.6

Example 3 Using Standard Form to Graph a Parabola

Sketch the graph of $f(x) = -x^2 + 6x - 8$. Identify the vertex.

SOLUTION As in Example 2, begin by writing the quadratic function in standard form.

$$\begin{aligned} f(x) &= -x^2 + 6x - 8 && \text{Write original function.} \\ &= (-x^2 + 6x) - 8 && \text{Group } x\text{-terms.} \\ &= -(x^2 - 6x) - 8 && \text{Factor } -1 \text{ out of } x\text{-terms.} \\ &= -(x^2 - 6x + 9 - 9) - 8 && \text{Add and subtract 9 within parentheses to complete the square.} \\ &\qquad\quad\ \uparrow \\ &\qquad\quad (-6/2)^2 \\ &= -(x^2 - 6x + 9) - (-9) - 8 && \text{Regroup terms.} \\ &= -(x - 3)^2 + 1 && \text{Standard form} \end{aligned}$$

So, the graph of f is a parabola that opens downward with vertex at $(3, 1)$, as shown in Figure 3.6.

Checkpoint 3

Sketch the graph of $f(x) = -3x^2 + 12x + 1$. Identify the vertex. ■

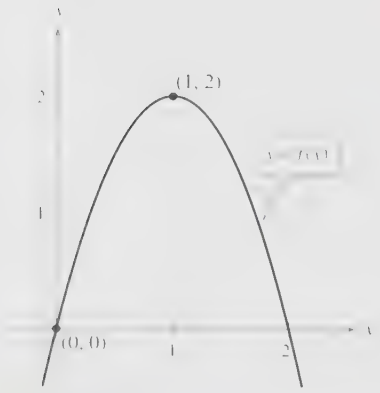


FIGURE 3.7

Example 4 Finding an Equation of a Parabola

Find an equation of the parabola whose vertex is $(1, 2)$ and that passes through the point $(0, 0)$, as shown in Figure 3.7.

SOLUTION Because the parabola has a vertex at $(h, k) = (1, 2)$, the equation must have the form

$$f(x) = a(x - 1)^2 + 2. \qquad \text{Standard form}$$

Because the parabola passes through the point $(0, 0)$, it follows that when $x = 0$, $f(x)$ must equal 0. Substitute 0 for x and 0 for $f(x)$ to obtain the equation

$$0 = a(0 - 1)^2 + 2.$$

By solving this equation for a, you find that $a = -2$. You can now write an equation of the parabola.

$$\begin{aligned} f(x) &= -2(x - 1)^2 + 2 && \text{Substitute for } a, h, \text{ and } k \text{ in standard form.} \\ &= -2x^2 + 4x && \text{Simplify.} \end{aligned}$$

Checkpoint 4

Find an equation of the parabola whose vertex is $(3, 4)$ and that passes through the point $(2, 5)$. ■

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, you must solve the equation

$$ax^2 + bx + c = 0.$$

If the expression $ax^2 + bx + c$ does not factor, then use the Quadratic Formula to determine the x-intercepts. Remember, however, that a parabola may have no x-intercepts.

TECH TUTOR

Your graphing utility may have *minimum* and *maximum* features that determine the minimum and maximum points of the graph of a function. You can use these features to find the vertex of a parabola. For instructions on how to use the *minimum* and *maximum* features, see Appendix A. Consult the user's guide for your graphing utility for more information.

Applications

Many applications involve finding the maximum or minimum value of a quadratic function. By writing $f(x) = ax^2 + bx + c$ in standard form, you can determine that the x-coordinate of the vertex is $-b/2a$. The y-coordinate of the vertex can then be found by evaluating the function at $x = -b/2a$.

Example 5 The Maximum Height of a Baseball

A baseball is hit 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by

$$f(x) = -0.0032x^2 + x + 3$$

where $f(x)$ is the height of the baseball (in feet) and x is the distance from home plate (in feet). What is the maximum height reached by the baseball?

SOLUTION For this quadratic function, you have

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= -0.0032x^2 + x + 3. \end{aligned}$$

So, $a = -0.0032$ and $b = 1$. Because the function has a maximum when $x = -b/2a$, the baseball reaches its maximum height when it is

$$x = -\frac{b}{2a} = -\frac{1}{2(-0.0032)} = 156.25 \text{ feet}$$

from home plate. At this distance, the maximum height is

$$\begin{aligned} f(156.25) &= -0.0032(156.25)^2 + 156.25 + 3 \\ &= 81.125 \text{ feet.} \end{aligned}$$

The path of the baseball is shown in Figure 3.8.

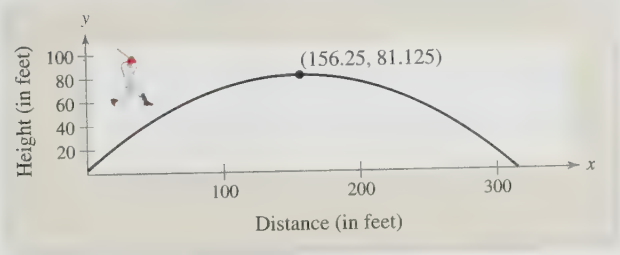


FIGURE 3.8



Checkpoint 5

In Example 5, the baseball is hit at a velocity of 70 feet per second. The path of the baseball is given by

$$f(x) = -0.007x^2 + x + 4$$

where $f(x)$ is the height of the baseball (in feet) and x is the distance from home plate (in feet). What is the maximum height reached by the baseball?

In Section 2.3, you plotted data points in the coordinate plane and estimated the best-fitting line. Fitting a quadratic model using this same process would be complicated. Most graphing utilities have a built-in statistical program that easily calculates the best-fitting quadratic model for a set of data points. Refer to the user's guide of your graphing utility for the required steps.

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Example 6 Fitting a Quadratic Function to Data

Each year, grizzly bears in Yellowstone National Park feed on moths along the mountain slopes of the park. The bears dig into the mountains and eat the moths they uncover. The table shows the number N of confirmed moth sites in the Greater Yellowstone Ecosystem each year from 1986 to 2008. Use a graphing utility to plot the data and find the quadratic model that best fits the data. Find the vertex of the graph of the quadratic model and interpret its meaning in the context of the problem. Let $x = 6$ represent the year 1986. *(Source: Yellowstone Grizzly Bear Investigations 2008)*

x	6	7	8	9	10	11	12	13	14	15	16	17
N	3	5	5	10	14	17	19	19	22	25	26	28

x	18	19	20	21	22	23	24	25	26	27	28
N	30	30	30	31	31	32	32	33	34	35	35

TECH TUTOR

For instructions on how to use the *regression* feature, see Appendix A. Consult the user's guide for your graphing utility for more information.

SOLUTION Begin by entering the data into your graphing utility and displaying the scatter plot. From the scatter plot that is shown in Figure 3.9(a), you can see that the points have a parabolic trend. Use the *quadratic regression* feature to find the quadratic function that best fits the data. The quadratic equation that best fits the data is

$$N = -0.076x^2 + 4.04x - 19.5, \quad 6 \le x \le 28.$$

Graph the data and the equation in the same viewing window, as shown in Figure 3.9(b). By using the *maximum* feature of your graphing utility, you can see that the vertex of the graph is approximately (26.6, 34.2), as shown in Figure 3.9(c). The vertex corresponds to the year in which the number of confirmed moth sites was the greatest. So, in 2006, the number of confirmed moth sites reached a maximum.

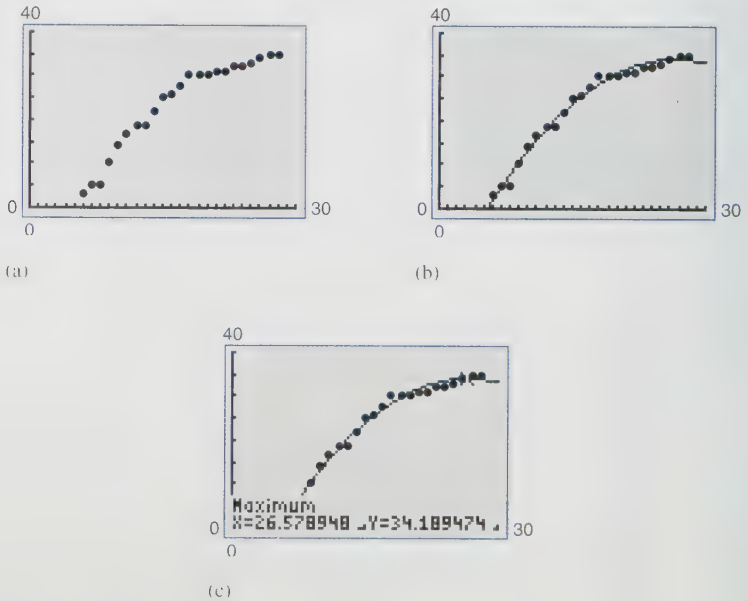


FIGURE 3.9

Checkpoint 6

In Example 6, use the model to predict the number of confirmed moth sites in 2014. What do you notice? ■

Example 7 Charitable Contributions

The percent of a family's income that is given to charities is related to the income of the family. For families with annual incomes between \$5000 and \$100,000, the percent P can be modeled by

$$P(x) = 0.0014x^2 - 0.1529x + 5.855, \quad 5 \le x \le 100$$

where x is the annual income (in thousands of dollars). Use the model to estimate the income that corresponds to the minimum percent of income given to charities.

SOLUTION One way to answer the question is to sketch the graph of the quadratic function, as shown in Figure 3.10. From this graph, it appears that the minimum percent corresponds to an income of about \$55,000.

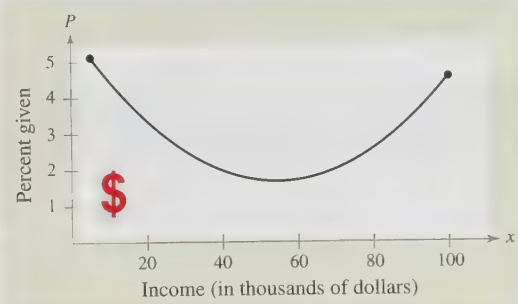


FIGURE 3.10

Another way to answer the question is to use the fact that the minimum point of the parabola occurs when $x = -b/2a$.

$$x = -\frac{b}{2a}$$

$$= -\frac{-0.1529}{2(0.0014)}$$

$$\approx 54.6$$

From this x-value, you can conclude that the minimum percent corresponds to an income of about \$54,600.



Checkpoint 7

A manufacturer has daily production costs C (in dollars per unit) of $C = 0.15x^2 - 9x + 700$, where x is the number of units produced. How many units should be produced each day to yield a minimum cost per unit?

SUMMARIZE (Section 3.1)

1. State the definition of a quadratic function and describe its graph *(pages 250–252)*. For an example of sketching the graph of a quadratic function, see Example 1.
2. State the standard form of a quadratic function *(page 253)*. For examples that use the standard form of a quadratic function, see Examples 2, 3, and 4.
3. Describe a real-life example of how a quadratic function can be used to analyze the path of a moving object *(page 255, Example 5)*.

George Dolgikh/www.shutterstock.com

SKILLS WARM UP 3.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.3 and 1.4.

In Exercises 1–4, solve the quadratic equation by factoring.

1. $2x^2 + 11x - 6 = 0$
2. $5x^2 - 12x - 9 = 0$
3. $3 + x - 2x^2 = 0$
4. $x^2 + 20x + 100 = 0$

In Exercises 5–10, use the Quadratic Formula to solve the quadratic equation.

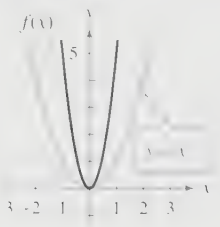
5. $x^2 - 6x + 4 = 0$
6. $x^2 + 4x + 1 = 0$
7. $2x^2 - 16x + 25 = 0$
8. $3x^2 + 30x + 74 = 0$
9. $x^2 + 3x + 1 = 0$
10. $x^2 + 3x - 3 = 0$

Exercises 3.1

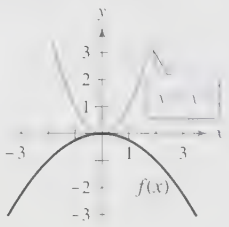
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Describing a Transformation In Exercises 1 and 2, compare the graph of the quadratic function with the graph of $y = x^2$. *See Example 1.*

1. $f(x) = 5x^2$



2. $f(x) = -\frac{1}{4}x^2$

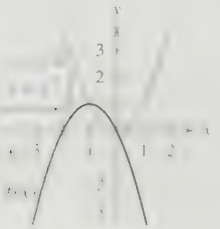


Sketching the Graph of a Quadratic Function In Exercises 3–6, sketch the graph of the quadratic function and compare it with the graph of $y = x^2$. *See Example 1.*

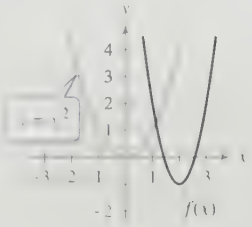
3. $f(x) = 3x^2$
4. $f(x) = -2x^2$
5. $g(x) = -\frac{1}{2}x^2$
6. $g(x) = \frac{3}{4}x^2$

Describing a Transformation In Exercises 7 and 8, compare the graph of the quadratic function with the graph of $y = x^2$.

7. $f(x) = -(x + 1)^2 + 1$

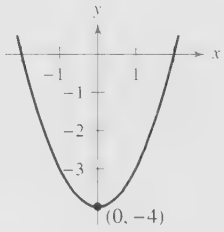


8. $f(x) = 3(x - 2)^2 - 1$

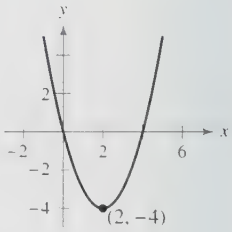


Matching In Exercises 9–14, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

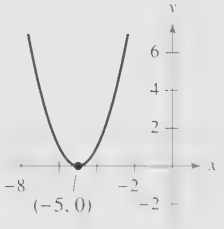
(a)



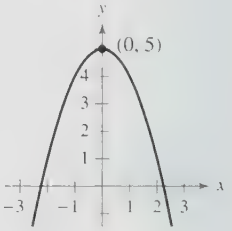
(b)



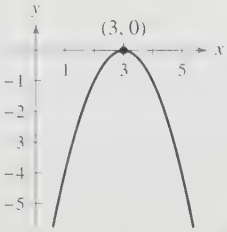
(c)



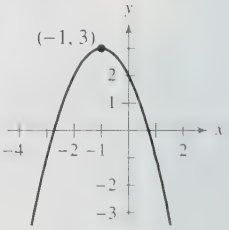
(d)



(e)



(f)



9. $f(x) = -(x - 3)^2$
10. $f(x) = (x + 5)^2$
11. $f(x) = x^2 - 4$
12. $f(x) = 5 - x^2$
13. $f(x) = -(x + 1)^2 + 3$
14. $f(x) = (x - 2)^2 - 4$

Using Standard Form to Graph a Parabola In Exercises 15–30, write the quadratic function in standard form (if necessary) and sketch its graph. Identify the vertex. *See Examples 2 and 3.*

15. $f(x) = 16 - x^2$

16. $h(x) = x^2 - 9$

17. $f(x) = (x + 5)^2 - 6$

18. $f(x) = (x - 6)^2 + 3$

19. $g(x) = x^2 + 2x + 1$

20. $h(x) = x^2 - 4x + 2$

21. $f(x) = -(x^2 + 2x - 3)$

22. $f(x) = -(x^2 + 6x - 3)$

23. $f(x) = x^2 - x + \frac{5}{4}$

24. $f(x) = x^2 + 3x + \frac{1}{4}$

25. $f(x) = -x^2 + 2x + 5$

26. $f(x) = -x^2 - 4x + 1$

27. $h(x) = 4x^2 - 4x + 21$

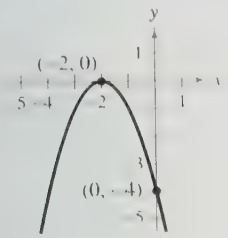
28. $f(x) = 2x^2 - x + 1$

29. $f(x) = \frac{1}{4}(x^2 - 16x + 32)$

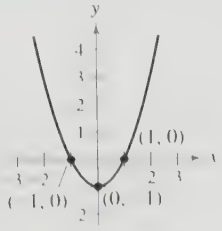
30. $g(x) = \frac{1}{2}(x^2 + 4x - 2)$

Finding an Equation of a Parabola In Exercises 31–36, find an equation of the parabola. *See Example 4.*

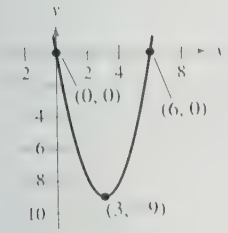
31.



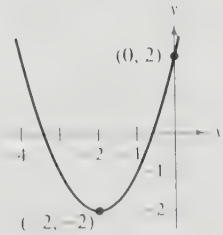
32.



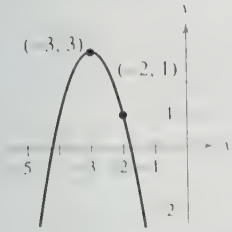
33.



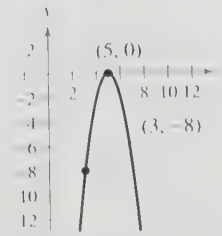
34.



35.



36.



Finding an Equation of a Parabola In Exercises 37–40, find an equation of the parabola that has the indicated vertex and whose graph passes through the given point. *See Example 4.*

37. Vertex: $(2, -1)$; point: $(4, -3)$

38. Vertex: $(-3, 5)$; point: $(-6, -1)$

39. Vertex: $(5, 12)$; point: $(7, 15)$

40. Vertex: $(-2, -2)$; point: $(-1, 0)$

Finding the Slope of a Line In Exercises 41–46, find two quadratic functions whose graphs have the given x-intercepts. Find one function whose graph opens upward and another whose graph opens downward. (There are many correct answers.)

41. $(2, 0), (-1, 0)$

42. $(-4, 0), (0, 0)$

43. $(0, 0), (10, 0)$

44. $(4, 0), (8, 0)$

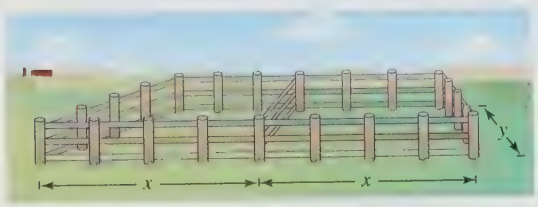
45. $(-3, 0), \left(-\frac{1}{2}, 0\right)$

46. $\left(-\frac{5}{2}, 0\right), (2, 0)$

47. Optimal Area The perimeter of a rectangle is 200 feet. Let x represent the width of the rectangle. Write a quadratic function for the area of the rectangle in terms of its width. Find the vertex of the graph of the quadratic function and interpret its meaning in the context of the problem.

48. Optimal Area The perimeter of a rectangle is 540 feet. Let x represent the width of the rectangle. Write a quadratic function for the area of the rectangle in terms of its width. Find the vertex of the graph of the quadratic function and interpret its meaning in the context of the problem.

49. Optimal Area A rancher has 1200 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What measurements will produce a maximum enclosed area?



50. Optimal Area An indoor physical-fitness room consists of a rectangular region with a semicircle on each end (see figure). The perimeter of the room is to be a 200-meter running track. What measurements will produce a maximum area of the rectangle?



Optimal Revenue In Exercises 51 and 52, find the number of units that produces a maximum revenue. The revenue R is measured in dollars and x is the number of units produced.

51. $R = 1000x - 0.02x^2$

52. $R = 80x - 0.0001x^2$

53. Optimal Cost The daily production costs C (in dollars per unit) for a manufacturer of lighting fixtures are given by the quadratic function

$$C(x) = 800 - 10x + 0.25x^2$$

where x is the number of units produced. How many fixtures should be produced each day to yield a minimum cost per unit?

54. Optimal Profit The profit P (in dollars) for a manufacturer of sound systems is given by the quadratic function

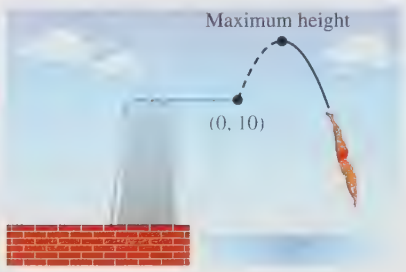
$$P(x) = -0.0003x^2 + 150x - 375{,}000$$

where x is the number of units produced. What production level will yield a maximum profit?

55. Maximum Height of a Diver The path of a diver is given by

$$y = -\frac{4}{9}x^2 + \frac{24}{9}x + 10$$

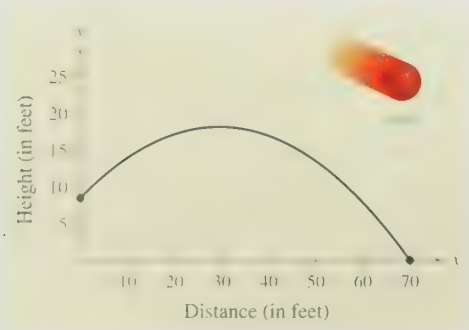
where y is the height (in feet) and x is the horizontal distance from the end of the diving board (in feet). (See figure.) Use the *trace* or *maximum* feature of a graphing utility to find the maximum height of the diver.



56. Maximum Height The path of the winning shot at a shot put event can be approximated by the quadratic function

$$y = -0.011x^2 + 0.65x + 8.3$$

where y is the height of the shot (in feet) and x is the horizontal distance (in feet). (See figure.) Use the *maximum* and *trace* features of a graphing utility to find the maximum height reached by the shot and its winning distance.



57. Revenue The table shows the total yearly revenues R (in millions of dollars) for golf courses and country clubs in the United States from 2003 through 2009. *(Source: U.S. Census Bureau)*

Year	2003	2004	2005	2006
Revenue, R	17,291	18,469	19,356	20,523

Year	2007	2008	2009
Revenue, R	21,195	21,044	20,326

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 3$ corresponding to 2003.

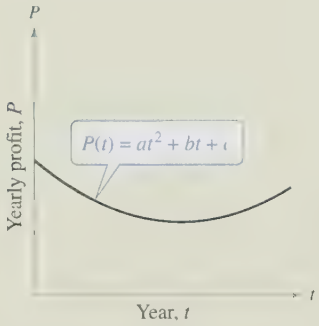
(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data. Graph the model with the scatter plot from part (a).

(c) Use the graph of the model from part (b) to estimate when the yearly revenue was the greatest. Does this result agree with the actual data?

58. HOW DO YOU SEE IT? The graph shows a quadratic function of the form

$$P(t) = at^2 + bt + c$$

which represents the yearly profits for a company, where $P(t)$ is the profit in year t.



(a) Is the value of a positive, negative, or zero? Explain.

(b) Write an expression in terms of a and b that represents the year t when the company made the least profit.

(c) The company made the same yearly profits in 2004 and 2012. Estimate the year in which the company made the least profit.

(d) Assume that the model is still valid today. Are the yearly profits currently increasing, decreasing, or constant? Explain.

59. Osteopathic Physicians The table shows the numbers N (in thousands) of female osteopathic physicians from 2001 through 2010. *(Source: American Osteopathic Association)*

Year	2001	2002	2003	2004	2005
Physicians, N	10.9	11.8	12.9	14.0	15.1

Year	2006	2007	2008	2009	2010
Physicians, N	16.4	17.7	19.1	20.8	22.5

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model from part (b) in the same viewing window as the scatter plot of the data.

(d) Use the graph of the model from part (c) to predict the number of female osteopathic physicians in 2012.

60. Kidney Transplants The table shows the numbers K of kidney transplants performed in the United States in the years 1999 through 2009. *(Source: U.S. Department of Health and Human Services)*

Year	1999	2000	2001	2002
Transplants, K	12,455	13,258	14,152	14,741

Year	2003	2004	2005	2006
Transplants, K	15,129	16,000	16,481	17,094

Year	2007	2008	2009
Transplants, K	16,624	16,517	16,829

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 9$ corresponding to 1999.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model from part (b) in the same viewing window as the scatter plot of the data.

(d) Use the graph of the model from part (c) to predict the number of kidney transplants performed in 2010. Does your answer seem reasonable? Explain.

61. Analyzing Profits Let x be the number of units (in tens of thousands) that a computer company produces and let $p(x)$ be the profit (in hundreds of thousands of dollars). The table shows the profits for different levels of production.

Units, x	2	4	6	8	10
Profit, $p(x)$	270.5	307.8	320.1	329.2	325.0

Units, x	12	14	16	18	20
Profit, $p(x)$	311.2	287.8	254.8	212.2	160.0

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for $p(x)$.

(c) Use the graphing utility to graph your model for $p(x)$ with the scatter plot of the data.

(d) Find the vertex of the graph of the model from part (c). Interpret its meaning in the context of the problem.

(e) With these data and this model, the profit begins to decrease. Discuss how it is possible for production to increase and profit to decrease.

62. Baseball Distance Let x be the angle (in degrees) at which a baseball is hit with no spin at an initial speed of 40 meters per second, and let $d(x)$ be the distance (in meters) the ball travels. The table shows the distances for the different angles at which the ball is hit. *(Source: The Physics of Sports)*

Angle, x	10	15	30	36	42
Distance, $d(x)$	58.3	79.7	126.9	136.6	140.6

Angle, x	44	45	48	54	60
Distance, $d(x)$	140.9	140.9	139.3	132.5	120.5

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for $d(x)$.

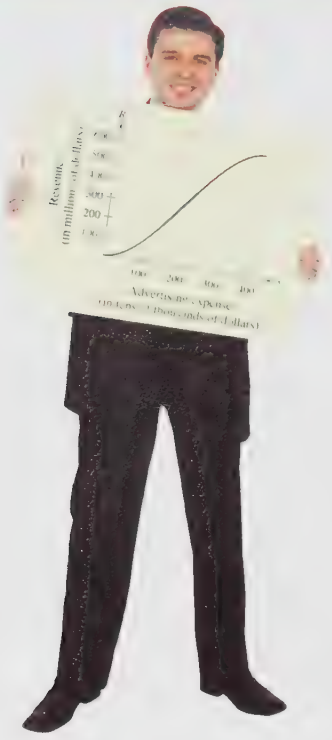
(c) Use the graphing utility to graph your model for $d(x)$ with the scatter plot of the data.

(d) Find the vertex of the graph of the model from part (c). Interpret its meaning in the context of the problem.

63. Verifying a Formula Write the quadratic function $f(x) = ax^2 + bx + c$ in standard form to verify that the vertex occurs at $(-b/2a, f(-b/2a))$.

3.2 Polynomial Functions of Higher Degree

- Sketch a transformation of a monomial function.
- Apply the Leading Coefficient Test to determine right-hand and left-hand behavior of graphs of polynomial functions.
- Find the real zeros and sketch the graph of a polynomial function.
- Use a polynomial model to solve an application problem.



In Exercise 78 on page 271, you will use the graph of a cubic function to find the point of diminishing returns for the advertising expenses of a company.

Graphs of Polynomial Functions

The graphs of polynomial functions of degree greater than 2 are more complicated than those of degree 0, 1, or 2. However, using the characteristics presented in this section, together with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*. Of course, a graphing utility makes the task easier.

In this section, you will study basic characteristics of the graphs of polynomial functions. The first characteristic is that the graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, as shown in Figure 3.11(a). Functions with graphs that are not continuous are not polynomial functions, as shown in Figure 3.11(b).



(a) Continuous



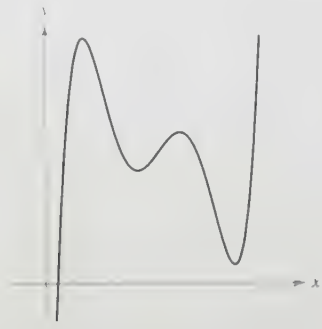
(b) Not continuous

FIGURE 3.11

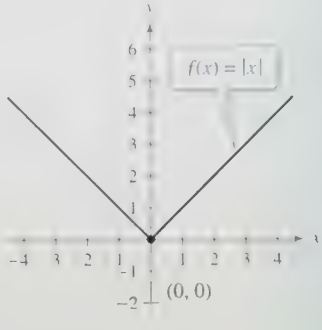
The second characteristic is that the graph of a polynomial function has only smooth, rounded turns, as shown in Figure 3.12(a). A polynomial function cannot have a sharp turn. For instance, the function given by

$$f(x) = |x|$$

which has a sharp turn at the point $(0, 0)$, as shown in Figure 3.12(b), is not a polynomial function.



(a) Polynomial functions have smooth, rounded turns.

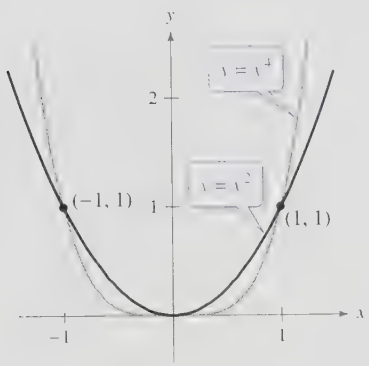


(b) Polynomial functions cannot have sharp turns.

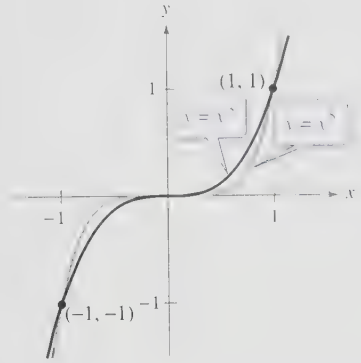
FIGURE 3.12

2010/Used under license from Shutterstock.com

The polynomial functions that have the simplest graphs are monomial functions of the form $f(x) = x^n$, where n is an integer greater than zero. From Figure 3.13, you can see that when n is *even*, the graph is similar to the graph of $f(x) = x^2$, and when n is *odd*, the graph is similar to the graph of $f(x) = x^3$. Moreover, the greater the value of n, the flatter the graph near the origin.



(a) When n is even, the graph of $y = x^n$ touches the axis at the x-intercept.



(b) When n is odd, the graph of $y = x^n$ crosses the axis at the x-intercept.

FIGURE 3.13

Example 1 Sketching Transformations of Monomial Functions

Sketch the graph of each function.

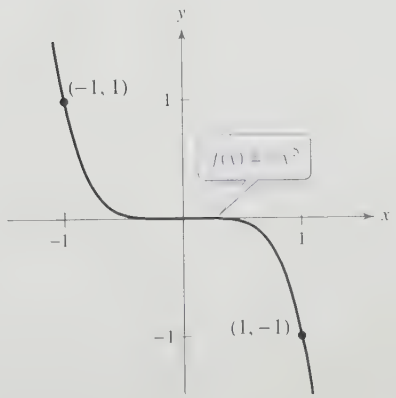
a. $f(x) = -x^5$

b. $h(x) = (x + 1)^4$

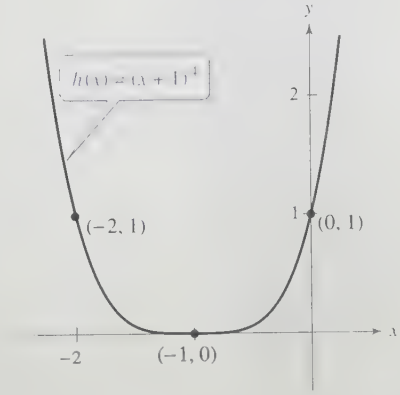
SOLUTION

a. The degree of $f(x) = -x^5$ is odd. So, its graph is similar to the graph of $y = x^3$. In Figure 3.14(a), note that the negative coefficient has the effect of reflecting the graph about the x-axis.

b. The graph of $h(x) = (x + 1)^4$ is a left shift, by one unit, of the graph of $y = x^4$, as shown in Figure 3.14(b).



(a)



(b)

FIGURE 3.14

Checkpoint 1

Sketch the graph of $f(x) = (x - 3)^3$.

The Leading Coefficient Test

In Example 1, note that both graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial function eventually rises or falls can be determined by the function's degree (even or odd) and by its leading coefficient (positive or negative), as indicated in the **Leading Coefficient Test.**

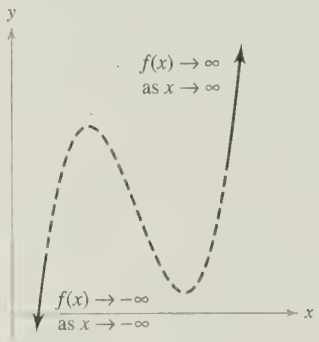
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function given by

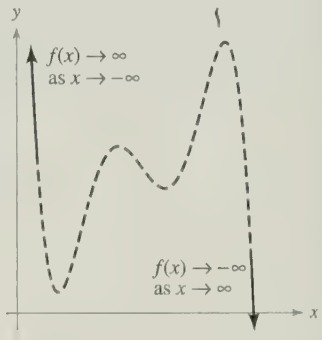
$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

eventually rises or falls in the following manner.

1. When n is *odd:*

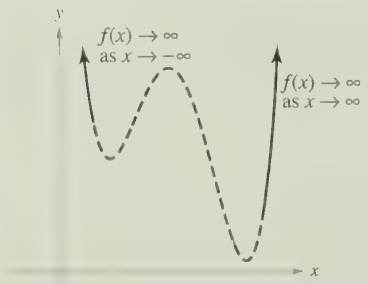


If the leading coefficient is positive ($a_n > 0$), then the graph falls to the left and rises to the right.

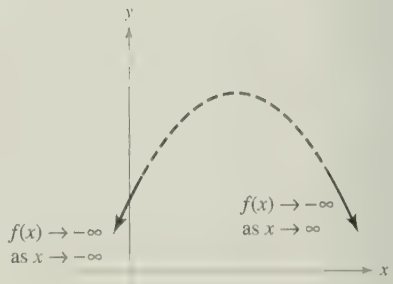


If the leading coefficient is negative ($a_n < 0$), then the graph rises to the left and falls to the right.

2. When n is *even:*



If the leading coefficient is positive ($a_n > 0$), then the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), then the graph falls to the left and right.

The dashed portions of the graphs indicate that the test determines *only* the right-hand and left-hand behavior of the graph.

STUDY TIP

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.



As you continue to study polynomial functions and their graphs, you will notice that the degree of a polynomial plays an important role in determining other characteristics of the polynomial function and its graph.

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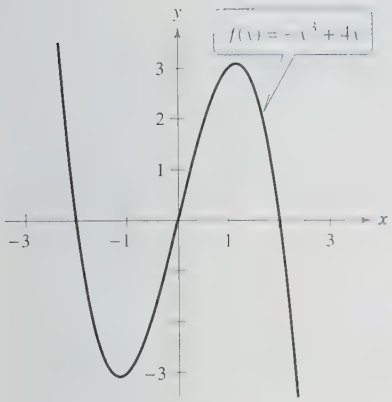


FIGURE 3.15

Example 2 **Applying the Leading Coefficient Test**

Describe the right-hand and left-hand behavior of the graph of

$$f(x) = -x^3 + 4x.$$

SOLUTION Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 3.15.

Checkpoint 2

Describe the right-hand and left-hand behavior of the graph of each function.

a. $f(x) = 3 + 2x + x^2$

b. $h(x) = -2x^4 + x$

In Example 2, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts, relative minima, and relative maxima, must be determined by other tests. For example, later you will use the number of real zeros of a polynomial function to determine how many times the graph of the function crosses the x-axis.

STUDY TIP

The functions in Example 3 are fourth- and fifth-degree polynomial functions, respectively. A fourth-degree polynomial function can also be referred to as a *quartic* function and a fifth-degree polynomial function as a *quintic* function.

Example 3 **Applying the Leading Coefficient Test**

Describe the right-hand and left-hand behavior of the graph of each function.

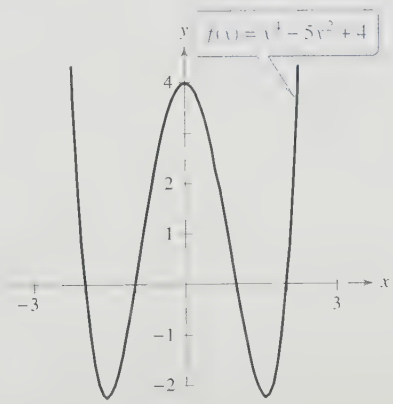
a. $f(x) = x^4 - 5x^2 + 4$

b. $f(x) = x^5 - x$

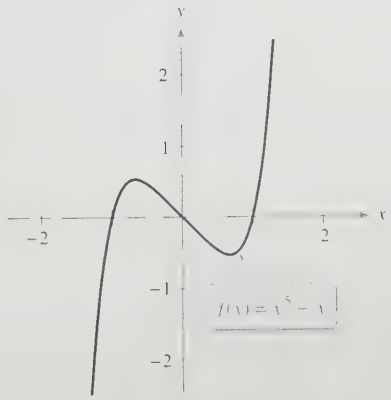
SOLUTION

a. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 3.16(a).

b. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 3.16(b).



(a)



(b)

FIGURE 3.16

Checkpoint 3

Describe the right-hand and left-hand behavior of the graph of

$$g(x) = -x^5 + 6x^2.$$

Real Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true. Remember that the **zeros** of a function are the x-values for which the function is zero.

1. The graph of f has, at most, $n - 1$ *turning points*. Turning points are points at which the graph changes from increasing to decreasing, or vice versa. For instance, the graph of

   $$f(x) = x^4 - 1$$

   has at most $4 - 1 = 3$ turning points.
2. The function f has, at most, n real zeros. For instance, the function given by

   $$f(x) = x^4 - 1$$

   has at most $n = 4$ real zeros. (You will study this result in detail in Section 3.6 on the Fundamental Theorem of Algebra.)

Finding the zeros of polynomial functions is one of the most important problems in algebra. There is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros, and in other cases you can use information about the zeros of a function to help sketch its graph.

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, then the following statements are equivalent.

1. $x = a$ is a zero of the function f.
2. $x = a$ is a solution of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a factor of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

In the equivalent statements above, notice that finding zeros of polynomial functions is closely related to factoring and finding x-intercepts.

Example 4 Finding Zeros of a Polynomial Function

Find all real zeros of $g(x) = x^3 - x^2 - 2x$.

SOLUTION By factoring, you obtain

$$\begin{aligned} g(x) &= x^3 - x^2 - 2x && \text{Write original function.} \\ &= x(x^2 - x - 2) && \text{Remove common monomial factor.} \\ &= x(x - 2)(x + 1). && \text{Factor completely.} \end{aligned}$$

So, the real zeros are $x = 0$, $x = 2$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(2, 0)$, and $(-1, 0)$, as shown in Figure 3.17. Note that the graph in the figure has two turning points. This is consistent with the fact that the graph of a third-degree polynomial function can have *at most* $3 - 1 = 2$ turning points.

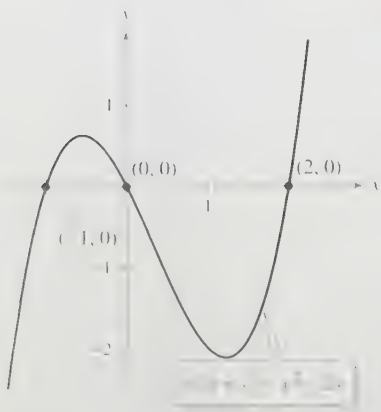


FIGURE 3.17

Checkpoint 4

Find all real zeros of $f(x) = x^2 - 4$.

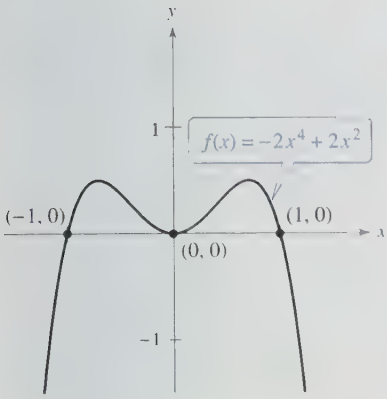


FIGURE 3.18

Example 5 Finding Zeros of a Polynomial Function

Find all real zeros of

$$f(x) = -2x^4 + 2x^2.$$

SOLUTION In this case, the polynomial factors as

$$f(x) = -2x^2(x^2 - 1) = -2x^2(x - 1)(x + 1).$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 3.18. Note that the graph in the figure has three turning points, which is consistent with the fact that the graph of a fourth-degree polynomial function can have *at most* $4 - 1 = 3$ turning points.

Checkpoint 5

Find all real zeros of $f(x) = x^3 - x$.

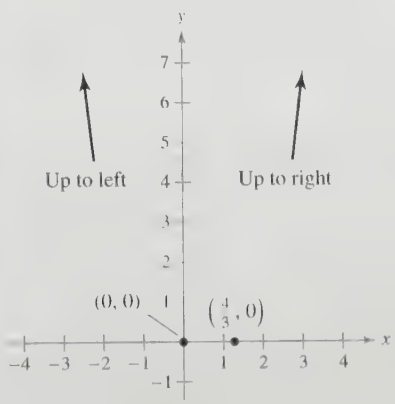
In Example 5, the real zero arising from $-2x^2 = 0$ is called a **repeated zero.** In general, a factor $(x - a)^k$ yields a repeated zero $x = a$ of **multiplicity** k. If k is odd, then the graph *crosses* the x-axis at $x = a$. If k is even, then the graph *touches* (but does not cross) the x-axis at $x = a$. This is illustrated in Figure 3.18.

Example 6 Sketching the Graph of a Polynomial Function

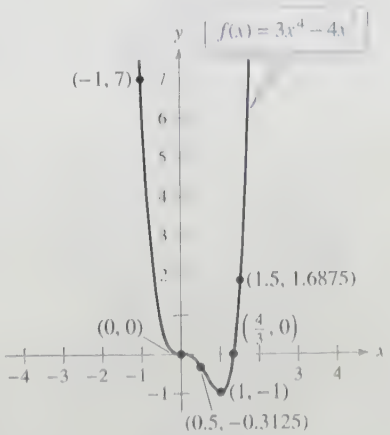
Sketch the graph of

$$f(x) = 3x^4 - 4x^3.$$

SOLUTION Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and right, as shown in Figure 3.19(a). By factoring $f(x) = 3x^4 - 4x^3$ as $f(x) = x^3(3x - 4)$, you can see that the zeros of f are $x = 0$ and $x = \frac{4}{3}$ (both of odd multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. To sketch the graph by hand, find a few additional points, plot the points, and draw a continuous curve through the points to complete the graph, as shown in Figure 3.19(b). If you are unsure of the shape of a portion of a graph, then plot some additional points.



(a)



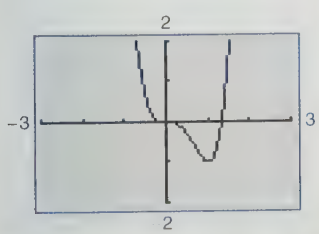
(b)

FIGURE 3.19

TECH TUTOR



Example 6 uses an algebraic approach to describe the graph of the function. A graphing utility is a valuable complement to this approach. Remember that when using a graphing utility, it is important that you find a viewing window that shows all important parts of the graph. For instance, the graph below shows the important parts of the graph of the function in Example 6.



Checkpoint 6

Sketch the graph of $h(x) = 2x^3 - 3x^2$.

Application

Example 7 Charitable Contributions Revisited

Example 7 in Section 3.1 discussed the model

$$P(x) = 0.0014x^2 - 0.1529x + 5.855, \quad 5 \le x \le 100$$

where P is the percent of annual income given to charities and x is the annual income (in thousands of dollars). Note that this model gives the charitable contributions as a *percent* of annual income. To find the *amount* that a family gives to charity, you can multiply the given model by the income $1000x$ (and divide by 100 to change from percent to decimal form) to obtain

$$A(x) = 0.014x^3 - 1.529x^2 + 58.55x, \quad 5 \le x \le 100$$

where A represents the amount of charitable contributions (in dollars). Sketch the graph of the function A. Then use the graph to estimate the annual income of a family that gives \$1000 per year to charities.

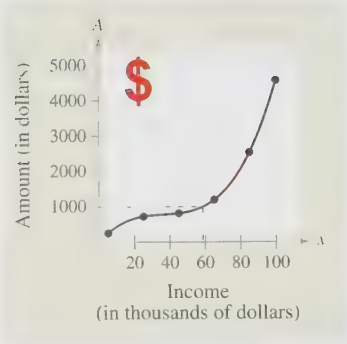


FIGURE 3.20

SOLUTION Because the leading coefficient is positive and the degree is odd, you know that the graph eventually falls to the left and rises to the right. To sketch the graph by hand, find a few points, as shown in the table. Then plot the points and complete the graph, as shown in Figure 3.20.

x	5	25	45	65	86	100
$A(x)$	256.28	726.88	814.28	1190.48	2527.48	4565.00

From the graph, you can see that an annual contribution of \$1000 corresponds to an annual income of about \$59,000.

✓ Checkpoint 7

The net profits P (in millions of dollars) for the Colgate-Palmolive Company for the years 2000 to 2009 can be approximated by the function

$$P(t) = 2.864t^3 - 27.93t^2 + 156.0t + 1062$$

where t represents the year, with $t = 0$ corresponding to 2000. Sketch the graph of the function. Then use the graph to estimate the year in which the net profit was about \$1600 million. *(Source: Colgate-Palmolive Company)*



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SUMMARIZE (Section 3.2)

1. Describe the graphs of monomial functions of the form $f(x) = x^n$ *(page 263)*. For an example of sketching a transformation of a monomial function, see Example 1.
2. Describe the Leading Coefficient Test *(page 264)*. For examples of applying the Leading Coefficient Test, see Examples 2 and 3.
3. Describe the real zeros of polynomial functions *(page 266)*. For examples of finding zeros of a polynomial function, see Examples 4 and 5.

SKILLS WARM UP 3.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.6, 1.3, 1.4, and 1.5.

In Exercises 1–6, factor the expression completely.

1. $12x^2 + 7x - 10$
2. $25x^3 - 60x^2 + 36x$
3. $12z^4 + 17z^3 + 5z^2$
4. $y^3 + 125$
5. $x^3 + 3x^2 - 4x - 12$
6. $x^3 + 2x^2 + 3x + 6$

In Exercises 7–10, find all real solutions of the equation.

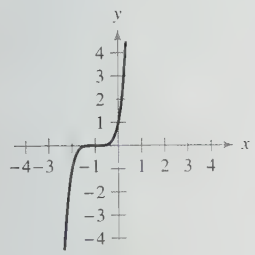
7. $5x^2 + 8 = 0$
8. $x^2 - 6x + 4 = 0$
9. $4x^2 + 4x - 11 = 0$
10. $x^4 - 18x^2 + 81 = 0$

Exercises 3.2

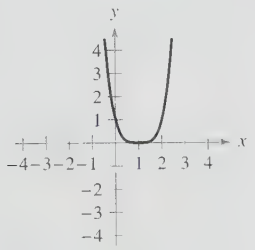
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–4, match the polynomial function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

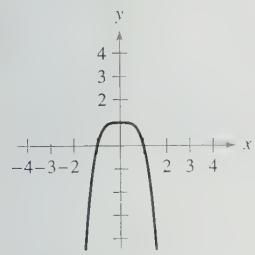
(a)



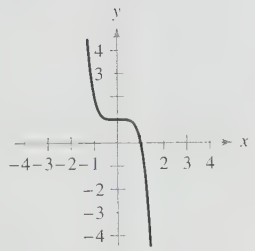
(b)



(c)



(d)



1. $f(x) = (x - 1)^4$
2. $f(x) = -x^5 + 1$
3. $f(x) = -x^4 + 1$
4. $f(x) = (x + 1)^5$

Sketching a Transformation of a Monomial Function In Exercises 5–8, use the graph of $y = x^3$ to sketch the graph of the function. *See Example 1.*

5. $f(x) = x^3 - 2$
6. $f(x) = (x + 3)^3$
7. $f(x) = (x + 1)^3 - 4$
8. $f(x) = -(x - 2)^3 + 2$

Sketching a Transformation of a Monomial Function In Exercises 9–12, use the graph of $y = x^4$ to sketch the graph of the function. *See Example 1.*

9. $f(x) = (x + 3)^4$
10. $f(x) = x^4 - 4$
11. $f(x) = 3 - x^4$
12. $f(x) = \frac{1}{2}(x - 1)^4$



Analyzing End Behavior In Exercises 13–20, (a) identify the degree of the function and state whether the degree is even or odd, (b) identify the leading coefficient and state whether it is positive or negative, (c) use a graphing utility to graph the function, and (d) describe the right-hand and left-hand behavior of the graph.

13. $y = 2x^5 + 2x^2 - 5x + 1$
14. $y = x^3 - 2x^2 - x + 1$
15. $y = -2x^5 - x^2 + 5x + 3$
16. $y = -x^3 + 5x - 2$
17. $y = 2x^2 + 3x - 4$
18. $y = x^4 - 3x^2 + 2x - 1$
19. $y = -x^6 - x^2 - 5x + 4$
20. $y = -x^2 + 3x + 4$

21. **Discovery** Using your results in Exercises 13–16, describe how the right-hand and left-hand behavior of the graph of a polynomial function of odd degree is related to the sign of the leading coefficient.

22. **Discovery** Using your results in Exercises 17–20, describe how the right-hand and left-hand behavior of the graph of a polynomial function of even degree is related to the sign of the leading coefficient.

Applying the Leading Coefficient Test In Exercises 23–32, describe the right-hand and left-hand behavior of the graph of the polynomial function. *See Examples 2 and 3.*

23. $f(x) = -x^3 + 1$
24. $f(x) = \frac{1}{3}x^3 + 5x$
25. $g(x) = 6 - 4x^2 + x - 3x^5$
26. $f(x) = 2x^5 - 5x + 7.5$
27. $f(x) = 4x^8 - 2$
28. $h(x) = 1 - x^6$
29. $f(x) = 2 + 5x - x^2 - x^3 + 2x^4$
30. $f(x) = \dfrac{3x^4 - 2x + 5}{4}$
31. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$
32. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

The Numbers of Turning Points and Real Zeros In Exercises 33–36, determine (a) the maximum number of turning points of the graph of the function and (b) the maximum number of real zeros of the function.

33. $f(x) = x^2 - 4x + 1$ **34.** $f(x) = -3x^4 + 1$

35. $f(x) = -x^5 + x^4 - x$ **36.** $f(x) = 2x^3 + x^2 + 1$

Finding Zeros of a Polynomial Function In Exercises 37–52, find all real zeros of the function algebraically. Then use a graphing utility to confirm your results. *See Examples 4 and 5.*

37. $f(x) = 9 - x^2$ **38.** $f(x) = x^2 - 25$

39. $h(t) = t^2 + 8t + 16$ **40.** $f(x) = x^2 - 12x + 36$

41. $f(x) = \frac{1}{3}x^2 + \frac{1}{3}x + \frac{2}{3}$

42. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$

43. $g(x) = -5x^2 - 10x + 20$

44. $f(x) = 2x^2 + 4x + 6$

45. $f(t) = t^3 - 4t^2 + 4t$

46. $f(x) = x^4 - x^3 - 20x^2$

47. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$

48. $f(x) = \frac{1}{3} - \frac{1}{3}x^4$

49. $f(x) = 2x^4 - 2x^2 - 40$

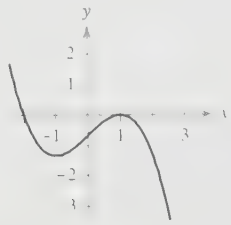
50. $g(t) = t^5 - 6t^3 + 9t$

51. $f(x) = x^3 - 3x^2 + 2x - 6$

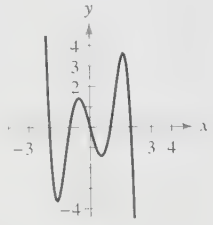
52. $f(x) = x^3 - 4x^2 - 25x + 100$

Matching In Exercises 53–60, match the polynomial function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

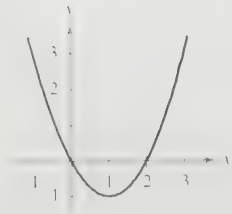
(a)



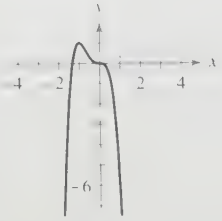
(b)



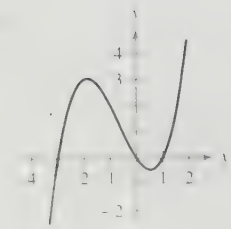
(c)



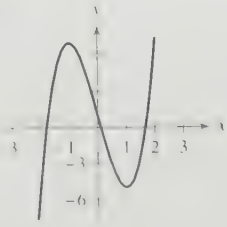
(d)



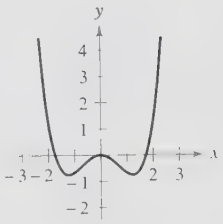
(e)



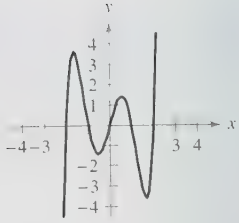
(f)



(g)



(h)



53. $f(x) = \frac{1}{2}(x^3 + 2x^2 - 3x)$

54. $f(x) = x^2 - 2x$ **55.** $f(x) = \frac{1}{3}x^4 - x^2$

56. $f(x) = -3x^4 - 4x^3$ **57.** $f(x) = 3x^3 - 9x + 1$

58. $f(x) = x^5 - 5x^3 + 4x$ **59.** $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$

60. $f(x) = -x^5 + 5x^3 - 4x$

Sketching the Graph of a Polynomial Function In Exercises 61–72, analyze the graph of the function algebraically and use the results to sketch the graph *by hand*. Then use a graphing utility to confirm your sketch. *See Example 6.*

61. $f(x) = \frac{2}{3}x + 5$ **62.** $h(x) = -\frac{3}{4}x + 2$

63. $f(t) = \frac{1}{2}(t^2 - 4t - 1)$ **64.** $g(x) = -x^2 + 10x - 16$

65. $f(x) = 4x^2 - x^3$ **66.** $f(x) = 1 - x^3$

67. $f(x) = x^3 - 9x$ **68.** $f(x) = \frac{1}{4}x^3 - 2x^2$

69. $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$

70. $f(x) = x(x - 2)^2(x + 1)$

71. $f(x) = 1 - x^6$ **72.** $g(x) = 1 - (x + 1)^6$

73. Modeling Polynomials Sketch the graph of a polynomial function that is of fourth degree, has a zero of multiplicity 2, and has a negative leading coefficient. Sketch the graph of another polynomial function with the same characteristics except the leading coefficient is positive.

74. Modeling Polynomials Sketch the graph of a polynomial function that is of fifth degree, has a zero of multiplicity 2, and has a negative leading coefficient. Sketch the graph of another polynomial function with the same characteristics except the leading coefficient is positive.

75. Modeling Polynomials Determine the equation of the fourth-degree polynomial function f whose graph is shown.

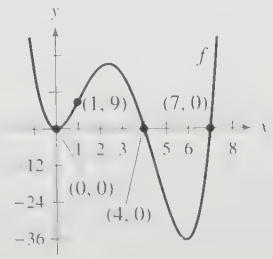


Figure for 75

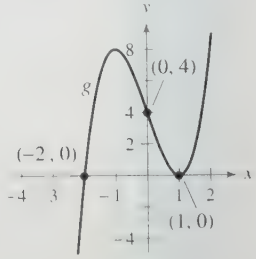


Figure for 76

76. Modeling Polynomials Determine the equation of the third-degree polynomial function g whose graph is shown.

77. Population The table shows the populations P (in millions) of foreign-born people living in the United States in 2009 and every 10 years from 1900 through 2000. *(Source: U.S. Census Bureau)*

Year	1900	1910	1920	1930
Population, P	10.3	13.5	13.9	14.2

Year	1940	1950	1960	1970
Population, P	11.6	10.3	9.7	9.7

Year	1980	1990	2000	2009
Population, P	14.1	19.8	31.1	38.5

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 0$ and $t = 10$ correspond to 1900 and 1910, respectively.

(b) Use what you know about end behavior and the scatter plot from part (a) to predict the sign of the leading coefficient of a cubic model for P.

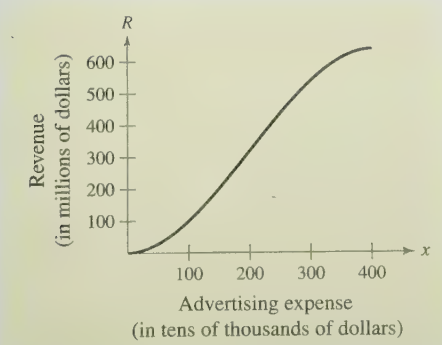
(c) Use the *regression* feature of the graphing utility to find a cubic model for P. Does your model agree with your answer from part (b)?

(d) Use the graphing utility to graph the model from part (c). Use the graph to predict the year in which the immigrant population will be about 45 million. Is your prediction reasonable?

78. Advertising Expenses The total revenue R (in millions of dollars) for a soft drink company is related to its advertising expenses by the function

$$R = \frac{1}{50{,}000}(-x^3 + 600x^2), \quad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of R to estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expenditure above this amount will yield less return per dollar invested in advertising.

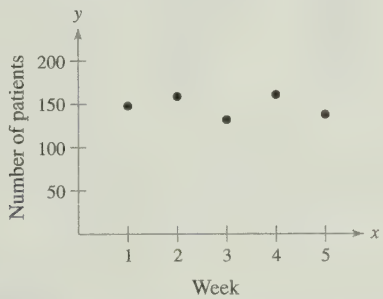


79. Advertising Expenses The total revenue R (in millions of dollars) for a hotel corporation is related to its advertising expenses by the function

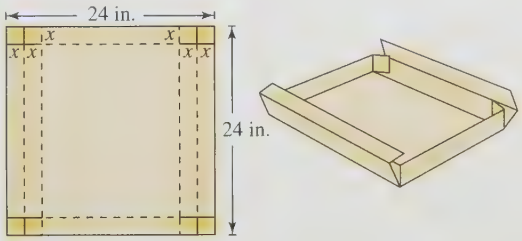
$$R = -0.148x^3 + 4.89x^2 - 17.8x + 125, \quad 0 \le x \le 20$$

where x is the amount spent on advertising (in millions of dollars). Use a graphing utility to graph the function and estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expenditure above this amount will yield less return per dollar invested in advertising.

80. HOW DO YOU SEE IT? The scatter plot shows the number y of patients in a hospital at the start of each week x for five consecutive weeks. What is the least possible degree of a polynomial function whose graph passes through all five data points? Explain your reasoning.



81. Maximum Value You are making an open box with locking tabs from a square piece of material 24 inches on a side. You cut squares of the same size from the corners and fold along the dashed lines, as shown in the figure.



(a) Verify that the volume of the box is given by the function

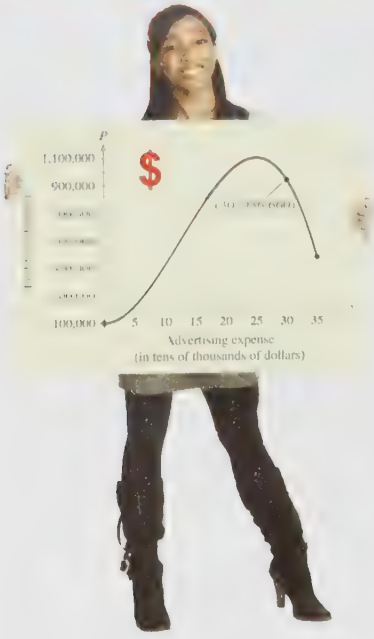
$$V(x) = 8x(6 - x)(12 - x).$$

(b) Determine the domain of the function V.

(c) Sketch a graph of the function and estimate the value of x for which the volume $V(x)$ is maximum.

3.3 Polynomial Division

- Divide one polynomial by a second polynomial using long division.
- Use synthetic division to divide two polynomials.
- Use the Remainder Theorem and synthetic division to evaluate a polynomial.
- Use the Factor Theorem to factor a polynomial.
- Use polynomial division to solve an application problem.



In Exercise 82 on page 281, you will use a polynomial function to estimate advertising expenses of a company.

Long Division of Polynomials

Suppose you are given the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice that a zero of f occurs at $x = 2$, as shown in Figure 3.21. Because $x = 2$ is a zero of the polynomial function f, you know that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 2) \cdot q(x).$$

To find $q(x)$, you can use **long division,** as illustrated in Example 1.

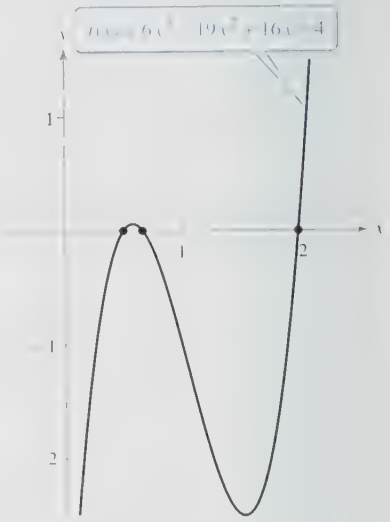


FIGURE 3.21

Example 1 Long Division of Polynomials

Divide the polynomial $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

SOLUTION

$$\begin{array}{rl} & 6x^2 - 7x + 2 \\ x - 2 \overline{)} & 6x^3 - 19x^2 + 16x - 4 \\ & \underline{6x^3 - 12x^2} \\ & \quad -7x^2 + 16x \\ & \quad \underline{-7x^2 + 14x} \\ & \qquad 2x - 4 \\ & \qquad \underline{2x - 4} \\ & \qquad\quad 0 \end{array}$$

Multiply $6x^2$ by $x - 2$.
Subtract and bring down $16x$.
Multiply $-7x$ by $x - 2$.
Subtract and bring down -4.
Multiply 2 by $x - 2$
Subtract

So, you can conclude that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2).$$

Checkpoint 1

Divide $x^3 - 6x^2 + 5x + 12$ by $x - 4$, and use the result to factor the polynomial completely.

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Note that the factorization shown in Example 1 agrees with the graph in Figure 3.21 in that the three x-intercepts occur at

$$x = 2, \quad x = -\frac{1}{2}, \quad \text{and} \quad x = \frac{2}{3}.$$

In Example 1, $x - 2$ is a factor of the polynomial

$$6x^3 - 19x^2 + 16x - 4$$

and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, divide $x^2 + 3x + 5$ by $x + 1$, as shown.

$$\begin{array}{rl} & x + 2 \quad \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow x + 1 \,\overline{)}\, & x^2 + 3x + 5 \quad \leftarrow \text{Dividend} \\ & \underline{x^2 + x} \\ & 2x + 5 \\ & \underline{2x + 2} \\ & 3 \quad \leftarrow \text{Remainder} \end{array}$$

In fractional form, you can write this result as shown.

$$\underbrace{\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}}}_{} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overbrace{3}^{\text{Remainder}}}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following well-known theorem called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

Dividend, Divisor, Quotient, Remainder

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ **divides evenly** into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression

$$f(x)/d(x)$$

is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression

$$r(x)/d(x)$$

is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Note how these steps are applied in the next example.

Example 2 Long Division of Polynomials

Divide $x^3 - 1$ by $x - 1$.

SOLUTION Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$\begin{array}{rl} & x^2 + x + 1 \\ x - 1 \overline{)\, x^3 + 0x^2 + 0x - 1} & \\ \underline{x^3 - x^2} & \\ x^2 + 0x & \\ \underline{x^2 - x} & \\ x - 1 & \\ \underline{x - 1} & \\ 0 & \end{array}$$

Insert $0x^2$ and $0x$.
Multiply x^2 by $x - 1$.
Subtract and bring down $0x$.
Multiply x by $x - 1$.
Subtract and bring down -1.
Multiply 1 by $x - 1$.
Subtract.

So, $x - 1$ divides evenly into $x^3 - 1$ and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1.$$

STUDY TIP

You can check the result of a division problem by multiplying. For instance, in Example 2, try checking that

$$(x - 1)(x^2 + x + 1) = x^3 - 1.$$

Checkpoint 2

Divide $x^3 + 8$ by $x + 2$.

In each of the long division examples, the divisor has been a first-degree polynomial. The Division Algorithm works just as well with polynomial divisors of degree two or more, as shown in Example 3.

Example 3 Long Division of Polynomials

Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$.

SOLUTION

$$\begin{array}{rl} & 2x^2 + 1 \\ x^2 + 2x - 3 \overline{)\, 2x^4 + 4x^3 - 5x^2 + 3x - 2} & \\ \underline{2x^4 + 4x^3 - 6x^2} & \\ x^2 + 3x - 2 & \\ \underline{x^2 + 2x - 3} & \\ x + 1 & \end{array}$$

Multiply $2x^2$ by $x^2 + 2x - 3$.
Subtract and bring down $3x - 2$.
Multiply 1 by $x^2 + 2x - 3$.
Subtract.

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. So, you can write

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

Checkpoint 3

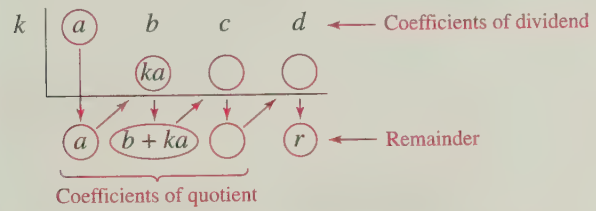
Divide $5x^4 + 10x^3 - 7x^2 + 28x - 39$ by $x^2 + 2x - 4$.

Synthetic Division

There is a nice shortcut for long division of polynomials when dividing by divisors of the form $x - k$. This shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is shown below. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (for a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms in columns.
Diagonal pattern: Multiply results by k.

This algorithm for synthetic division works only for divisors of the form $x - k$. Remember that

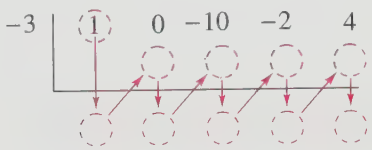
$$x + k = x - (-k).$$



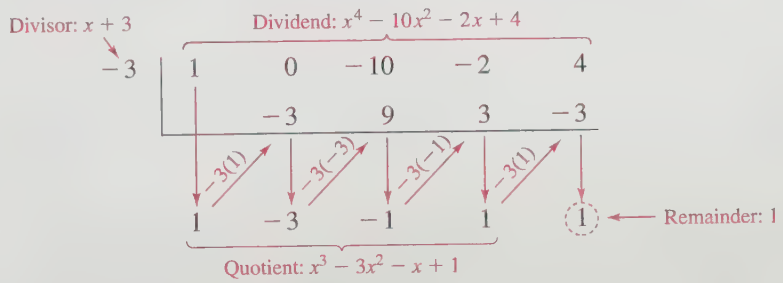
Example 4 **Using Synthetic Division**

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

SOLUTION Set up the array as shown. Note that a zero is included for the missing x^3-term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.



So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

Checkpoint 4

Use synthetic division to divide $2x^3 - 7x^2 - 80$ by $x - 5$.

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Remainder Theorem

The remainder obtained in the synthetic division process has an important interpretation, as described in the Remainder Theorem.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function f at $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

Example 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

when $x = -2$.

SOLUTION Use synthetic division.

$$\begin{array}{r|cccc} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9.$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

CHECK

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 \\ &= -9 \end{aligned}$$

Checkpoint 5

Use the Remainder Theorem to evaluate $f(x) = 4x^3 + 6x^2 + 4x + 5$ when $x = -1$.



TECH TUTOR

Remember, you can also evaluate a function with your graphing utility by entering the function in the equation editor and using the *table* feature in ASK mode. For instructions on how to use the *table* feature, see Appendix A. Consult the user's guide for your graphing utility for more information.

Factor Theorem

Another important theorem is the Factor Theorem, which is stated below.

Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

You can think of the Factor Theorem as stating that if $(x - k)$ is a factor of $f(x)$, then $f(k) = 0$. Conversely, if $f(k) = 0$, then $(x - k)$ is a factor of $f(x)$.

Example 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of the polynomial

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

SOLUTION Use synthetic division with the factor $(x - 2)$.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array}$$

0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

0 remainder, so $f(-3) = 0$ and $(x + 3)$ is a factor.

Quadratic: $2x^2 + 5x + 3$

Because the resulting quadratic expression factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Note that this factorization implies that f has four real zeros: $2, -3, -\frac{3}{2}$, and -1. This is confirmed by the graph of f, which is shown in Figure 3.22.

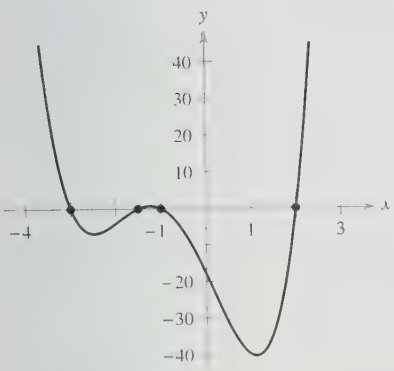


FIGURE 3.22

✓Checkpoint 6

Show that $(x + 2)$ and $(x + 4)$ are factors of the polynomial $f(x) = x^4 + 6x^3 + 7x^2 - 6x - 8$. Then find the remaining factors of $f(x)$.

STUDY TIP

Try using more than one strategy to solve some of the exercises in this section. For instance, if you find that $x - k$ divides evenly into $f(x)$ (with no remainder), then try sketching the graph of f. You should see that $(k, 0)$ is an x-intercept.

Uses of the Remainder in Synthetic Division

The remainder r obtained in the synthetic division of $f(x)$ by $x - k$ provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, then $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, then $(k, 0)$ is an x-intercept of the graph of f.

Application

Example 7 Tax Liability

The 2009 federal income tax liability for an employee who was single and claimed no dependents is given by the function

$$y = 0.00000077x^2 + 0.096x - 1426.86, \quad 20{,}000 \le x \le 100{,}000$$

where y represents the tax liability (in dollars) and x represents the employee's yearly salary (in dollars) (see Figure 3.23). *(Source: U.S. Department of the Treasury)*

a. Find a function that gives the tax liability as a *percent* of the yearly salary.

b. Graph the function in part (a). What conclusions can you make from the graph?



FIGURE 3.23

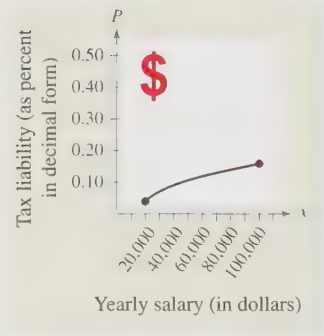


FIGURE 3.24

SOLUTION

a. Because the yearly salary is given by x and the tax liability is given by y, the percent (in decimal form) of yearly salary that a person owes in federal income tax is

$$P = \frac{y}{x}$$

$$= \frac{0.00000077x^2 + 0.096x - 1426.86}{x}$$

$$= 0.00000077x + 0.096 - \frac{1426.86}{x}.$$

b. The graph of the function P is shown in Figure 3.24. From the graph you can see that as a person's yearly salary increases, the percent that he or she must pay in federal income tax also increases.

Checkpoint 7

Use the function P from part (a) of Example 7.

a. What percent of a \$50,000 yearly salary does a person owe in federal income tax?

b. What percent of a \$100,000 yearly salary does a person owe in federal income tax?



SUMMARIZE (Section 3.3)

1. Describe how to divide one polynomial by another polynomial using long division *(pages 272–274)*. For examples of long division of polynomials, see Examples 1, 2, and 3.
2. Describe the algorithm for synthetic division *(page 275)*. For an example of using synthetic division, see Example 4.
3. State the Remainder Theorem *(page 276)*. For an example of using the Remainder Theorem, see Example 5.
4. State the Factor Theorem *(page 277)*. For an example of using the Factor Theorem, see Example 6.
5. Describe a real-life example of how polynomial division is used to write a function that gives the tax liability as a percent of an employee's yearly salary *(page 278, Example 7)*.

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SKILLS WARM UP 3.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.5 and 0.6.

In Exercises 1–4, write the expression in standard polynomial form.

1. $(x - 1)(x^2 + 2) + 5$
2. $(x^2 - 3)(2x + 4) + 8$
3. $(x^2 + 1)(x^2 - 2x + 3) - 10$
4. $(x + 6)(2x^3 - 3x) - 5$

In Exercises 5–10, factor the polynomial.

5. $x^2 - 4x + 3$
6. $8x^2 - 24x - 80$
7. $3x^2 + 2x - 5$
8. $9x^2 - 24x + 16$
9. $4x^3 - 10x^2 + 6x$
10. $6x^3 + 7x^2 + 2x$

Exercises 3.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Long Division of Polynomials In Exercises 1–22, use long division to divide. *See Examples 1, 2, and 3.*

	Dividend	*Divisor*
1.	$x^2 - x - 6$	$x - 3$
2.	$x^2 + 2x - 35$	$x + 7$
3.	$3x^2 - 7x + 4$	$x - 1$
4.	$5x^2 - 17x - 12$	$x - 4$
5.	$2x^2 + 10x + 12$	$x + 3$
6.	$2x^2 + x - 11$	$x + 5$
7.	$x^3 + 3x^2 + 5x + 3$	$x + 1$
8.	$x^3 - 6x^2 + 7x - 2$	$x - 1$
9.	$2x^3 + 6x^2 - x - 3$	$2x^2 - 1$
10.	$3x^3 - 12x^2 - 2x + 8$	$3x^2 - 2$
11.	$x^4 + 5x^3 + 6x^2 - x - 2$	$x + 2$
12.	$x^4 + 2x^3 - 3x^2 - 8x - 4$	$x^2 - 4$
13.	$7x + 3$	$x + 4$
14.	$8x - 5$	$2x + 3$
15.	$6x^3 + 10x^2 + x + 8$	$2x^2 + 1$
16.	$2x^3 - 8x^2 + 3x - 9$	$2x^2 + 3$
17.	$x^3 - 27$	$x^2 - 1$
18.	$x^3 - 9$	$x^2 + 1$
19.	$x^3 - 4x^2 + 5x - 2$	$x + 2$
20.	$x^3 - x^2 + 2x - 8$	$x - 2$
21.	$2x^5 - 8x^3 + 4x - 1$	$x^2 - 2x + 1$
22.	$x^4 + 3x^2 + 1$	$x^2 - 2x + 3$

Using Synthetic Division In Exercises 23–42, use synthetic division to divide. *See Example 4.*

	Dividend	*Divisor*
23.	$x^2 + 6x - 7$	$x - 1$
24.	$x^2 - 4x - 21$	$x + 3$
25.	$2x^3 + 5x^2 - 7x + 20$	$x + 4$
26.	$3x^3 - 23x^2 - 12x + 32$	$x - 8$
27.	$4x^3 - 9x + 8x^2 - 18$	$x + 2$
28.	$9x^3 - 16x - 18x^2 + 32$	$x - 2$
29.	$-x^3 + 75x - 250$	$x + 10$
30.	$3x^3 - 16x^2 - 72$	$x - 6$
31.	$x^4 - 4x^3 - 7x^2 + 22x + 24$	$x + 3$
32.	$6x^4 - 15x^3 - 11x$	$x + 2$
33.	$10x^4 - 50x^3 - 800$	$x - 6$
34.	$x^5 - 13x^4 - 120x + 80$	$x + 3$
35.	x^5	$x + 1$
36.	$5x^3$	$x + 3$
37.	$-3x^4$	$x - 2$
38.	$2x^5$	$x + 3$
39.	$5 - 3x + 2x^2 - x^3$	$x + 1$
40.	$180x - x^4$	$x - 6$
41.	$4x^3 + 16x^2 - 23x - 15$	$x + \frac{1}{2}$
42.	$3x^3 - 4x^2 + 5$	$x - \frac{3}{2}$

Using the Remainder Theorem In Exercises 43–50, write the function in the form

$f(x) = (x - k)q(x) + r$

for the given value of k, and demonstrate that $f(k) = r$.

43. $f(x) = x^3 + x^2 - 12x + 20, \quad k = 2$

44. $f(x) = x^3 - 2x^2 - 15x + 7, \quad k = -4$

45. $f(x) = 3x^3 + 2x^2 + 5x - 2, \quad k = \frac{1}{3}$

46. $f(x) = 4x^4 + 6x^3 + 4x^2 - 5x + 13, \quad k = -\frac{1}{2}$

47. $f(x) = x^3 + 2x^2 - 3x - 12, \quad k = \sqrt{3}$

48. $f(x) = x^3 + 3x^2 - 7x - 6, \quad k = -\sqrt{2}$

49. $f(x) = 2x^3 + x^2 - 14x - 10, \quad k = 1 + \sqrt{3}$

50. $f(x) = 3x^3 - 19x^2 + 27x - 7, \quad k = 3 - \sqrt{2}$

Using the Remainder Theorem In Exercises 51–54, use the Remainder Theorem to find each function value. *See Example 5.*

51. $f(x) = 2x^5 - 3x^2 - 4x - 1$

(a) $f(-2)$ (b) $f(-4)$

(c) $f(1)$ (d) $f(3)$

52. $g(x) = x^6 - 4x^4 + 3x^2 + 2$

(a) $g(2)$ (b) $g(-4)$

(c) $g(7)$ (d) $g(-1)$

53. $f(x) = 2x^3 + 3x^2 + 8x - 14$

(a) $f(2)$ (b) $f(-1)$

(c) $f(1.1)$ (d) $f\left(\frac{1}{2}\right)$

54. $f(x) = 3x^4 - 7x^3 + 5x - 12$

(a) $f(1)$ (b) $f(4)$

(c) $f\left(\frac{1}{3}\right)$ (d) $f(-1.2)$

Factoring a Polynomial In Exercises 55–60, (a) verify the given factors of $f(x)$, (b) find the remaining factor of $f(x)$, (c) use your results to write the complete factorization of $f(x)$, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function. *See Example 6.*

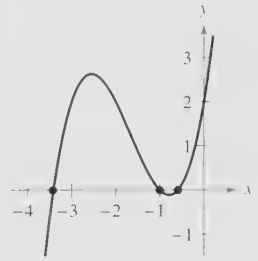
Function	*Factors*
55. $f(x) = x^3 - 12x - 16$	$(x + 2), (x - 4)$
56. $f(x) = x^3 - 28x - 48$	$(x + 4), (x - 6)$
57. $f(x) = 3x^3 + 10x^2 - 27x - 10$	$(3x + 1), (x - 2)$
58. $f(x) = 5x^3 - 11x^2 - 38x + 8$	$(5x - 1), (x - 4)$
59. $f(x) = x^3 + 2x^2 - 3x - 6$	$(x - \sqrt{3}), (x + 2)$
60. $f(x) = x^3 + 2x^2 - 2x - 4$	$(x - \sqrt{2}), (x + 2)$

61. Reasoning You divide a polynomial by another polynomial. The remainder is zero. What conclusion(s) can you make?

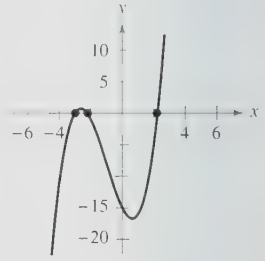
62. Reasoning A remainder obtained in a polynomial division by $x - k$ is zero. How is the divisor related to the graph of the dividend?

Finding Real Zeros In Exercises 63–68, match the function with its graph and use the result to find all real solutions of $f(x) = 0$. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

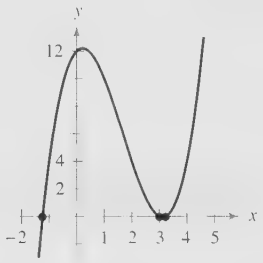
(a)



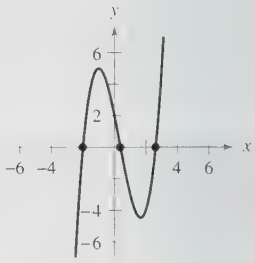
(b)



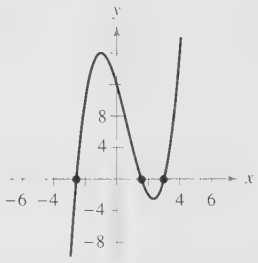
(c)



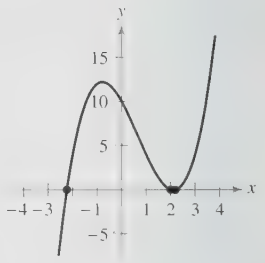
(d)



(e)



(f)



63. $f(x) = x^3 - 2x^2 - 7x + 12$

64. $f(x) = x^3 - x^2 - 5x + 2$

65. $f(x) = x^3 + 5x^2 + 6x + 2$

66. $f(x) = x^3 - 5x^2 + 2x + 12$

67. $f(x) = x^3 + 3x^2 - 5x - 15$

68. $f(x) = x^3 - 2x^2 - 5x + 10$

69. Modeling Polynomials A third-degree polynomial function f has real zeros -1, 2, and $\frac{10}{3}$. Find two different polynomial functions, one with a positive leading coefficient and one with a negative leading coefficient, that could be f. How many different polynomial functions are possible for f?

70. Modeling Polynomials A fourth-degree polynomial function g has real zeros -2, 0, 1, and 5. Find two different polynomial functions, one with a positive leading coefficient and one with a negative leading coefficient, that could be g. How many different polynomial functions are possible for g?

Simplifying a Rational Expression In Exercises 71–78, simplify the rational expression.

71. $\dfrac{x^3 - 10x^2 + 31x - 30}{x - 3}$

72. $\dfrac{x^3 + 15x^2 + 68x + 96}{x + 4}$

73. $\dfrac{6x^3 + x^2 - 21x - 10}{2x + 1}$

74. $\dfrac{3x^3 - 5x^2 - 34x + 24}{3x - 2}$

75. $\dfrac{x^4 - 5x^3 + 14x^2 - 120x}{x^2 + x + 20}$

76. $\dfrac{x^4 + x^3 + 3x^2 + 10x}{x^2 - x + 5}$

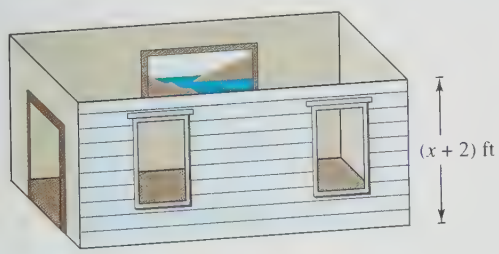
77. $\dfrac{x^4 + 4x^3 - 6x^2 - 36x - 27}{x^2 - 9}$

78. $\dfrac{x^4 + x^3 - 13x^2 - x + 12}{x^2 + x - 12}$

79. Examination Room A rectangular examination room in a veterinary clinic has a volume of

$$x^3 + 11x^2 + 34x + 32$$

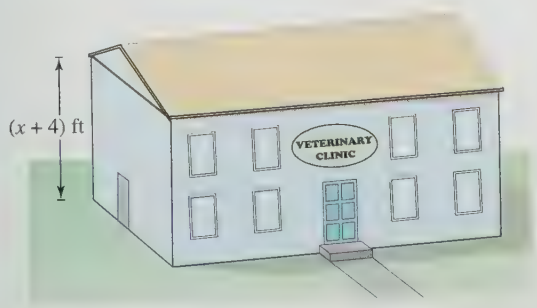
cubic feet. The height of the room is $(x + 2)$ feet (see figure). Find the number of square feet of floor space in the examination room.



80. Veterinary Clinic A rectangular veterinary clinic has a volume of

$$x^3 + 55x^2 + 650x + 1784$$

cubic feet (the space in the attic is not included). The height of the clinic is $(x + 4)$ feet (see figure). Find the number of square feet of floor space *on the first floor* of the clinic.



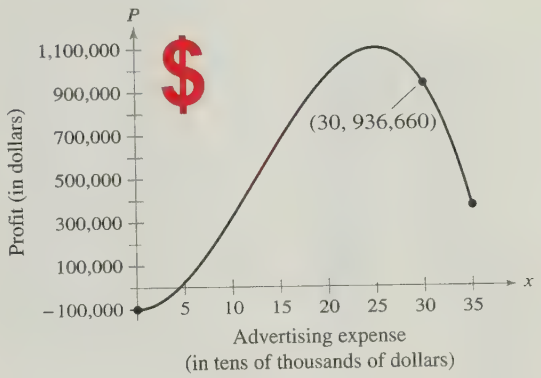
81. Writing Briefly explain how to check polynomial division, and justify your answer. Give an example.

82. HOW DO YOU SEE IT? A company making fishing poles estimated that the profit P (in dollars) from selling a particular fishing pole was

$$P = -140.75x^3 + 5348.3x^2 - 76{,}560, \quad 0 \le x \le 35$$

where x was the advertising expense (in tens of thousands of dollars). For this fishing pole, the advertising expense was \$300,000 ($x = 30$) and the profit was \$936,660.

(a) From the graph shown in the figure, it appears that the company could have obtained the same profit by spending less on advertising. Use the graph to estimate another amount the company could have spent on advertising that would have produced the same profit.



(b) Explain how you could verify the amount from part (a) algebraically.

83. Profit A company that produces calculators estimated that the profit P (in dollars) from selling a particular model of calculator was

$$P = -152x^3 + 7545x^2 - 169{,}625, \quad 0 \le x \le 45$$

where x was the advertising expense (in tens of thousands of dollars). For this model of calculator, the advertising expense was \$400,000 ($x = 40$) and the profit was \$2,174,375.

(a) Use a graphing utility to graph the profit function.

(b) Use the graph from part (a) to estimate another amount the company could have spent on advertising that would have produced the same profit.

(c) Use synthetic division to confirm the result of part (b) algebraically.

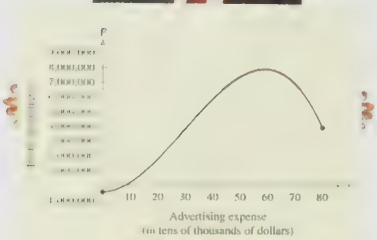
Exploration In Exercises 84 and 85, find the constant c such that the denominator will divide evenly into the numerator.

84. $\dfrac{x^3 + 4x^2 - 3x + c}{x - 4}$

85. $\dfrac{x^5 - 2x^2 + x + c}{x + 2}$

3.4 Real Zeros of Polynomial Functions







In Exercise 68 on page 294, you will use a graph to determine how much money a company should spend on advertising to obtain a certain profit.

- Find all possible rational zeros of a function using the Rational Zero Test.
- Approximate the real zeros of a polynomial function using the Intermediate Value Theorem.
- Approximate the real zeros of a polynomial function using a graphing utility.
- Apply techniques for approximating real zeros to solve an application problem.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial function (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

The Rational Zero Test

If the polynomial function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

has *integer* coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{\text{a factor of the constant term } a_0}{\text{a factor of the leading coefficient } a_n} = \frac{p}{q}$$

where p and q have no common factors other than 1. When the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Make a list of *possible rational zeros*. Then use a trial-and-error method to determine which, if any, are actual zeros of the polynomial function.

Example 1 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of

$$f(x) = x^3 + x + 1.$$

SOLUTION Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: 1 and -1

By testing these possible zeros, you can see that neither checks.

$$f(1) = (1)^3 + 1 + 1 = 3$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, you can conclude that the given function has *no* rational zeros. Note from the graph of f in Figure 3.25 that f does have one real zero (between -1 and 0). By the Rational Zero Test, you know that this real zero is *not* a rational number.

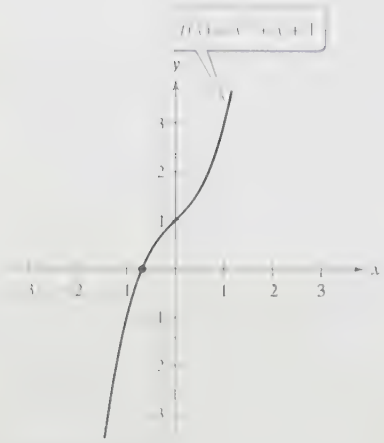


FIGURE 3.25

Checkpoint 1

Find the rational zeros of

$$f(x) = x^3 + 2x^2 + 2.$$

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Example 2 **Rational Zero Test with Leading Coefficient of 1**

Find the rational zeros of

$$f(x) = x^4 - x^3 + x^2 - 3x - 6.$$

SOLUTION Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

Test each possible rational zero. The test shows $x = -1$ and $x = 2$ are the only two rational zeros of the function.

Checkpoint 2

Find the rational zeros of

$$f(x) = x^4 + 2x^3 + x^2 - 4.$$

When the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways.

- A programmable calculator can be used to speed up the calculations.
- A graph, created either by hand or with a graphing utility, can give a good estimate of the locations of the zeros.
- Synthetic division can be used to test the possible rational zeros.



TECH TUTOR

There are several ways to use your graphing utility to locate the zeros of a polynomial function after listing the possible rational zeros. You can use the *table* feature by setting the increments of x to the smallest difference between possible rational zeros, or use the *table* feature in ASK mode. In either case, the value in the function column will be 0 when x is a zero of the function. Another way to locate zeros is to graph the function. Be sure that your viewing window contains all the possible rational zeros.

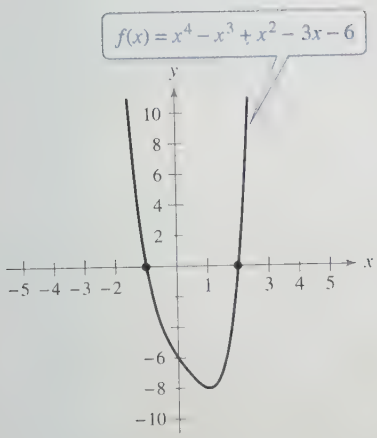


FIGURE 3.26

To see how to use synthetic division to test the possible rational zeros, let's take another look at the function given by

$$f(x) = x^4 - x^3 + x^2 - 3x - 6$$

from Example 2. To test that $x = -1$ and $x = 2$ are zeros of f, you can apply synthetic division, as shown.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

So, you have

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

Because the factor $(x^2 + 3)$ produces no real zeros, you can conclude that $x = -1$ and $x = 2$ are the only *real* zeros of f. This is verified in the graph of f shown in Figure 3.26.

Finding the first zero is often the hardest part. After that, the search is simplified by using the lower-degree polynomial obtained in synthetic division. Once the lower-degree polynomial is quadratic, either factoring or the Quadratic Formula can be used to find the remaining zeros.

Example 3 Using the Rational Zero Test

Find the rational zeros of

$$f(x) = 2x^3 + 3x^2 - 8x + 3.$$

SOLUTION The leading coefficient is 2 and the constant term is 3.

$$\text{Possible rational zeros: } \frac{\text{Factors of 3}}{\text{Factors of 2}} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|cccc} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

Checkpoint 3

Find the rational zeros of $f(x) = 2x^3 + 5x^2 + x - 2$.

Example 4 Using the Rational Zero Test

Find all the real zeros of

$$f(x) = 10x^3 - 15x^2 - 16x + 12.$$

SOLUTION The leading coefficient is 10 and the constant term is 12.

$$\text{Possible rational zeros: } \frac{\text{Factors of 12}}{\text{Factors of 10}} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

There are 32 possibilities. So, it is worth your time to stop and sketch a graph. From Figure 3.27, it looks like three reasonable choices would be $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$. Testing these by synthetic division shows that only $x = 2$ checks. So, you have

$$f(x) = (x - 2)(10x^2 + 5x - 6).$$

Using the Quadratic Formula, you find that the two additional zeros are irrational numbers.

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.5639 \quad \text{and} \quad x = \frac{-5 - \sqrt{265}}{20} \approx -1.0639$$

You can conclude that the real zeros of f are $x = 2$, $x \approx 0.5639$, and $x \approx -1.0639$.

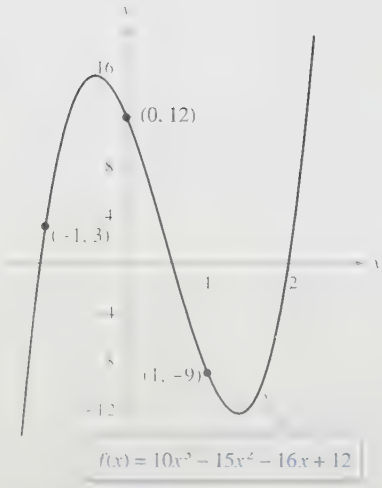


FIGURE 3.27

Checkpoint 4

Find the rational zero of $f(x) = 3x^3 + 2x^2 - 5x + 6$.

The Intermediate Value Theorem

The next theorem, called the **Intermediate Value Theorem,** tells you of the existence of real zeros of polynomial functions. The theorem implies that if

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 3.28.)

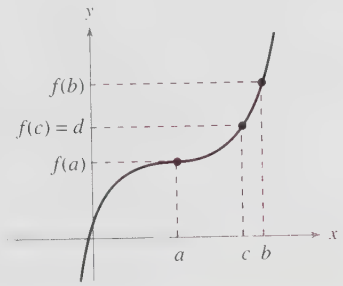


FIGURE 3.28



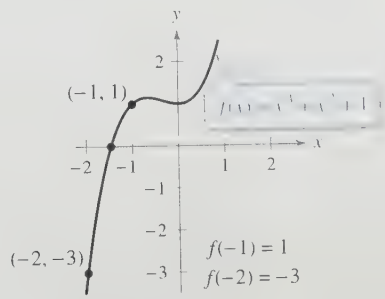
Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

The Intermediate Value Theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ where a polynomial function is positive, and another value $x = b$ where it is negative, then you can conclude that the function has at least one real zero between these two values. For example, the function given by

$$f(x) = x^3 + x^2 + 1$$

is negative when $x = -2$ and positive when $x = -1$. So, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1, as shown in Figure 3.29.



The function f must have a real zero somewhere between -2 and -1.

FIGURE 3.29

By continuing this line of reasoning, you can approximate any real zeros of a polynomial function to any desired level of accuracy. This concept is further demonstrated in Example 5.

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Example 5 **Using the Intermediate Value Theorem**

Use the Intermediate Value Theorem to approximate a real zero of

$$f(x) = x^3 - x^2 + 1.$$

SOLUTION Compute a few function values.

x	-2	-1	0	1
$f(x)$	-11	-1	1	1

Because $f(-1)$ is negative and $f(0)$ is positive, you can apply the Intermediate Value Theorem to conclude that the function has a zero between -1 and 0. To approximate this zero more closely, divide the interval $[-1, 0]$ into tenths and evaluate the function at each value of x. A few function values in the interval $[-1, 0]$ are shown below.

x	-0.9	-0.8	-0.7	-0.6	-0.5
$f(x)$	-0.539	-0.152	0.167	0.424	0.625

x	-0.4	-0.3	-0.2	-0.1
$f(x)$	0.776	0.883	0.952	0.989

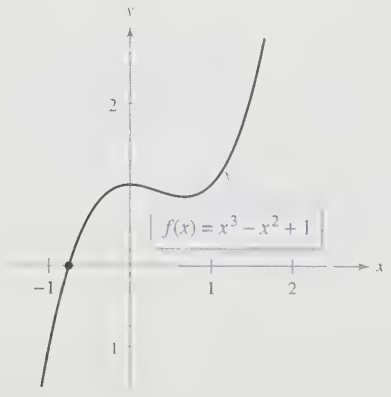
You can see that

$$f(-0.8) = -0.152$$

and

$$f(-0.7) = 0.167.$$

So, f must have a zero between -0.8 and -0.7, as shown in Figure 3.30. By continuing this process, you can approximate this zero to any desired level of accuracy.



f has a zero between -0.8 and -0.7.

FIGURE 3.30

Checkpoint 5

Use the Intermediate Value Theorem to approximate a real zero of

$$f(x) = x^3 + x - 4.$$

Approximating Zeros Using a Graphing Utility

In this section, you will study two techniques for approximating the zeros of a polynomial function using a graphing utility. The first is called the **zoom-and-trace** technique.

STUDY TIP

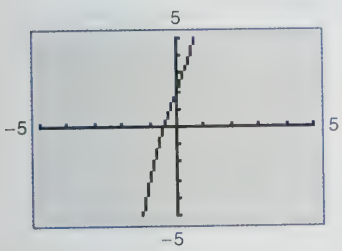
To help you visually determine when you have zoomed in enough times to reach the desired level of accuracy, set the X-scale of the viewing window to the accuracy you need and zoom in repeatedly. For instance, to approximate the zero to the nearest hundredth, set the X-scale to 0.01.

Zoom-and-Trace Technique

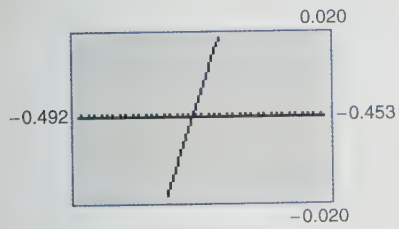
To approximate a real zero of a function with a graphing utility, use the following steps.

1. Graph the function so that the real zero you want to approximate appears as an x-intercept on the screen.
2. Move the cursor near the x-intercept and use the *zoom* feature to zoom in to get a better look at the intercept.
3. Use the *trace* feature to find the x-values that occur just before and just after the x-intercept. If the difference between these values is sufficiently small, then use their average as the approximation. If not, continue zooming in until the approximation reaches the desired level of accuracy.

The amount that a graphing utility zooms in is determined by the *zoom factor*. The zoom factor is a positive number greater than or equal to 1 that gives the ratio of the larger screen to the smaller screen. For instance, if you zoom in with a zoom factor of 2, you will obtain a screen in which the x- and y-values are half their original values. This text uses a zoom factor of 4.



(a)



(b)

FIGURE 3.31

Example 6 Approximating a Zero of a Polynomial Function

Approximate a real zero of $f(x) = x^3 + 4x + 2$ to the nearest thousandth.

SOLUTION To begin, use a graphing utility to graph the function, as shown in Figure 3.31(a). Set the X-scale to 0.001 and zoom in several times until the tick marks on the x-axis become visible. The final screen should be similar to the one shown in Figure 3.31(b). At this point, you can use the *trace* feature to determine that the x-values just to the left and right of the x-intercept are

$$x \approx -0.4735 \quad \text{and} \quad x \approx -0.4733.$$

So, to the nearest thousandth, you can approximate the zero of the function to be

$$x \approx -0.473.$$

Checkpoint 6

Approximate a real zero of $f(x) = 2x^3 - x + 3$ to the nearest thousandth.

STUDY TIP

Remember that when substituting a real zero that is an approximation into the original function, you usually obtain a result that is not exactly zero. For instance, in Example 6,

$$f(-0.473) \approx 0.002.$$

In Example 6, the cubic polynomial function has only one real zero. Remember that functions can also have two or more real zeros. In such cases, you can use the zoom-and-trace technique for each zero separately. For instance, the function given by $f(x) = x^3 - 4x^2 + x + 2$ has three real zeros, as shown in Figure 3.32. Using the zoom-and-trace technique for each real zero, you can approximate the real zeros to be

$$x \approx -0.562, \quad x = 1.000, \quad \text{and} \quad x \approx 3.562.$$

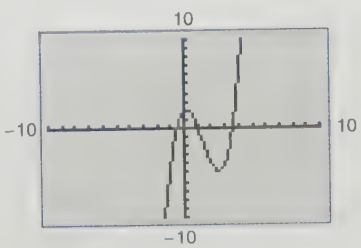


FIGURE 3.32

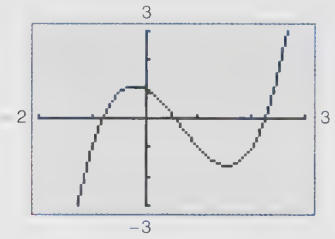
The second technique that can be used with some graphing utilities is to employ the graphing utility's *zero* or *root* feature. The name of this feature differs with different calculators. Consult your user's guide to determine if this feature is available.

Example 7 **Approximating the Zeros of a Polynomial Function**

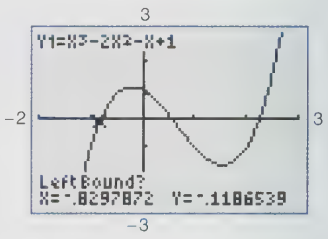
Approximate the real zeros of

$$f(x) = x^3 - 2x^2 - x + 1.$$

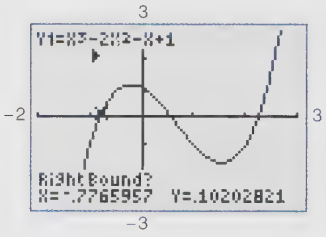
SOLUTION To begin, use a graphing utility to graph the function, as shown in the first screen in Figure 3.33. Notice that the graph has three x-intercepts. To approximate the leftmost intercept, find an appropriate viewing window and use the zero feature, as shown below. The calculator should display an approximation of $x \approx -0.8019377$, which is accurate to seven decimal places.



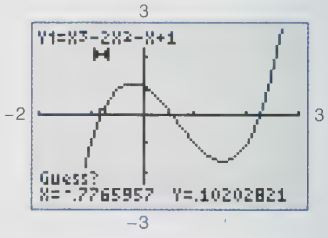
Find an appropriate viewing window, and then use the zero feature.



Move the cursor to the left of the intercept and press "Enter."



Move the cursor to the right of the intercept and press "Enter."



Move the cursor near the intercept and press "Enter."

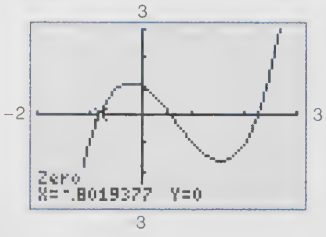


FIGURE 3.33



TECH TUTOR

For instructions on how to use the *zoom*, *trace*, *zero*, and *root* features, see Appendix A. Consult your user's guide for your graphing utility for more information.

By repeating this process, you can determine that the other two zeros are $x \approx 0.555$ and $x \approx 2.247$.

Checkpoint 7

Approximate the real zeros of

$$f(x) = x^3 - 4x^2 + 3x + 1.$$



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You may be wondering why so much time is spent in algebra trying to find the zeros of a function. The reason is that if you have a technique that will enable you to solve the equation $f(x) = 0$, you can use the same technique to solve the more general equation

$$f(x) = c$$

where c is any real number. This procedure is demonstrated in Example 8.

Example 8 **Solving the Equation $f(x) = c$**

Find a value of x such that $f(x) = 30$ for the function given by

$$f(x) = x^3 - 4x + 4.$$

SOLUTION The graph of $f(x) = x^3 - 4x + 4$ is shown in Figure 3.34. Note from the graph that $f(x) = 30$ when x is about 3.5.

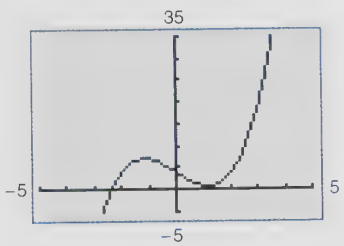


FIGURE 3.34

To use the zoom-and-trace technique to approximate this x-value more closely, consider the equation

$$x^3 - 4x + 4 = 30$$

$$x^3 - 4x - 26 = 0.$$

So, the *solutions* of the equation $f(x) = 30$ are precisely the same x-values as the *zeros* of

$$g(x) = x^3 - 4x - 26.$$

Using the graph of g, as shown in Figure 3.35, you can approximate the zero of g to be $x \approx 3.41$. You can check this value by substituting $x = 3.41$ into the original function.

$$f(3.41) = (3.41)^3 - 4(3.41) + 4$$

$$\approx 30.01 \checkmark$$

Remember that with decimal approximations, a check usually will not produce an exact value.

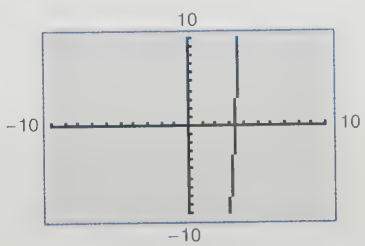


FIGURE 3.35

Checkpoint 8

Find a value of x such that $f(x) = 20$ for the function given by

$$f(x) = x^3 + 4x^2 - 1.$$

Application

Example 9 Profit and Advertising Expenses

A company that produces sports clothes estimates that the profit from selling a particular line of sportswear is given by

$$P = -0.014x^3 + 0.752x^2 - 40, \quad 0 \le x \le 50$$

where P is the profit (in tens of thousands of dollars) and x is the advertising expense (in tens of thousands of dollars). According to this model, how much money should the company spend on advertising to obtain a profit of \$2,750,000?

SOLUTION From Figure 3.36, it appears that there are two different values of x between 0 and 50 that will produce a profit of \$2,750,000.

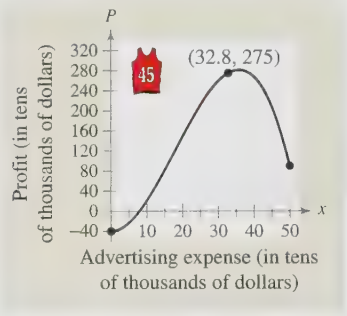


FIGURE 3.36

However, because of the context of the problem, it is clear that the better answer is the smaller of the two numbers. So, to solve the equation

$$-0.014x^3 + 0.752x^2 - 40 = 275$$

$$-0.014x^3 + 0.752x^2 - 315 = 0$$

find the zeros of the function $g(x) = -0.014x^3 + 0.752x^2 - 315$. Using the zoom-and-trace technique, you can find that the leftmost zero is $x \approx 32.8$. You can check this solution by substituting $x = 32.8$ into the original function.

$$P = -0.014(32.8)^3 + 0.752(32.8)^2 - 40 \approx 275$$

The company should spend about \$328,000 on advertising for the line of sportswear.



Checkpoint 9

In Example 9, how much money should the company spend on advertising to obtain a profit of \$2,500,000? ■

SUMMARIZE (Section 3.4)

1. Describe the Rational Zero Test *(page 282)*. For examples of using the Rational Zero Test, see Examples 1, 2, 3, and 4.
2. State the Intermediate Value Theorem *(page 285)*. For an example of using the Intermediate Value Theorem, see Example 5.
3. Describe the zoom-and-trace technique *(page 287)*. For an example of using the zoom-and-trace technique, see Example 6.

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SKILLS WARM UP 3.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.5 and 3.3.

In Exercises 1 and 2, find a polynomial function with integer coefficients having the given zeros.

1. $-1, \frac{2}{3}, 3$

2. $-2, 0, \frac{3}{4}, 2$

In Exercises 3 and 4, use synthetic division to divide.

3. $\dfrac{x^5 - 9x^3 + 5x + 18}{x + 3}$

4. $\dfrac{3x^4 + 17x^3 + 10x^2 - 9x - 8}{x + \frac{2}{3}}$

In Exercises 5–8, use the given zero to find all the real zeros of f.

5. $f(x) = 2x^3 + 11x^2 + 2x - 4,\ x = \frac{1}{2}$

6. $f(x) = 6x^3 - 47x^2 - 124x - 60,\ x = 10$

7. $f(x) = 4x^3 - 13x^2 - 4x + 6,\ x = -\frac{3}{4}$

8. $f(x) = 10x^3 + 51x^2 + 48x - 28,\ x = \frac{2}{5}$

In Exercises 9 and 10, find all real solutions of the equation.

9. $x^4 - 3x^2 + 2 = 0$

10. $x^4 - 7x^2 + 12 = 0$

Exercises 3.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Rational Zero Test with Leading Coefficient of 1 In Exercises 1–6, use the Rational Zero Test to list all possible rational zeros of f. Test each possible rational zero to determine whether it is an actual zero of f. *See Examples 1 and 2.*

1. $f(x) = x^3 + 2x^2 - 1$

2. $f(x) = x^3 + 3x^2 - x - 3$

3. $f(x) = x^3 - 4x^2 + 1$

4. $f(x) = x^4 + x^3 + 2x^2 - 1$

5. $f(x) = x^3 - 7x - 6$

6. $f(x) = x^4 + 2x^3 + 3x^2 + 2x - 8$



Using the Rational Zero Test In Exercises 7 and 8, use the Rational Zero Test to list all possible rational zeros of f. Then use a graphing utility to graph the function. Use the graph to help determine which of the possible rational zeros are actual zeros of the function.

7. $f(x) = x^3 + x^2 - 4x - 4$

8. $f(x) = 2x^4 - x^2 - 6$

Using the Rational Zero Test In Exercises 9–12, use the Rational Zero Test to list all possible rational zeros of f. Use synthetic division to determine which of the possible rational zeros are actual zeros of the function. *See Example 3.*

9. $f(x) = x^3 - 9x^2 + 20x - 12$

10. $f(x) = x^3 - 9x^2 + 27x - 27$

11. $f(x) = 2x^3 - 3x^2 - 11x + 6$

12. $f(x) = 5x^3 - 7x^2 - 58x + 24$

Finding Rational Zeros In Exercises 13–16, find the rational zeros of the polynomial function. *See Examples 1, 2, and 3.*

13. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6$

14. $f(x) = x^3 + 3x^2 - x - 3$

15. $f(x) = 4x^4 - 17x^2 + 4$

16. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

Using the Rational Zero Test In Exercises 17–24, find all the real zeros of the function. *See Example 4.*

17. $f(x) = x^3 - 6x^2 + 11x - 6$

18. $g(x) = x^3 - 4x^2 - x + 4$

19. $h(t) = t^3 + 12t^2 + 21t + 10$

20. $f(x) = x^3 - 4x^2 + 5x - 2$

21. $C(x) = 2x^3 + 3x^2 - 1$

22. $f(x) = 3x^3 - 19x^2 + 33x - 9$

23. $f(x) = x^4 - 11x^2 + 18$

24. $P(t) = t^4 - 19t^2 + 48$

Finding Real Solutions In Exercises 25–30, find all real solutions of the polynomial equation.

25. $z^4 - z^3 - 2z - 4 = 0$

26. $x^4 - 13x^2 - 12x = 0$

27. $2y^4 + 7y^3 - 26y^2 + 23y - 6 = 0$

28. $2x^4 - 11x^3 - 6x^2 + 64x + 32 = 0$

29. $x^5 - x^4 - 3x^3 + 5x^2 - 2x = 0$

30. $x^5 - 7x^4 + 10x^3 + 14x^2 - 24x = 0$

Using the Rational Zero Test In Exercises 31 and 32, (a) list all possible rational zeros of f, (b) sketch the graph of f so that some of the possible zeros in part (a) can be discarded, and (c) determine all the real zeros of f. *See Example 4.*

31. $f(x) = 32x^3 - 52x^2 + 17x + 3$

32. $f(x) = 4x^3 + 7x^2 - 11x - 18$

Using the Intermediate Value Theorem In Exercises 33–36, use the Intermediate Value Theorem to show that the function has at least one zero in the interval $[a, b]$. (You do not have to approximate the zero.) *See Example 5.*

33. $f(x) = x^3 + 2x - 5, \quad [1, 2]$

34. $f(x) = x^5 - 3x + 3, \quad [-2, -1]$

35. $f(x) = x^4 - 3x^2 - 10, \quad [2, 3]$

36. $f(x) = -x^3 + 2x^2 + 7x - 3, \quad [3, 4]$

Using the Intermediate Value Theorem In Exercises 37–40, use the Intermediate Value Theorem to approximate the zero of f in the interval $[a, b]$. Give your approximation to the nearest tenth. *See Example 5.*

37. $f(x) = x^3 + x - 1, \quad [0, 1]$

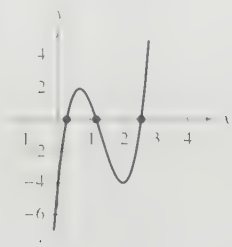
38. $f(x) = x^5 + x + 1, \quad [-1, 0]$

39. $f(x) = x^4 - 10x^2 - 11, \quad [3, 4]$

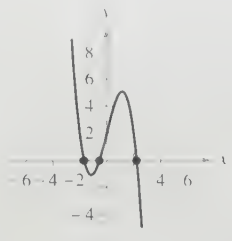
40. $f(x) = -x^3 + 3x^2 + 9x - 2, \quad [4, 5]$

Matching In Exercises 41–46, match the function with its graph. Then approximate the real zeros of the function to three decimal places. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

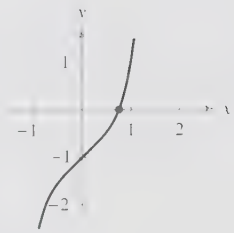
(a)



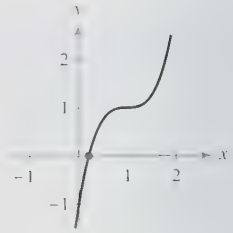
(b)



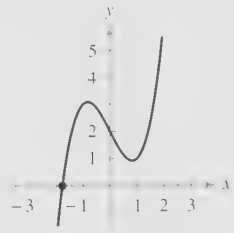
(c)



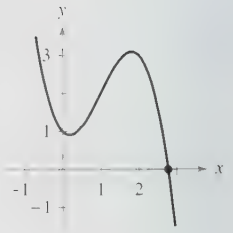
(d)



(e)



(f)



41. $f(x) = x^3 - 2x + 2$

42. $f(x) = x^5 + x - 1$

43. $f(x) = 2x^3 - 6x^2 + 6x - 1$

44. $f(x) = 5x^3 - 20x^2 + 20x - 4$

45. $f(x) = -x^3 + 3x^2 - x + 1$

46. $f(x) = -x^3 + 4x + 2$



Approximating Zeros of a Polynomial Function In Exercises 47–50, use the *zoom* and *trace* features of a graphing utility to approximate the real zeros of f. Give your approximations to the nearest thousandth. *See Example 6.*

47. $f(x) = x^4 - x - 3$ **48.** $f(x) = 4x^3 + 14x - 8$

49. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

50. $f(x) = -x^3 + 2x^2 + 4x + 5$



Approximating Zeros of a Polynomial Function In Exercises 51–54, use the *zero* or *root* feature of a graphing utility to approximate the real zeros of f. Give your approximations to the nearest thousandth. *See Example 7.*

51. $f(x) = x^4 + x - 3$

52. $f(x) = -x^4 + 2x^3 + 4$

53. $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$

54. $f(x) = 3x^4 - 12x^3 + 27x^2 + 4x - 4$

Rational and Irrational Zeros In Exercises 55–58, match the cubic function with the numbers of rational and irrational zeros.

(a) Rational zeros: 0; Irrational zeros: 1

(b) Rational zeros: 3; Irrational zeros: 0

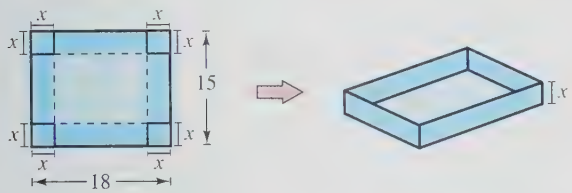
(c) Rational zeros: 1; Irrational zeros: 2

(d) Rational zeros: 1; Irrational zeros: 0

55. $f(x) = x^3 - 1$ **56.** $f(x) = x^3 - 2$

57. $f(x) = x^3 - x$ **58.** $f(x) = x^3 - 2x$

59. Dimensions of a Box An open box is to be made from a rectangular piece of material, 18 inches by 15 inches, by cutting equal squares from the corners and turning up the sides (see figure).



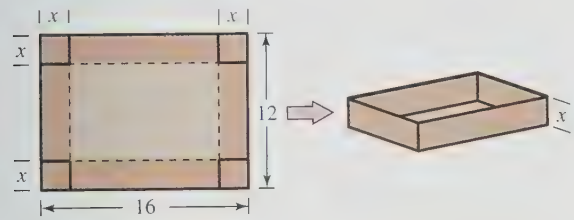
(a) Write the volume V of the box as a function of x. Determine the domain of the function.

(b) Sketch the graph of the function and approximate the dimensions of the box that yield a maximum volume.

(c) Find values of x such that $V = 108$. Which of these values is a physical impossibility in the construction of the box? Explain.

(d) What value of x should you use to make the tallest possible box with a volume of 108 cubic inches?

60. Dimensions of a Box An open box is to be made from a rectangular piece of material, 16 inches by 12 inches, by cutting equal squares from the corners and turning up the sides (see figure).



(a) Write the volume V of the box as a function of x. Determine the domain of the function.

(b) Sketch the graph of the function and approximate the dimensions of the box that yield a maximum volume.

(c) Find values of x such that $V = 120$. Which of these values is a physical impossibility in the construction of the box? Explain.

(d) What value of x should you use to make the tallest possible box with a volume of 120 cubic inches?

61. Dimensions of a Terrarium A rectangular terrarium with a square cross section has a combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the terrarium, given that the volume is 11,664 cubic inches.

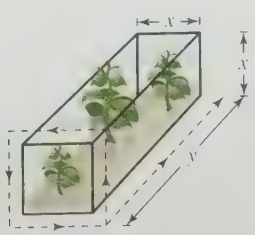


Figure for 61 and 62

62. Dimensions of a Terrarium A rectangular terrarium has a combined length and girth (perimeter of a cross section) of 120 inches (see figure). Find the dimensions of the terrarium, given that the volume is 16,000 cubic inches.

63. Geometry A bulk food storage bin with dimensions 2 feet by 3 feet by 4 feet needs to be increased in size to hold five times as much food as the current bin. (Assume each dimension is increased by the same amount.)

(a) Write a function that represents the volume V of the new bin.

(b) Find the dimensions of the new bin.

64. Medical Lab A medical examiner wants to enlarge an existing medical lab such that the total area of the new lab is 1.5 times that of the original lab. The original lab's dimensions are 120 feet by 75 feet. The medical examiner wants to increase each dimension by the same amount.

(a) Write a function that represents the area A of the new lab.

(b) Find the dimensions of the new lab.

(c) The medical examiner wants to add a length to the sides of the lab that are 120 feet, and twice this length to the sides that are 75 feet, such that the total area of the new lab is 1.5 times that of the original lab. Repeat parts (a) and (b). Explain your results.

65. Medicine The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t \geq 0.$$

The concentration is greatest when

$$3t^4 + 2t^3 - 300t - 50 = 0.$$

Approximate this time to the nearest hundredth of an hour.

66. Transportation Cost The transportation cost C (in thousands of dollars) of the components used in manufacturing prefabricated homes is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where x is the order size (in hundreds). The cost is a minimum when

$$3x^3 - 40x^2 - 2400x - 36{,}000 = 0.$$

Approximate the optimal order size to the nearest unit.

67. Online Sales The revenues per share R (in dollars) for Amazon.com for the years 2001 through 2010 are shown in the table. *(Source: Amazon.com)*

Year	Revenue per share, R
2001	8.37
2002	10.14
2003	13.05
2004	17.16
2005	20.41

Year	Revenue per share, R
2006	25.87
2007	35.66
2008	44.78
2009	55.20
2010	75.84

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the *regression* feature of the graphing utility to find a linear model, a quadratic model, a cubic model, and a quartic model for the data.

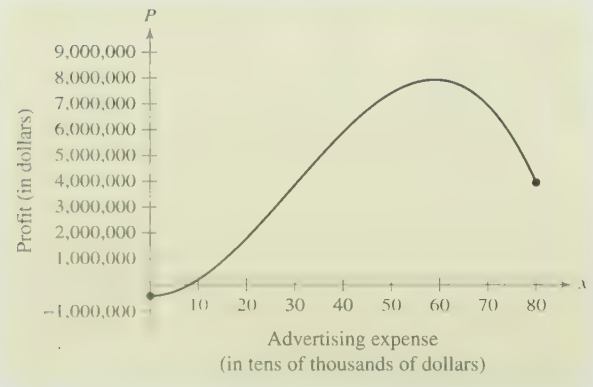
(c) Use the graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the revenue per share will be about \$100. Explain any differences in the predictions.

68. Advertising Cost A company that produces video games estimates that the profit P (in dollars) from selling a new game is given by

$$P = -82x^3 + 7250x^2 - 450{,}000, \quad 0 \le x \le 80$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, how much money should the company spend on advertising to obtain a profit of \$5,900,000?



69. Solar Energy Photovoltaic cells convert light energy into electricity. The domestic shipments S (in peak kilowatts) of photovoltaic cells and modules for the years 2000 through 2009 are shown in the table. *(Source: U.S. Energy Information Administration)*

Year	Shipments, S
2000	19,838
2001	36,310
2002	45,313
2003	48,664
2004	78,346

Year	Shipments, S
2005	134,465
2006	206,511
2007	280,475
2008	524,252
2009	601,133

(a) Use a spreadsheet software program to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the spreadsheet software program to find a quadratic model and a cubic model for the data.

(c) Use each model to predict the year in which the shipments will be about 1,000,000 peak kilowatts. Then discuss the appropriateness of each model for predicting future values.

70. Advertising Cost A company that manufactures hydroponic gardening systems estimates that the profit P (in dollars) from selling a new system is given by

$$P = -35x^3 + 2700x^2 - 300{,}000, \quad 0 \le x \le 70$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, how much money should the company spend on advertising to obtain a profit of \$1,800,000?

71. Demand Function A company that produces cell phones estimates that the demand D for a new model of phone is given by

$$D = -x^3 + 54x^2 - 140x - 3000, \quad 10 \le x \le 50$$

where x is the price of the phone (in dollars).

(a) Use a graphing utility to graph D. Use the *trace* feature to determine the values of x for which the demand is 14,400 phones.

(b) You may also determine the values of x for which the demand is 14,400 phones by setting D equal to 14,400 and solving for x with a graphing utility. Discuss this alternative solution method. Of the solutions that lie within the given interval, what price would you recommend the company charge for the phones?

72. Demand Function A company that produces handheld organizers estimates that the demand D for a new model of organizer is given by

$$D = -0.005x^3 + 2.65x^2 - 70x - 2500, \quad 50 \le x \le 500$$

where x is the price of the organizer (in dollars).

(a) Use a graphing utility to graph D. Use the *trace* feature to determine the values of x for which the demand will be 80,000 organizers.

(b) You may also determine the values of x for which the demand will be 80,000 organizers by setting D equal to 80,000 and solving for x with a graphing utility. Discuss this alternative solution method. Of the solutions that lie within the given interval, what price would you recommend the company charge for the new organizers?

73. Height of a Baseball A baseball is launched upward from ground level with an initial velocity of 48 feet per second, and its height h (in feet) is

$$h(t) = -16t^2 + 48t, \quad 0 \le t \le 3$$

where t is the time (in seconds). You are told the ball reaches a height of 64 feet. Is this possible?

74. Exploration Use a graphing utility to graph the function $f(x) = x^4 - 4x^2 + k$ for different values of k. Find the values of k such that the zeros of f satisfy the specified characteristics. (Some parts do not have unique answers.)

(a) Four real zeros

(b) Two real zeros and two complex roots

75. Reasoning Is it possible that a second-degree polynomial function with integer coefficients has one rational zero and one irrational zero? If so, give an example.

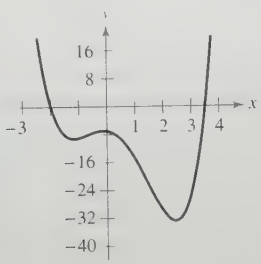
76. Graphical Reasoning The graph of one of the following functions is shown below. Identify the function shown in the graph. Explain why each of the others is not the correct function. Use a graphing utility to verify your result.

(a) $f(x) = x^2(x + 2)(x - 3.5)$

(b) $g(x) = (x + 2)(x - 3.5)$

(c) $h(x) = (x + 2)(x - 3.5)(x^2 + 1)$

(d) $k(x) = (x + 1)(x + 2)(x - 3.5)$



77. Reasoning Is it possible that a third-degree polynomial function with integer coefficients has one rational zero and two irrational zeros? If so, give an example.

78. HOW DO YOU SEE IT? Use the information in the table.

Interval	Value of $f(x)$
$(-\infty, -2)$	Positive
$(-2, 1)$	Negative
$(1, 4)$	Negative
$(4, \infty)$	Positive

(a) What are the three real zeros of the polynomial function f?

(b) What can be said about the behavior of the graph of f at $x = 1$?

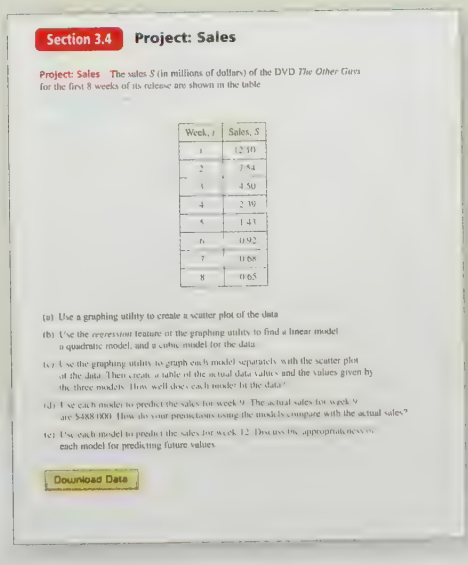
(c) What is the least possible degree of f? Explain. Can the degree of f ever be odd? Explain.

(d) Is the leading coefficient of f positive or negative? Explain.

(e) Write an equation for f. (There are many correct answers.)

(f) Sketch a graph of the equation you wrote in part (e).

79. Project: Sales For a project involving the sales of the DVD *The Other Guys* for the first 8 weeks of its release, visit this text's website at *www.cengagebrain.com.* *(Source: Nash Information Services)*


Section 3.4 Project: Sales

Project: Sales The sales S (in millions of dollars) of the DVD *The Other Guys* for the first 8 weeks of its release are shown in the table

Week, t	Sales, S
1	12.30
2	7.54
3	4.50
4	2.39
5	1.43
6	0.92
7	0.68
8	0.65

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a linear model, a quadratic model, and a cubic model for the data.

(c) Use the graphing utility to graph each model separately with the scatter plot of the data. Then create a table of the actual data values and the values given by the three models. How well does each model fit the data?

(d) Use each model to predict the sales for week 9. The actual sales for week 9 are $488,000. How do your predictions using the models compare with the actual sales?

(e) Use each model to predict the sales for week 12. Discuss the appropriateness of each model for predicting future values.

Download Data

QUIZ YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, sketch the graph of the quadratic function. Identify the vertex and the intercepts.

1. $f(x) = (x + 1)^2 - 2$
2. $f(x) = 25 - x^2$

In Exercises 3 and 4, describe the right-hand and left-hand behavior of the graph of the polynomial function. Verify with a graphing utility.

3. $f(x) = -2x^3 + 7x^2 - 9$
4. $f(x) = x^4 + 7x^2 - 8$
5. Use synthetic division to evaluate $f(x) = 2x^4 + x^3 + 18x^2 - 4$ when $x = -3$.

In Exercises 6 and 7, write the function in the form $f(x) = (x - k)q(x) + r$ for the given value of k, and demonstrate that $f(k) = r$.

6. $f(x) = x^4 - 5x^2 + 4, \quad k = -1$
7. $f(x) = x^3 + 5x^2 - 2x - 24, \quad k = 2$
8. Simplify $\dfrac{2x^4 + 9x^3 - 32x^2 - 99x + 180}{x^2 + 2x - 15}$.

In Exercises 9–12, find all the real zeros of the function.

9. $f(x) = -2x^3 - 7x^2 + 10x + 35$
10. $f(x) = 4x^4 - 37x^2 + 9$
11. $f(x) = 3x^4 + 4x^3 - 3x - 4$
12. $f(x) = 2x^3 - 3x^2 + 2x - 3$
13. The profit P (in dollars) for a clothing company is



$P = -95x^3 + 5650x^2 - 250{,}000, \quad 0 \le x \le 55$

where x is the advertising expense (in tens of thousands of dollars). What is the profit for an advertising expense of $450,000? Use a graphing utility to approximate another advertising expense that would yield the same profit.

14. The worldwide land areas A (in millions of hectares) of transgenic crops for the years 2000 through 2010 are shown in the table. *(Source: International Service for the Acquisition of Agri-Biotech Applications)*



(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a linear model, a quadratic model, a cubic model, and a quartic model for the data.

(c) Use the graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the land area will be about 170 million hectares. Explain any differences in the predictions.

Year	Area, A
2000	44.2
2001	52.6
2002	58.7
2003	67.7
2004	81.0
2005	90.0
2006	102.0
2007	114.3
2008	125.0
2009	134.0
2010	148.0

Table for 14

3.5 Complex Numbers



In Exercises 79–84 on page 305, you will determine whether various complex numbers are members of the Mandelbrot Set.

- Develop an understanding of the imaginary unit i and complex numbers.
- Perform operations with complex numbers and write the results in standard form.
- Find the complex conjugate of a complex number.
- Solve a polynomial equation that has complex solutions.
- Plot a complex number in the complex plane.

The Imaginary Unit i

Some quadratic equations have no real solutions. For instance, the quadratic equations

$$x^2 + 1 = 0 \quad \text{and} \quad x^2 = -5$$ Equations with no real solutions

have no real solutions because there is no real number x that can be squared to produce a negative number. To overcome this deficiency, mathematicians utilized an expanded system of numbers that used the **imaginary unit i,** which is defined as

$$i = \sqrt{-1}$$ Imaginary unit

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, the set of **complex numbers** is obtained. Each complex number can be written in the **standard form $a + bi$.**

Definition of a Complex Number

Let a and b be real numbers. The number $a + bi$ is called a **complex number,** and it is said to be written in **standard form.** When $b = 0$, the number $a + bi = a$ is a real number. When $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 3.37. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

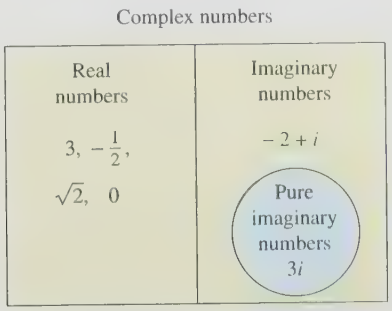


FIGURE 3.37

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$ written in standard form are **equal** to each other,

$$a + bi = c + di$$ Equality of two complex numbers

if and only if $a = c$ and $b = d$.

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Operations with Complex Numbers

To add (or subtract) two complex numbers, add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, then their sum and difference are defined as shown.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$-(a + bi) = -a - bi.$ Additive inverse

So, you have

$(a + bi) + (-a - bi) = 0 + 0i = 0.$

Example 1 Adding and Subtracting Complex Numbers

Perform the operation(s) and write each result in standard form.

a. $(3 - i) + (2 + 3i)$

b. $2i + (-4 - 2i)$

c. $3 - (-2 + 3i) + (-5 + i)$

SOLUTION

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$ Remove parentheses.

$= 3 + 2 - i + 3i$ Group like terms.

$= (3 + 2) + (-1 + 3)i$

$= 5 + 2i$ Write in standard form.

b. $2i + (-4 - 2i) = 2i - 4 - 2i$ Remove parentheses.

$= -4 + 2i - 2i$ Group like terms.

$= -4$ Write in standard form.

Note that the sum of two imaginary numbers can be a real number.

c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$

$= 3 + 2 - 5 - 3i + i$

$= 0 - 2i$

$= -2i$

Checkpoint 1

Perform the operation(s) and write each result in standard form.

a. $(4 + 7i) + (1 - 6i)$ **b.** $8i - (-10 + i) - (10 + 6i)$

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication

Commutative Properties of Addition and Multiplication

Distributive Property of Multiplication Over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive Property}\\ &= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property}\\ &= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1\\ &= ac - bd + (ad)i + (bc)i && \text{Commutative Property}\\ &= (ac - bd) + (ad + bc)i && \text{Associative Property}\end{aligned}$$

Rather than trying to memorize this multiplication rule, you should simply remember how the Distributive Property is used to multiply two complex numbers. The procedure is similar to multiplying two binomials and combining like terms (as in the FOIL Method).

Example 2 Multiplying Complex Numbers

Find each product.

a. $(i)(-6i)$ **b.** $(2 - i)(4 + 3i)$ **c.** $(3 + 2i)(3 - 2i)$ **d.** $(7 + 5i)^2$

SOLUTION

a. $(i)(-6i) = -6i^2$ Multiply.

$= -6(-1)$ $i^2 = -1$

$= 6$ Simplify.

b. $(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2$ Distributive Property

$= 8 + 6i - 4i - 3(-1)$ $i^2 = -1$

$= 8 + 3 + 6i - 4i$ Group like terms.

$= 11 + 2i$ Write in standard form.

c. $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$ Distributive Property

$= 9 - 4(-1)$ $i^2 = -1$

$= 9 + 4$ Simplify.

$= 13$ Write in standard form.

d. $(7 + 5i)^2 = 49 + 35i + 35i + 25i^2$ Distributive Property

$= 49 + 70i + 25i^2$ Simplify.

$= 49 + 70i + 25(-1)$ $i^2 = -1$

$= 49 - 25 + 70i$ Commutative Property

$= 24 + 70i$ Write in standard form.



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✓Checkpoint 2

Find each product.

a. $4(-2 + 3i)$ **b.** $(5 - 3i)^2$

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.** In general, the product of two complex conjugates can be written as shown.

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2(-1) = a^2 + b^2\end{aligned}$$

Complex conjugates can be used to write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero. To do this, multiply the numerator and denominator by the complex conjugate of the denominator to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

TECH TUTOR

Some graphing utilities can perform operations with complex numbers. Consult the user's guide for your graphing utility for more information.

Example 3 Writing Quotients of Complex Numbers in Standard Form

Write each quotient in standard form.

a. $\dfrac{1}{1 + i}$ **b.** $\dfrac{2 + 3i}{4 - 2i}$

SOLUTION

a. $\dfrac{1}{1 + i} = \dfrac{1}{1 + i}\left(\dfrac{1 - i}{1 - i}\right)$ Multiply numerator and denominator by complex conjugate of denominator.

$= \dfrac{1 - i}{1^2 - i^2}$ Expand.

$= \dfrac{1 - i}{1 - (-1)}$ $i^2 = -1$

$= \dfrac{1 - i}{2}$ Simplify.

$= \dfrac{1}{2} - \dfrac{1}{2}i$ Write in standard form.

b. $\dfrac{2 + 3i}{4 - 2i} = \dfrac{2 + 3i}{4 - 2i}\left(\dfrac{4 + 2i}{4 + 2i}\right)$ Multiply numerator and denominator by complex conjugate of denominator.

$= \dfrac{8 + 4i + 12i + 6i^2}{16 - 4i^2}$ Expand.

$= \dfrac{8 - 6 + 16i}{16 + 4}$ $i^2 = -1$

$= \dfrac{2 + 16i}{20}$ Simplify.

$= \dfrac{1}{10} + \dfrac{4}{5}i$ Write in standard form.

✓ Checkpoint 3

Write $\dfrac{6 - 7i}{1 - 2i}$ in standard form.

Complex Solutions

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of -3.

Principal Square Root of a Negative Number

If a is a positive number, the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

STUDY TIP

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid if *both* a and b are negative. For example,

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5$$

whereas

$$\sqrt{-5}\sqrt{-5} = 5i^2 = -5.$$

To avoid problems with multiplying square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

Example 4 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}i = 4\sqrt{3}i - 3\sqrt{3}i = \sqrt{3}i$

c. $(-1 + \sqrt{-3})^2 = (-1 + \sqrt{3}i)^2$

$= (-1)^2 - 2\sqrt{3}i + (\sqrt{3})^2(i^2)$

$= 1 - 2\sqrt{3}i + 3(-1)$

$= -2 - 2\sqrt{3}i$

ALGEBRA TUTOR

For help in simplifying radical expressions, see the *Chapter 3 Algebra Tutor* on page 327.

Checkpoint 4

Write $4 + \sqrt{-9}$ in standard form.

Example 5 Complex Solutions of a Quadratic Equation

Solve $3x^2 - 2x + 5 = 0$.

SOLUTION

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$ Quadratic Formula

$$= \frac{2 \pm \sqrt{-56}}{6}$$ Simplify.

$$= \frac{2 \pm 2\sqrt{14}i}{6}$$ Write in i-form.

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i$$ Write in standard form.

Checkpoint 5

Solve $x^2 + 3x + 4 = 0$.

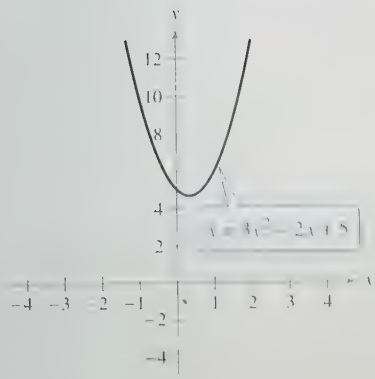


FIGURE 3.38

The graph of $f(x) = 3x^2 - 2x + 5$, shown in Figure 3.38, does not touch or cross the x-axis. This confirms that the equation in Example 5 has no real solution.

Applications

Most applications involving complex numbers are either theoretical (see the next section) or very technical, and so are not appropriate for inclusion in this text. However, to give you some idea of how complex numbers can be used in applications, a general description of their use in **fractal geometry** is presented.

To begin, consider a coordinate system called the **complex plane.** Just as every real number corresponds to a point on the real number line, every complex number corresponds to a point in the complex plane, as shown in Figure 3.39. In this figure, note that the vertical axis is the **imaginary axis** and the horizontal axis is the **real axis.** The point that corresponds to the complex number $a + bi$ is (a, b).

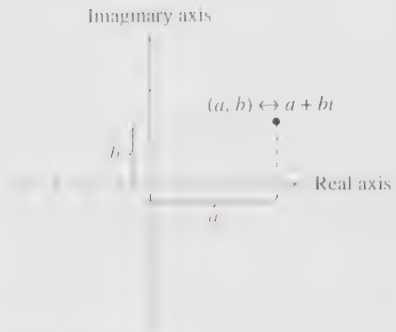


FIGURE 3.39

Complex number		Ordered pair
$a + bi$	⇨	(a, b)

From Figure 3.39, you can see that i is called the imaginary unit because it is located one unit from the origin on the imaginary axis of the complex plane.

Example 6 Plotting Complex Numbers in the Complex Plane

Plot each complex number in the complex plane.

a. $2 + 3i$

b. $-1 + 2i$

c. 4

SOLUTION



FIGURE 3.40

a. To plot the complex number $2 + 3i$, move (from the origin) two units to the right on the real axis and then three units upward. See Figure 3.40. In other words, plotting the complex number $2 + 3i$ in the complex plane is comparable to plotting the point $(2, 3)$ in the Cartesian plane.

b. The complex number $-1 + 2i$ corresponds to the point $(-1, 2)$. See Figure 3.40.

c. The complex number 4 corresponds to the point $(4, 0)$. See Figure 3.40.

✓Checkpoint 6

Plot each complex number in the complex plane.

a. $-3i$ **b.** $1 + 4i$ **c.** 1

In the hands of a person who understands "fractal geometry," the complex plane can become an easel on which stunning pictures, called **fractals,** can be drawn. The most famous such picture is called the **Mandelbrot Set,** named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

$$c, \quad c^2 + c, \quad (c^2 + c)^2 + c, \quad [(c^2 + c)^2 + c]^2 + c, \quad \ldots$$

The behavior of this sequence depends on the value of the complex number c. For some values of c, this sequence is **bounded,** which means that the absolute value of each number $\left(|a + bi| = \sqrt{a^2 + b^2}\right)$ in the sequence is less than some fixed number N. For other values of c, this sequence is **unbounded,** which means that the absolute values of the terms of the sequence become infinitely large. If the sequence is bounded, then the complex number c is in the Mandelbrot Set. If the sequence is unbounded, then the complex number c is not in the Mandelbrot Set.

Example 7 **Members of the Mandelbrot Set**

Decide whether each complex number is in the Mandelbrot Set.

a. -2

b. i

c. $1 + i$

SOLUTION

a. For $c = -2$, the corresponding Mandelbrot sequence is

$$-2, \quad 2, \quad 2, \quad 2, \quad 2, \quad 2, \ldots.$$

Because the sequence is bounded, the complex number -2 is in the Mandelbrot Set.

b. For $c = i$, the corresponding Mandelbrot sequence is

$$i, \quad -1 + i, \quad -i, \quad -1 + i, \quad -i, \quad -1 + i, \ldots.$$

Because the sequence is bounded, the complex number i is in the Mandelbrot Set.

c. For $c = 1 + i$, the corresponding Mandelbrot sequence is

$$1 + i, \quad 1 + 3i, \quad -7 + 7i, \quad 1 - 97i, \quad -9407 - 193i,$$
$$88454401 + 3631103i, \ldots.$$

Because the sequence is unbounded, the complex number $1 + i$ is *not* in the Mandelbrot Set.



Mandelbrot Set

FIGURE 3.41

Checkpoint 7

Decide whether -3 is in the Mandelbrot Set. Explain your reasoning.

With this definition, a picture of the Mandelbrot Set would have only two colors: one color for points that are in the set (the sequence is bounded) and one color for points that are outside the set (the sequence is unbounded). Figure 3.41 shows a black and yellow picture of the Mandelbrot Set. The points that are black are in the Mandelbrot Set and the points that are yellow are not.



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SUMMARIZE (Section 3.5)

1. State the definition of a complex number and describe how to add, subtract, and multiply complex numbers *(pages 297–299)*. For examples of performing operations with complex numbers, see Examples 1 and 2.
2. State the definition of a complex conjugate and describe how it is used to find the quotient of two complex numbers *(page 300)*. For an example of writing quotients of complex numbers in standard form, see Example 3.
3. State the definition of the principal square root of a negative number *(page 301)*. For examples that use the principal square root of a negative number, see Examples 4 and 5.
4. Describe how to plot a complex number in the complex plane *(page 302)*. For an example of plotting complex numbers in the complex plane, see Example 6.

SKILLS WARM UP 3.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4 and 1.4.

In Exercises 1–8, simplify the expression.

1. $\sqrt{12}$
2. $\sqrt{500}$
3. $\sqrt{20} - \sqrt{5}$
4. $\sqrt{27} - \sqrt{243}$
5. $\sqrt{24}\sqrt{6}$
6. $2\sqrt{18}\sqrt{32}$
7. $\dfrac{1}{\sqrt{7}}$
8. $\dfrac{5}{\sqrt{5}}$

In Exercises 9 and 10, solve the quadratic equation.

9. $x^2 + x - 1 = 0$
10. $x^2 + 2x - 1 = 0$

Exercises 3.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. **Discovery** Write out the first 16 positive integer powers of i ($i, i^2, i^3, \ldots, i^{16}$), and write each as i, $-i$, 1, or -1. What pattern do you observe?
2. **Finding Powers of i** Use the pattern you found in Exercise 1 to help you write each power of i as i, $-i$, 1, or -1.
   (a) i^{28} (b) i^{37} (c) i^{127} (d) i^{82}

Equality of Complex Numbers In Exercises 3–6, find the real numbers a and b such that the equation is true.

3. $a + bi = 7 + 12i$
4. $a + bi = -2 - 5i$
5. $(a + 3) + 2bi = 7 - 4i$
6. $(a + 6) + 2bi = 6 - 5i$

Adding, Subtracting, or Multiplying Complex Numbers In Exercises 7–26, perform the indicated operation(s) and write the result in standard form. *See Examples 1 and 2.*

7. $(-4 + 3i) + (6 - 2i)$
8. $(13 - 2i) + (-5 + 6i)$
9. $(12 + 5i) - (7 - i)$
10. $(3 + 2i) - (6 + 13i)$
11. $\frac{5}{3}i + \left(\frac{5}{3} + \frac{11}{3}i\right)$
12. $\frac{3}{4}i + \left(\frac{3}{5} + \frac{4}{5}i\right)$
13. $(2.7 + 4.3i) + 6.0 - (7.4 - 4.7i)$
14. $9.8 - (1.6 + 3.2i) + (-5.8 + 4.3i)$
15. $(3 + 4i)(3 - 4i)$
16. $(8 + 3i)(8 - 3i)$
17. $(2 + 3i)(1 - i)$
18. $(6 - 5i)(1 + i)$
19. $5i(4 - 6i)$
20. $-2i(7 + 9i)$
21. $(5 + 6i)^2$
22. $(3 - 7i)^2$
23. $(3 + 4i)^2 + (3 - 4i)^2$
24. $(2 - 5i)^2 - (2 + 5i)^2$
25. $\left(\sqrt{5} - \sqrt{3}i\right)\left(\sqrt{5} + \sqrt{3}i\right)$
26. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

Writing Complex Numbers in Standard Form In Exercises 27–38, write the complex number in standard form and find its complex conjugate.

27. $9 + \sqrt{-16}$
28. $2 + \sqrt{-25}$
29. $-3 - \sqrt{-12}$
30. $1 - \sqrt{-8}$
31. -21
32. 45
33. $-6i + i^2$
34. $4i^2 - 2i^3$
35. $-5i^5$
36. $(-i)^3$
37. $\left(\sqrt{-6}\right)^2 + 3$
38. $\left(\sqrt{-4}\right)^2 - 5$

Writing Quotients of Complex Numbers in Standard Form In Exercises 39–50, write the quotient in standard form. *See Example 3.*

39. $\dfrac{3 - i}{3 + i}$
40. $\dfrac{8 - 5i}{1 - 3i}$
41. $\dfrac{5}{4 - 2i}$
42. $\dfrac{3}{1 + 2i}$
43. $\dfrac{7 + 10i}{2i}$
44. $\dfrac{8 + 15i}{3i}$
45. $\dfrac{1}{(2i)^3}$
46. $\dfrac{1}{(3i)^3}$
47. $\dfrac{4}{(1 - 2i)^3}$
48. $\dfrac{3}{(5 - 2i)^2}$
49. $\dfrac{(21 - 7i)(4 + 3i)}{2 - 5i}$
50. $\dfrac{(3 - i)(2 + 5i)}{4 + 3i}$

Writing Complex Numbers in Standard Form In Exercises 51–58, perform the indicated operation and write the result in standard form. *See Example 4.*

51. $\left(-2 + \sqrt{-8}\right) - \left(5 - \sqrt{-50}\right)$
52. $\left(5 + \sqrt{-18}\right) - \left(3 + \sqrt{-32}\right)$
53. $\sqrt{-3} \cdot \sqrt{-8}$
54. $\sqrt{-5} \cdot \sqrt{-10}$

55. $(\sqrt{-10})^2$
56. $(\sqrt{-75})^3$
57. $(2 - \sqrt{-8})(8 + \sqrt{-6})$
58. $(3 + \sqrt{-5})(7 - \sqrt{-10})$

Error Analysis In Exercises 59 and 60, a student has handed in the specified problem. Find the error(s) and discuss how to explain the error(s) to the student.

59. Write $\frac{5}{3 - 2i}$ in standard form.

$$\frac{5}{3 - 2i} \cdot \frac{3 + 2i}{3 + 2i} = \frac{15 + 10i}{9 - 4} = 3 + 2i$$

60. Multiply $(\sqrt{-4} + 3)(i - \sqrt{-3})$.

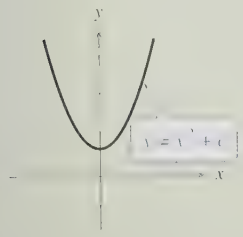
$$(\sqrt{-4} + 3)(i - \sqrt{-3})$$
$$= i\sqrt{-4} - \sqrt{-4}\sqrt{-3} + 3i - 3\sqrt{-3}$$
$$= -2i - \sqrt{12} + 3i - 3i\sqrt{3}$$
$$= (1 - 3\sqrt{3})i - 2\sqrt{3}$$

Complex Solutions of a Quadratic Equation In Exercises 61–70, solve the quadratic equation. *See Example 5.*

61. $x^2 + 9 = 0$
62. $x^2 + 36 = 0$
63. $x^2 - 2x + 2 = 0$
64. $x^2 + 6x + 10 = 0$
65. $4x^2 + 16x + 17 = 0$
66. $9x^2 - 6x + 37 = 0$
67. $4x^2 + 16x + 20 = 0$
68. $9x^2 - 6x + 2 = 0$
69. $16t^2 - 4t + 3 = 0$
70. $5s^2 + 6s + 3 = 0$

71. **Think About It** Can a quadratic equation have one real solution and one complex imaginary solution? Explain. (*Hint:* Consider the Quadratic Formula.)

72. **HOW DO YOU SEE IT?** Which statement is true for the function shown in the graph? Explain. For each of the other statements, describe a translation of the graph that represents a function for which the statement is true.



(a) The function has two different real zeros.

(b) The function has one repeated real zero.

(c) The function has two complex imaginary zeros.

Plotting Complex Numbers in the Complex Plane In Exercises 73–78, plot the complex number in the complex plane. *See Example 6.*

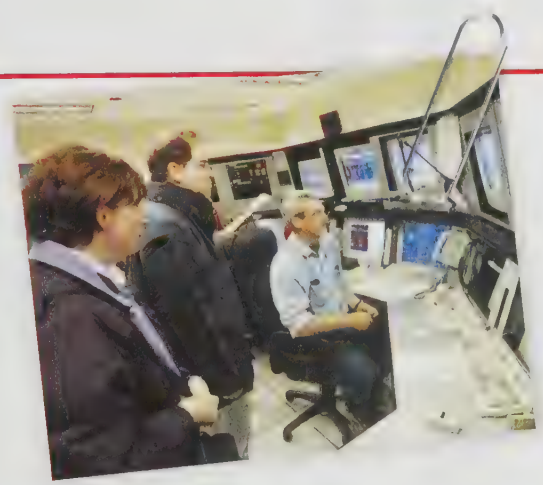
73. 3
74. i
75. $-2 + i$
76. $-2 - 3i$
77. $1 - 2i$
78. $4 + 3i$

Members of the Mandelbrot Set In Exercises 79–84, decide whether the complex number is in the Mandelbrot Set. Explain your reasoning. *See Example 7.*

79. $c = 0$
80. $c = 2$
81. $c = 1$
82. $c = -1$
83. $c = \frac{1}{2}i$
84. $c = -i$

True or False? In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.

85. There is no complex number that is equal to its conjugate.
86. The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two complex numbers.



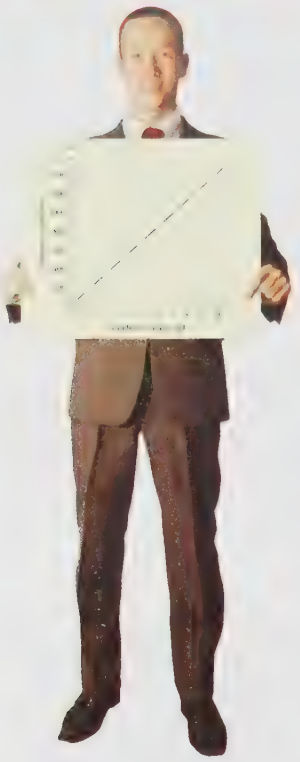
Business Capsule

Fractal Graphics, established in Australia in 1992, built a world-class reputation as a leader in application of 3-D visualization technology to the interpretation of complex geoscientific models. In 2002, Fractal Graphics split to form the software development group Fractal Technologies Pty Ltd and the geological consulting group Fractal Geoscience. Five years later, Fractal Technologies and its software, FracSIS, were acquired by global mining technology leader Runge Limited. FracSIS combines geological, geochemical, and geophysical data with an interactive 3-D visualization environment.

87. **Research Project** Use your campus library, the Internet, or some other reference source to find information about a company that uses algorithms to generate 3-D images or gaming software. Write a report about the company.

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3.6 The Fundamental Theorem of Algebra



In Exercise 69 on page 313, you will use a graph to determine the price of a product that yields a specific revenue amount.

- Use the Fundamental Theorem of Algebra and the Linear Factorization Theorem to write a polynomial as the product of linear factors.
- Find a polynomial with real coefficients whose zeros are given.
- Factor a polynomial over the rational, real, and complex numbers, and find all real and complex zeros of a polynomial function.

The Fundamental Theorem of Algebra

You have been using the fact that an nth-degree polynomial function can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely* n zeros. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the famous German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the following theorem.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n > 0$, then $f(x)$ has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers and a_n is the leading coefficient of $f(x)$.

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called **existence theorems.** To find the zeros of a polynomial function, you still must rely on the techniques developed in the earlier parts of the text.

Method	*Section*	*Example(s)*
Factoring	3.2	4, 5
Rational Zero Test	3.4	1–4
Intermediate Value Theorem	3.4	5
Graphing Utility	3.4	6, 7

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Example 1 illustrates several cases.

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Example 1 **Zeros of Polynomial Functions**

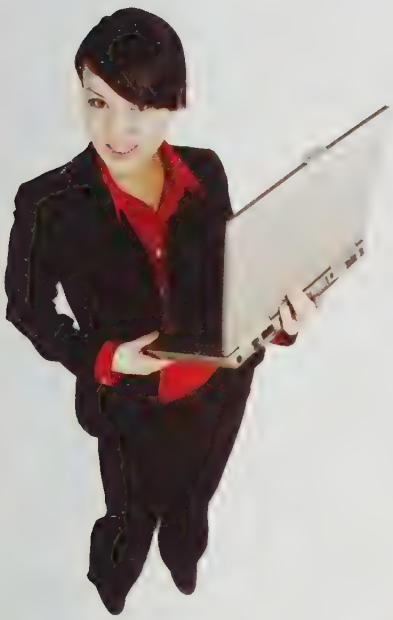
Determine the number of zeros of each polynomial function. Then list the zeros.

a. $f(x) = x - 2$ **b.** $f(x) = x^2 - 6x + 9$

c. $f(x) = x^3 + 4x$ **d.** $f(x) = x^4 - 1$

SOLUTION

a. The first-degree polynomial function given by $f(x) = x - 2$ has exactly *one* zero: $x = 2$.



b. Counting multiplicity, the second-degree polynomial function given by

$$f(x) = x^2 - 6x + 9$$
$$= (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$.

c. The third-degree polynomial function given by

$$f(x) = x^3 + 4x$$
$$= x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree polynomial function given by

$$f(x) = x^4 - 1$$
$$= (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

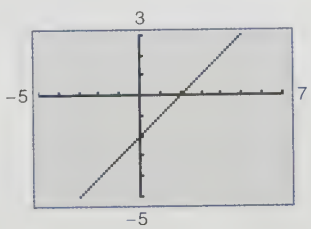
Checkpoint 1

Determine the number of zeros of $f(x) = x^4 - 36$. Then list the zeros.

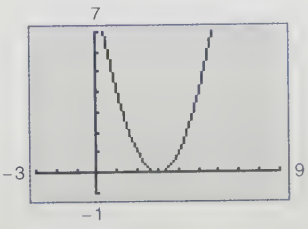
TECH TUTOR

Remember that when you use a graphing utility to locate the zeros of a function, the only zeros that appear as x-intercepts are the *real zeros*. Match the graphs below with the functions in Example 1. Which zeros appear on the graphs?

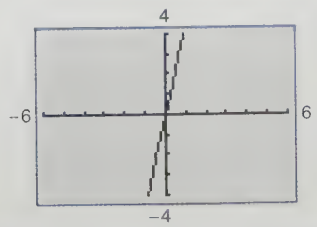
(a)



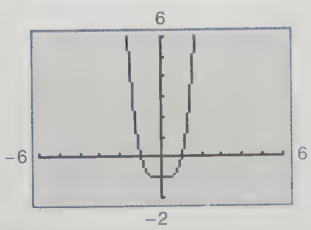
(b)



(c)



(d)



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Example 2 shows how you can use the methods described in Sections 3.3 and 3.4 (the Rational Zero Test, synthetic division, and factoring) to find all the zeros of a polynomial function, including the complex zeros.

Example 2 Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$

and write the polynomial as the product of the linear factors.

SOLUTION Using the Rational Zero Test, the possible rational zeros are

$$\pm 1, \quad \pm 2, \quad \pm 4, \quad \text{and} \quad \pm 8.$$

Synthetic division produces the results below.

1	1	0	1	2	−12	8
		1	1	2	4	−8
	1	1	2	4	−8	0

⇒ 1 is a zero.

1	1	1	2	4	−8
		1	2	4	8
	1	2	4	8	0

⇒ 1 is a repeated zero.

−2	1	2	4	8
		−2	0	−8
	1	0	4	0

⇒ −2 is a zero.

So, you have

$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x - 1)(x + 2)(x^2 + 4). \end{aligned}$$

By factoring $x^2 + 4$ as the difference of two squares over the imaginary numbers

$$\begin{aligned} x^2 - (-4) &= x^2 - \left(\sqrt{-4}\right)^2 \\ &= \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) \\ &= (x - 2i)(x + 2i) \end{aligned}$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i).$$

So, the zeros of f are

$$1, \quad 1, \quad -2, \quad 2i, \quad \text{and} \quad -2i.$$

Note from the graph of f shown in Figure 3.42 that the *real* zeros are the only ones that appear as x-intercepts.

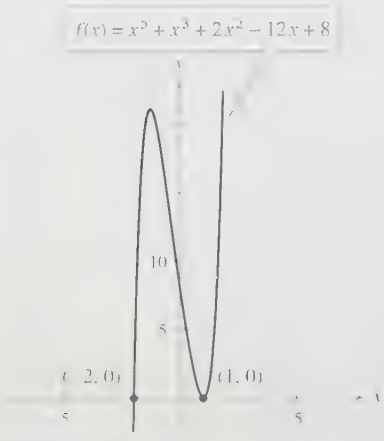


FIGURE 3.42

✓ Checkpoint 2

Find all the zeros of each function and write the polynomial as the product of linear factors.

a. $f(x) = x^4 + 8x^2 - 9$

b. $g(x) = x^5 - 5x^4 + 4x^3 + 4x^2 + 3x + 9$

Conjugate Pairs

In Example 2, note that the two imaginary zeros are **conjugates.** That is, they are of the form

$$a + bi \quad \text{and} \quad a - bi.$$

Complex Zeros Occur in Conjugate Pairs

Let f be a polynomial function that has *real coefficients.* If $a + bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only if the polynomial function has *real coefficients*. For instance, the result applies to the function given by

$$f(x) = x^2 + 1$$

but not to the function given by

$$g(x) = x - i.$$

You have been using the Rational Zero Test, synthetic division, and factoring to find the zeros of polynomial functions. The Linear Factorization Theorem enables you to reverse this process and find a polynomial function when its zeros are given.

Example 3 **Finding a Polynomial Function with Given Zeros**

Find a *fourth-degree* polynomial function with real coefficients that has

$$-1, \quad -1, \quad \text{and} \quad 3i$$

as zeros.

SOLUTION Because $3i$ is a zero *and* the function is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, the four zeros are

$$-1, \quad -1, \quad 3i, \quad \text{and} \quad -3i.$$

Then, using the Linear Factorization Theorem, $f(x)$ can be written as the product of linear factors, as shown.

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i)$$

For simplicity, let $a = 1$. Then multiply the factors with real coefficients to get

$$(x + 1)(x + 1) = (x^2 + 2x + 1)$$

and multiply the complex conjugates to get

$$(x - 3i)(x + 3i) = (x^2 + 9).$$

So, you obtain the following fourth-degree polynomial function.

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9 \end{aligned}$$

Checkpoint 3

Find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

a. $-3, 3, 2i$

b. $1, 5 + i, 5 - i$

Factoring a Polynomial

The Linear Factorization Theorem shows that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n)$$

However, this result includes the possibility that some of the values of c_i are complex. The following result implies that even if you do not want to get involved with "imaginary factors," you can still write $f(x)$ as the product of linear and/or quadratic factors.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is said to be **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals.* For example, the quadratic $x^2 + 1 = (x - i)(x + i)$ is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$$

is irreducible over the rationals, but it is *reducible* over the reals.

Example 4 Factoring a Polynomial

Complete parts (a)–(d) using the polynomial $x^4 - x^2 - 20$.

a. Write the polynomial as the product of factors that are irreducible over the *rationals.*

b. Write the polynomial as the product of linear factors and quadratic factors that are irreducible over the *reals.*

c. Write the polynomial in completely factored form.

d. How many of the zeros are rational, irrational, or imaginary?

SOLUTION

a. Factor the polynomial into the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = (x + \sqrt{5})(x - \sqrt{5})(x^2 + 4)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = (x + \sqrt{5})(x - \sqrt{5})(x - 2i)(x + 2i).$$

d. Using the completely factored form, you can conclude that there are no rational zeros, two irrational zeros $(\pm\sqrt{5})$, and two imaginary zeros $(\pm 2i)$.

✓ Checkpoint 4

In Example 4, complete parts (a)–(d) using the polynomial $x^4 + x^2 - 12$. ■

Example 5 Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

SOLUTION Because imaginary zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$[x - (1 + 3i)] \quad \text{and} \quad [x - (1 - 3i)]$$

are factors of $f(x)$. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into $f(x)$ as shown.

$$\begin{array}{r} x^2 - x - 6 \\ x^2 - 2x + 10 \overline{)\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\ \underline{x^4 - 2x^3 + 10x^2} \\ -x^3 - 4x^2 + 2x \\ \underline{-x^3 + 2x^2 - 10x} \\ -6x^2 + 12x - 60 \\ \underline{-6x^2 + 12x - 60} \\ 0 \end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are $1 + 3i$, $1 - 3i$, 3, and -2.

Checkpoint 5

Find all the zeros of $f(x) = 3x^3 - 5x^2 + 48x - 80$ given that $4i$ is a zero of f. ■



SUMMARIZE (Section 3.6)

1. State the Fundamental Theorem of Algebra and the Linear Factorization Theorem *(page 306)*. For examples that apply these theorems, see Examples 1 and 2.
2. State the definition of conjugates *(page 309)*. For an example of using conjugates to find a polynomial function with given zeros, see Example 3.
3. Describe what it means for a quadratic factor to be irreducible over the reals and irreducible over the rationals *(page 310)*. For an example of writing a polynomial as the product of factors that are irreducible over the rationals, and as the product of linear factors and quadratic factors that are irreducible over the reals, see Example 4.

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SKILLS WARM UP 3.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 3.5.

In Exercises 1–4, write the complex number in standard form and find its complex conjugate.

1. $4 - \sqrt{-29}$ **2.** $-5 - \sqrt{-144}$ **3.** $-1 + \sqrt{-32}$ **4.** $6 + \sqrt{-1/4}$

In Exercises 5–10, perform the indicated operation(s) and write the result in standard form.

5. $(-3 + 6i) - (10 - 3i)$ **6.** $(12 - 4i) + 20i$

7. $(4 - 2i)(3 + 7i)$ **8.** $(2 - 5i)(2 + 5i)$

9. $\dfrac{1 + i}{1 - i}$ **10.** $(3 + 2i)^3$

Exercises 3.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Zeros of Polynomial Functions In Exercises 1–6, determine the number of zeros of the polynomial function. *See Example 1.*

1. $f(x) = x - 7$ **2.** $g(x) = x^4 - 36$

3. $h(x) = -x^3 + 2x^2 - 5$ **4.** $f(t) = -2t^5 - 3t^3 + 1$

5. $f(x) = 6x - x^4$

6. $f(x) = 3 - 7x^2 - 5x^4 + 9x^6$

Finding the Zeros of a Polynomial Function In Exercises 7–36, find all the zeros of the function and write the polynomial as the product of linear factors. *See Example 2.*

7. $f(x) = x^2 + 25$ **8.** $f(x) = x^2 + 49$

9. $f(x) = x^2 - 12x + 26$ **10.** $f(x) = x^2 + 6x - 2$

11. $h(x) = x^2 - 4x + 1$ **12.** $g(x) = x^2 + 10x + 23$

13. $f(x) = x^4 - 81$ **14.** $f(t) = t^4 - 625$

15. $g(x) = x^3 + 5x$ **16.** $g(x) = x^3 + 7x$

17. $h(x) = x^3 - 11x^2 - 15x + 325$

18. $h(x) = x^3 - 3x^2 + 4x - 2$

19. $g(x) = x^3 - 6x^2 + 13x - 10$

20. $f(x) = x^3 - 2x^2 - 11x + 52$

21. $f(t) = t^3 - 3t^2 - 15t + 125$

22. $f(x) = x^3 + 8x^2 + 20x + 13$

23. $f(x) = x^3 + 24x^2 + 214x + 740$

24. $h(x) = x^3 - x + 6$

25. $h(x) = x^3 + 9x^2 + 27x + 35$

26. $f(s) = 2s^3 - 5s^2 + 12s - 5$

27. $f(x) = 16x^3 - 20x^2 - 4x + 15$

28. $f(x) = 9x^3 - 15x^2 + 11x - 5$

29. $f(x) = 5x^3 - 9x^2 + 28x + 6$

30. $g(x) = 3x^3 - 4x^2 + 8x + 8$

31. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

32. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

33. $f(x) = x^4 + 10x^2 + 9$

34. $f(x) = x^4 + 29x^2 + 100$

35. $f(t) = t^5 + 5t^4 - 7t^3 - 43t^2 - 8t - 48$

36. $g(x) = x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32$

Finding a Polynomial Function with Given Zeros In Exercises 37–46, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.) *See Example 3.*

37. $2, i, -i$ **38.** $5, 2i, -2i$

39. $1, 2 + i, 2 - i$

40. $6, -5 + 2i, -5 - 2i$

41. $-4, 3i, -3i, 2i, -2i$

42. $2, 2, 2, 4i, -4i$

43. $-5, -5, 1 + \sqrt{3}i$ **44.** $0, 0, 4, 1 + i$

45. $\frac{2}{3}, -1, 3 + \sqrt{2}i$ **46.** $\frac{3}{4}, -2, -\frac{1}{2} + i$

Factoring a Polynomial In Exercises 47–50, write the polynomial (a) as the product of factors that are irreducible over the rationals, (b) as the product of linear and quadratic factors that are irreducible over the reals, and (c) in completely factored form. *See Example 4.*

47. $x^4 - 7x^2 - 8$

48. $x^4 - 6x^2 - 72$

49. $x^4 - 5x^3 + 4x^2 + x - 15$
(*Hint:* One factor is $x^2 - 2x + 3$.)

50. $x^4 + x^3 + 8x^2 + 9x - 9$
(*Hint:* One factor is $x^2 + 9$.)

Finding the Zeros of a Polynomial Function In Exercises 51–60, use the given zero of f to find all the zeros of f. *See Example 5.*

51. $f(x) = 2x^3 + 3x^2 + 50x + 75, \quad 5i$

52. $f(x) = 3x^3 - x^2 + 27x - 9, \quad 3i$

53. $f(x) = x^4 - 2x^3 + 37x^2 - 72x + 36, \quad 6i$

54. $f(x) = x^3 - 7x^2 - x + 87, \quad 5 + 2i$

55. $f(x) = 4x^3 + 23x^2 + 34x - 10, \quad -3 + i$

56. $f(x) = x^4 - 2x^3 + 22x^2 + 54x + 29, \quad 2 + 5i$

57. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22, \quad -3 + \sqrt{2}i$

58. $f(x) = 2x^3 - 13x^2 + 34x - 35, \quad 2 - \sqrt{3}i$

59. $f(x) = 8x^3 - 14x^2 + 18x - 9, \quad \frac{1}{2}(1 - \sqrt{5}i)$

60. $f(x) = 25x^3 - 55x^2 - 54x - 18, \quad \frac{1}{5}(-2 + \sqrt{2}i)$

Graphical Analysis In Exercises 61–64, (a) use the *zero* or *root* feature of a graphing utility to find the real zeros of the function and (b) find the exact values of the remaining zeros.

61. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$

62. $h(x) = x^3 + 4x^2 + 14x + 20$

63. $f(x) = 8x^3 - 14x^2 + 18x - 9$

64. $h(x) = 25x^3 - 55x^2 - 54x - 18$

65. Graphical Reasoning Solve $x^4 - 5x^2 + 4 = 0$. Then use a graphing utility to graph

$$y = x^4 - 5x^2 + 4.$$

What is the connection between the solutions you found and the intercepts of the graph?

66. Graphical Reasoning Solve $x^4 + 5x^2 + 4 = 0$. Then use a graphing utility to graph

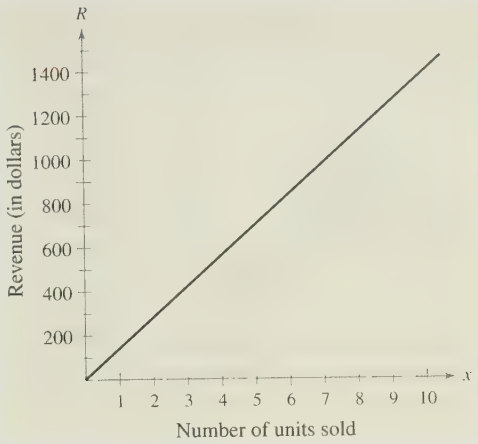
$$y = x^4 + 5x^2 + 4.$$

What is the connection between the solutions you found and the intercepts of the graph?

67. Graphical Analysis Find a fourth-degree polynomial function that has (a) four real zeros, (b) two real zeros, and (c) no real zeros. Use a graphing utility to graph the functions and describe the similarities and differences among them.

68. Graphical Analysis Find a sixth-degree polynomial function that has (a) six real zeros, (b) four real zeros, (c) two real zeros, and (d) no real zeros. Use a graphing utility to graph the functions and describe the similarities and differences among them.

69. Revenue The demand equation for a stethoscope is given by $p = 140 - 0.0001x$, where p is the unit price (in dollars) and x is the number of units. The total revenue R (in dollars) obtained by producing and selling x units is given by the model $R = xp = -0.0001x^2 + 140x$ (see figure).

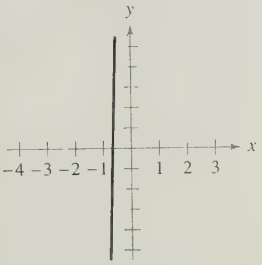


Write and solve an equation to determine a price that yields revenue of \$50 million. Discuss the meaning of your solution in the context of the problem. Use a graphing utility to verify your results.

70. HOW DO YOU SEE IT? Let f be a third-degree polynomial function with real coefficients. Two of the zeros of f are

$$-\frac{2}{3} \quad \text{and} \quad 2 + 3i.$$

The graph of f is shown below.



What is the third zero? Explain.

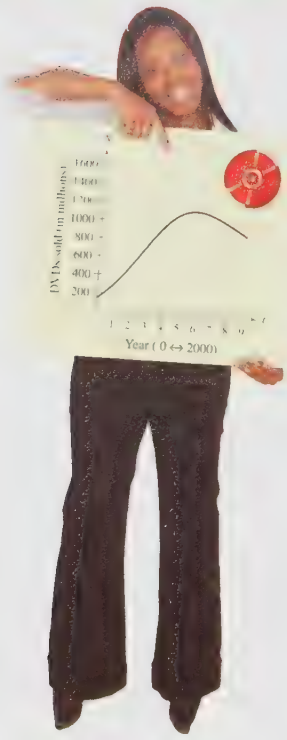
71. Reasoning Let f be a fourth-degree polynomial function with real coefficients. Three of the zeros of f are -1, $3 - 2i$, and $3 + 2i$. Explain why the fourth zero must be a real number.

72. Reasoning The imaginary number $2i$ is a zero of $f(x) = x^3 - 2ix^2 - 4x + 8i$, but the complex conjugate of $2i$ is not a zero of $f(x)$. Is this a contradiction of the conjugate pairs statement on page 309? Explain.

73. Reasoning Let f be a third-degree polynomial function with real coefficients. Explain how you know that f must have at least one zero that is a real number.

74. Think About It A student claims that the polynomial $x^4 - 7x^2 + 12$ may be factored over the rational numbers as $(x - \sqrt{3})(x + \sqrt{3})(x - 2)(x + 2)$. Do you agree with this claim? Explain your answer.

3.7 Rational Functions



In Exercise 79 on page 324, you will use a rational function to make predictions about the sales of DVDs.

- Find the domain of a rational function.
- Find the vertical and horizontal asymptotes of the graph of a rational function.
- Sketch the graph of a rational function.
- Sketch the graph of a rational function that has a slant asymptote.
- Use a rational function model to solve an application problem.

Introduction

A **rational function** is one that can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials, and $q(x)$ is not the zero polynomial. In this section, assume that $p(x)$ and $q(x)$ have no common factors. Unlike polynomial functions, whose domains consist of all real numbers, rational functions often have restricted domains. In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero.

Example 1 Finding the Domain of a Rational Function

Find the domain of the rational function $f(x) = \frac{1}{x}$ and discuss the behavior of f near any excluded x-values.

SOLUTION The domain of f is all real numbers except $x = 0$. To determine the behavior of f near this x-value, evaluate $f(x)$ to the left and right of $x = 0$.

x approaches 0 from the left

x	-1	-0.5	-0.1	-0.01	-0.001	$\rightarrow 0$
$f(x)$	-1	-2	-10	-100	-1000	$\rightarrow -\infty$

x approaches 0 from the right

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

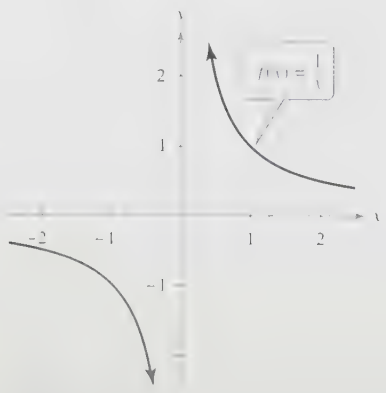


FIGURE 3.43

Note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. The graph of f is shown in Figure 3.43.

✓ Checkpoint 1

Find the domain of the rational function

$$f(x) = \frac{1}{x - 1}$$

and discuss the behavior of f near any excluded x-values. ■

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Horizontal and Vertical Asymptotes

In Example 1, the behavior of f near $x = 0$ is denoted as shown.

$f(x) \to -\infty$ as $x \to 0^-$ — $f(x)$ decreases without bound as x approaches 0 from the left.

$f(x) \to \infty$ as $x \to 0^+$ — $f(x)$ increases without bound as x approaches 0 from the right.

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 3.44. In this figure, note that the graph of f also has a **horizontal asymptote**—the line $y = 0$. The behavior of f near $y = 0$ is denoted as shown.

$f(x) \to 0$ as $x \to -\infty$ — $f(x)$ approaches 0 as x decreases without bound.

$f(x) \to 0$ as $x \to \infty$ — $f(x)$ approaches 0 as x increases without bound.

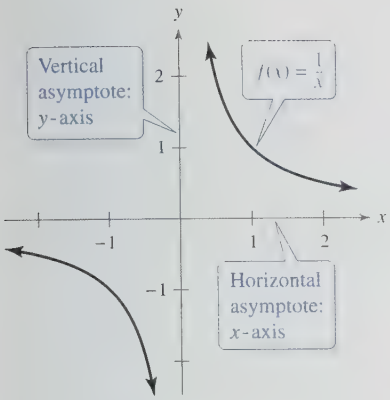


FIGURE 3.44

Definitions of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if

   $$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

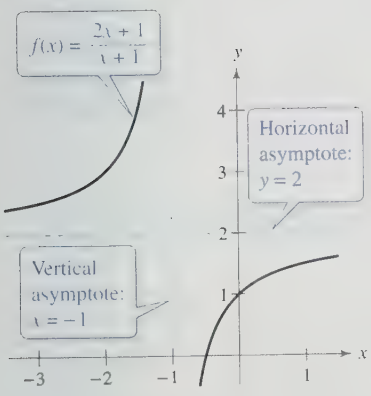
   as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f if

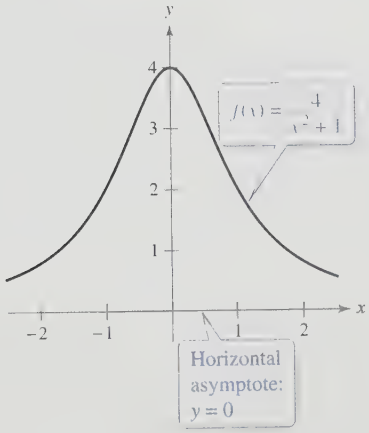
   $$f(x) \to b$$

   as $x \to \infty$ or $x \to -\infty$.

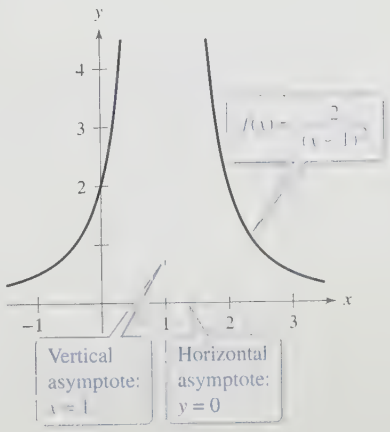
The graph of a rational function can never intersect its vertical asymptote. It may or may not intersect its horizontal asymptote. In either case, the distance between the horizontal asymptote and the points on the graph must approach zero (as $x \to \infty$ or $x \to -\infty$). Figure 3.45 shows the horizontal and vertical asymptotes of the graphs of three rational functions.



(a)



(b)



(c)

FIGURE 3.45

The graphs of $f(x) = 1/x$ in Figure 3.44 and

$$f(x) = \frac{2x + 1}{x + 1}$$

in Figure 3.45(a) are **hyperbolas.** You can learn more about hyperbolas in Appendix B.1.

Asymptotes of a Rational Function

Let f be the rational function given by

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \quad a_n \neq 0, b_m \neq 0.$$

Assume that $p(x)$ and $q(x)$ are polynomials with no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $q(x)$.
2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $p(x)$ and $q(x)$.
   a. When $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   b. When $n = m$, the graph of f has the line $y = a_n/b_m$ (ratio of the leading coefficients) as a horizontal asymptote.
   c. When $n > m$, the graph of f has no horizontal asymptote.

Example 2 **Finding Horizontal and Vertical Asymptotes**

Find all horizontal and vertical asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ b. $f(x) = \dfrac{2x^2}{x^2 - 1}$

SOLUTION

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x. Because the equation

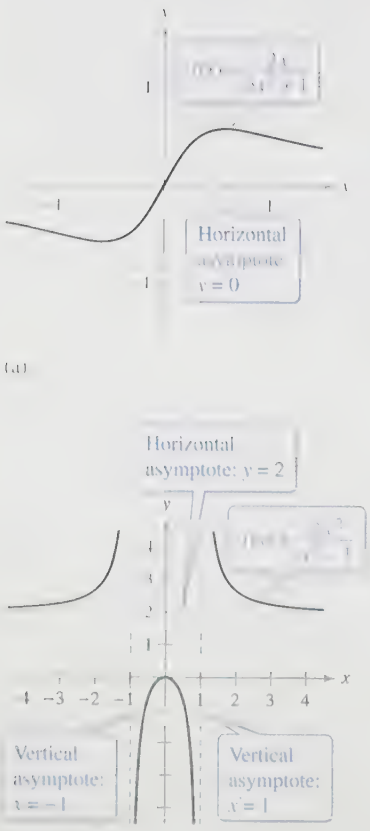
$$3x^2 + 1 = 0$$

has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 3.46(a).

b. For this rational function, the degree of the numerator is *equal* to the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$x^2 - 1 = 0 \qquad \text{Set denominator equal to zero.}$$
$$(x + 1)(x - 1) = 0 \qquad \text{Factor.}$$
$$x + 1 = 0 \Rightarrow x = -1 \qquad \text{Set 1st factor equal to 0.}$$
$$x - 1 = 0 \Rightarrow x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes. The graph of the function is shown in Figure 3.46(b).



(a)

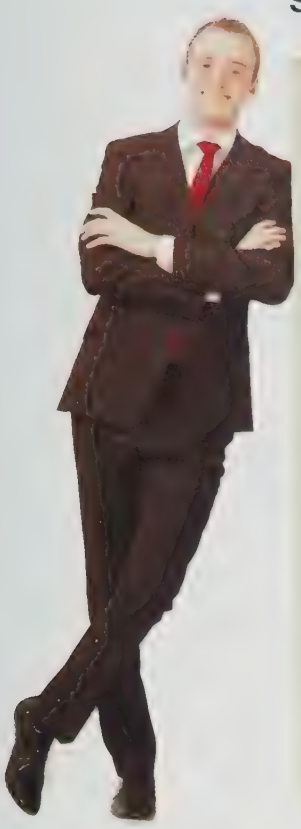
(b)

FIGURE 3.46

Checkpoint 2

Find all horizontal and vertical asymptotes of the graph of $f(x) = \dfrac{x^2}{x^2 + 4}$.

Sketching the Graph of a Rational Function



Guidelines for Graphing Rational Functions

Let

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials with no common factors.

1. Find and plot the y-intercept (if any) by evaluating $f(0)$.
2. Find the zeros of the numerator (if any) by solving the equation $p(x) = 0$. Then plot the corresponding x-intercepts.
3. Find the zeros of the denominator (if any) by solving the equation $q(x) = 0$. Then sketch the corresponding vertical asymptotes.
4. Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function.
5. Test for symmetry.
6. Plot at least one point both *between and beyond* each x-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.

Testing for symmetry can be useful, especially for simple rational functions. For example, the graph of $f(x) = 1/x$ is symmetric with respect to the origin, and the graph of $g(x) = 1/x^2$ is symmetric with respect to the y-axis.

STUDY TIP

Note that in the examples in this section, the vertical asymptotes are included in the table of additional points. This is done to emphasize numerically the behavior of the graph of the function.

Example 3 Sketching the Graph of a Rational Function

Sketch the graph of $g(x) = \dfrac{3}{x - 2}$.

SOLUTION Begin by noting that the numerator and denominator have no common factors.

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$.

x-intercept: None, numerator has no zeros.

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$, degree of $p(x)$ < degree of $q(x)$

Additional points:

x	-4	1	2	3	5
$g(x)$	-0.5	-3	Undefined	3	1

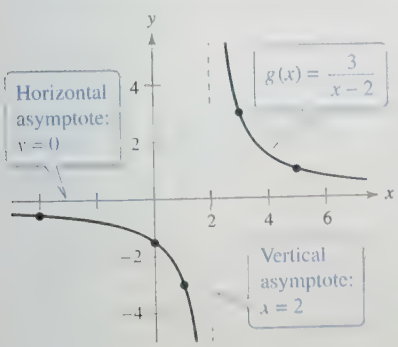


FIGURE 3.47

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 3.47. In the figure, note that the graph of g is a vertical stretch and a right shift of the graph of $y = 1/x$.

✓Checkpoint 3

Sketch the graph of $f(x) = \dfrac{4}{x + 3}$.

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Example 4 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x}{x^2 - x - 2}$.

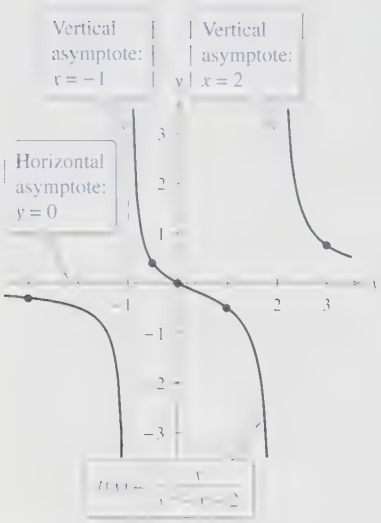


FIGURE 3.48

SOLUTION Factor the denominator to determine more easily the zeros of the denominator.

$$f(x) = \frac{x}{x^2 - x - 2}$$

$$= \frac{x}{(x + 1)(x - 2)}$$

y-intercept: $(0, 0)$, because $f(0) = 0$.

x-intercept: $(0, 0)$

Vertical asymptotes: $x = -1, x = 2$, zeros of denominator

Horizontal asymptote: $y = 0$, degree of $p(x) <$ degree of $q(x)$

Additional points:

x	-3	-1	-0.5	1	2	3
$f(x)$	-0.3	Undefined	0.4	-0.5	Undefined	0.75

The graph is shown in Figure 3.48. Confirm the graph with your graphing utility.

Checkpoint 4

Sketch the graph of $f(x) = \dfrac{3x}{x^2 + x - 6}$.

Example 5 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$.

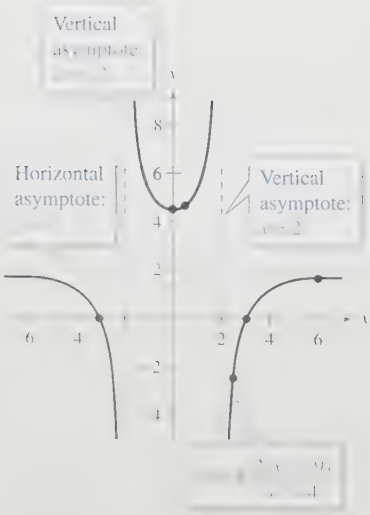


FIGURE 3.49

SOLUTION By factoring the numerator and denominator, you have

$$f(x) = \frac{2(x^2 - 9)}{x^2 - 4}$$

$$= \frac{2(x - 3)(x + 3)}{(x - 2)(x + 2)}.$$

y-intercept: $\left(0, \frac{9}{2}\right)$, because $f(0) = \frac{9}{2}$.

x-intercepts: $(-3, 0)$ and $(3, 0)$

Vertical asymptotes: $x = -2, x = 2$, zeros of denominator

Horizontal asymptote: $y = 2$, degree of $p(x) =$ degree of $q(x)$

Symmetry: With respect to y-axis, because $f(-x) = f(x)$.

Additional points:

x	-2	0.5	2	2.5	6
$f(x)$	Undefined	4.67	Undefined	-2.44	1.6875

The graph is shown in Figure 3.49.

Checkpoint 5

Sketch the graph of $f(x) = \dfrac{5(x^2 - 1)}{x^2 - 9}$.

Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 3.50. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you have

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\text{Slant asymptote } (y = x - 2)} + \frac{2}{x + 1}.$$

As x increases or decreases without bound, the remainder term

$$\frac{2}{x + 1}$$

approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 3.50.

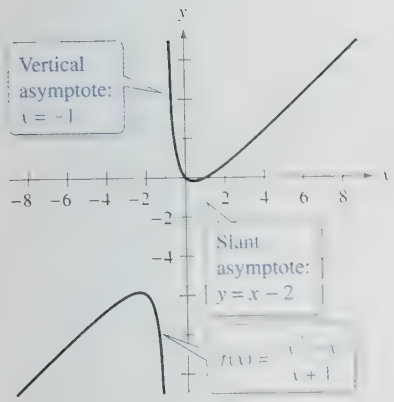


FIGURE 3.50

Example 6 A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

SOLUTION First write f in two different ways. Factoring the numerator enables you to recognize the x-intercepts.

$$f(x) = \frac{x^2 - x - 2}{x - 1}$$

$$= \frac{(x - 2)(x + 1)}{x - 1}$$

Then long division enables you to recognize that the line $y = x$ is a slant asymptote of the graph.

$$f(x) = \frac{x^2 - x - 2}{x - 1}$$

$$= x - \frac{2}{x - 1}$$

y-intercept: $(0, 2)$, because $f(0) = 2$.

x-intercepts: $(-1, 0)$ and $(2, 0)$

Vertical asymptote: $x = 1$, zero of denominator

Horizontal asymptote: None, degree of $p(x)$ > degree of $q(x)$

Slant asymptote: $y = x$

Additional points:

x	−2	0.5	1	1.5	3
$f(x)$	$-1.\overline{3}$	4.5	Undefined	2.5	2

The graph is shown in Figure 3.51.

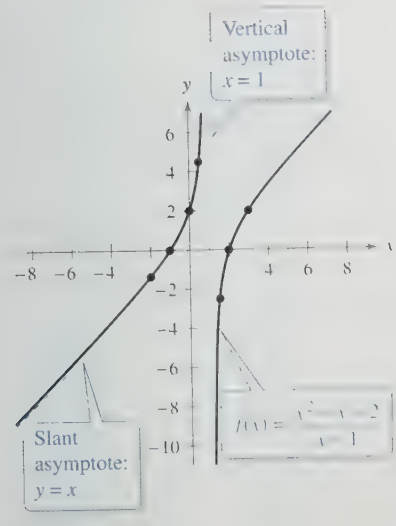


FIGURE 3.51

✓ Checkpoint 6

Sketch the graph of $f(x) = \dfrac{x^2 + 3x + 2}{x - 1}$.

Applications

There are many examples of asymptotic behavior in business and biology. For instance, the following example describes the asymptotic behavior related to the cost of removing smokestack emissions.

Example 7 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost of removing a certain *percent* of the pollutants from the stack emissions is typically not a linear function. That is, if it costs C dollars to remove 25% of the pollutants, then it would cost more than $2C$ dollars to remove 50% of the pollutants. As the percent of pollutants removed approaches 100%, the cost tends to become prohibitive. The cost C (in dollars) of removing p percent of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}.$$

You are a member of a state legislature that is considering a new law that will require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal.

a. How much additional expense is the new law asking the utility company to incur?

b. According to the model, would it be possible to remove 100% of the smokestack pollutants?

SOLUTION

a. The graph of this function is shown in Figure 3.52. Note that the graph has a vertical asymptote at

$$p = 100. \qquad \text{Vertical asymptote}$$

Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \qquad \text{Substitute 85 for } p.$$

$$\approx \$453{,}333. \qquad \text{Use a calculator.}$$

If the new law increases the percent removal to 90%, then the cost to the utility company will be

$$C = \frac{80{,}000(90)}{100 - 90} \qquad \text{Substitute 90 for } p.$$

$$= \$720{,}000. \qquad \text{Use a calculator.}$$

The new law would require the utility company to spend an additional

$$\$720{,}000 - \$453{,}333 = \$266{,}667.$$

b. From Figure 3.52, you can see that the graph has a vertical asymptote at $p = 100$. Because the graph of a rational function can never intersect its vertical asymptote, you can conclude that it is not possible for the company to remove 100% of the pollutants from the stack emissions.

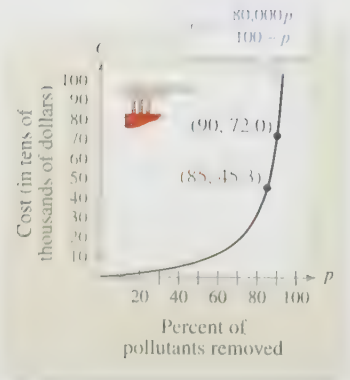


FIGURE 3.52

Checkpoint 7

In Example 7, a new law will require utility companies to remove 95% of the pollutants. Find the additional cost to the utility company. ■

Example 8 Per Capita Land Area

A model for the population P (in millions) of the United States from 1960 to 2009 is $P = 2.6174t + 176.112$, where t represents the year, with $t = 0$ corresponding to 1960. A model for the land area A (in millions of acres) of the United States from 1960 to 2009 is $A = 2263.960$. Construct a rational function for per capita land area L (in acres per person). Sketch a graph of the rational function. Use the model to predict the per capita land area in 2015. *(Source: U.S. Census Bureau)*

SOLUTION The rational function for the per capita land area L is

$$L = \frac{A}{P} = \frac{2263.960}{2.6174t + 176.112}.$$

The graph of the function is shown in Figure 3.53. To find the per capita land area in 2015, substitute $t = 55$ into L.

$$L = \frac{2263.960}{2.6174t + 176.112} = \frac{2263.960}{2.6174(55) + 176.112}$$

$$= \frac{2263.960}{320.069}$$

$$\approx 7.07$$

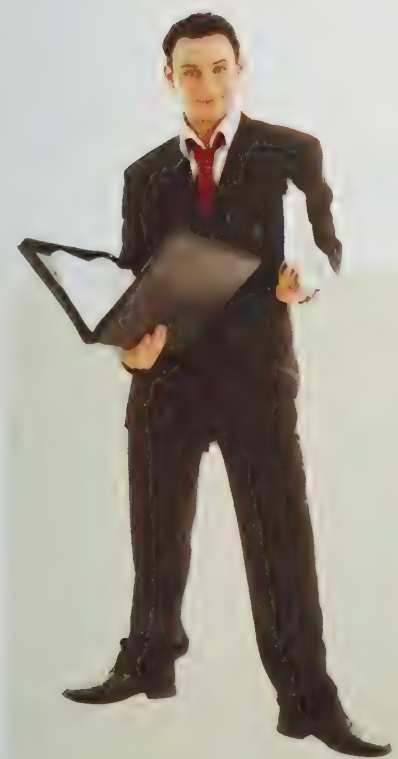
The per capita land area will be approximately 7.1 acres per person in 2015.



FIGURE 3.53

Checkpoint 8

In Example 8, use the model to predict the per capita land area in 2020.



SUMMARIZE (Section 3.7)

1. State the definition of a rational function and describe the domain of a rational function *(page 314)*. For an example of finding the domain of a rational function, see Example 1.
2. State the definitions of vertical and horizontal asymptotes *(page 315)*. For an example of finding vertical and horizontal asymptotes of rational functions, see Example 2.
3. State the guidelines for graphing a rational function *(page 317)*. For examples of sketching graphs of rational functions, see Examples 3, 4, and 5.
4. Describe when a rational function has a slant asymptote *(page 319)*. For an example of sketching the graph of a rational function with a slant asymptote, see Example 6.

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SKILLS WARM UP 3.7 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.6 and 2.2.

In Exercises 1–6, factor the polynomial.

1. $x^2 - 4x$
2. $2x^3 - 6x$
3. $x^2 - 3x - 10$
4. $x^2 - 7x + 10$
5. $x^3 + 4x^2 + 3x$
6. $x^3 - 4x^2 - 2x + 8$

In Exercises 7–10, sketch the graph of the equation.

7. $y = 2$
8. $x = -1$
9. $y = x - 2$
10. $y = -x + 1$

Exercises 3.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding the Domain of a Rational Function In Exercises 1–6, find the domain of the function f and discuss the behavior of f near any excluded x-values. *See Example 1.*

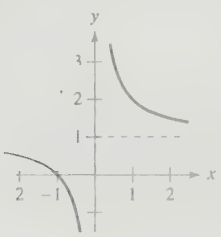
1. $f(x) = \dfrac{1}{x - 2}$
2. $f(x) = \dfrac{5}{x + 4}$
3. $f(x) = \dfrac{4x}{x + 1}$
4. $f(x) = \dfrac{5x}{x - 1}$
5. $f(x) = \dfrac{3x^2}{x^2 - 1}$
6. $f(x) = \dfrac{2x}{x^2 - 4}$

Finding Horizontal and Vertical Asymptotes In Exercises 7–14, find all horizontal and vertical asymptotes of the graph of the function. *See Example 2.*

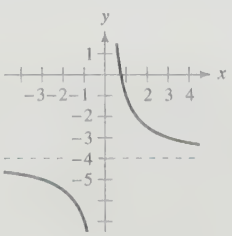
7. $f(x) = \dfrac{3x}{x + 1}$
8. $f(x) = \dfrac{x}{x - 2}$
9. $f(x) = \dfrac{x - 7}{5 - x}$
10. $f(x) = \dfrac{1 - 5x}{1 + 2x}$
11. $f(x) = \dfrac{3x^2 + 1}{x^2 + 9}$
12. $f(x) = \dfrac{3x^2 + x - 5}{x^2 + 1}$
13. $f(x) = \dfrac{5}{(x + 4)^2}$
14. $f(x) = \dfrac{1}{(x - 1)^2}$

Matching In Exercises 15–20, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

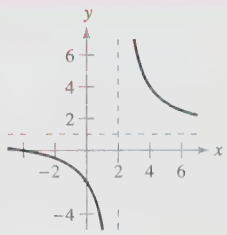
(a)



(b)



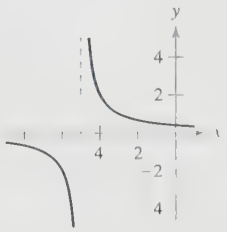
(c)



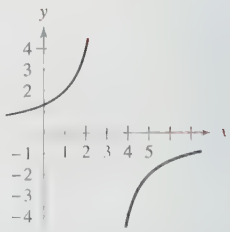
(d)



(e)



(f)



15. $f(x) = -\dfrac{4}{x - 3}$
16. $f(x) = \dfrac{2}{x + 5}$
17. $f(x) = \dfrac{x + 1}{x}$
18. $f(x) = \dfrac{3 - 4x}{x}$
19. $f(x) = \dfrac{x + 4}{x - 2}$
20. $f(x) = -\dfrac{x + 2}{x + 1}$

Describing a Transformation In Exercises 21–24, compare the graph of $f(x) = 1/x$ with the graph of g.

21. $g(x) = f(x) - 2 = \dfrac{1}{x} - 2$
22. $g(x) = f(x + 1) = \dfrac{1}{x + 1}$
23. $g(x) = -f(x) = -\dfrac{1}{x}$
24. $g(x) = -f(x + 1) = -\dfrac{1}{x + 1}$

Describing a Transformation In Exercises 25–28, compare the graph of $f(x) = 4/x^2$ with the graph of g.

25. $g(x) = f(x) + 3 = \frac{4}{x^2} + 3$

26. $g(x) = f(x - 1) = \frac{4}{(x - 1)^2}$

27. $g(x) = -f(x) = -\frac{4}{x^2}$

28. $g(x) = \frac{1}{8}f(x) = \frac{1}{2x^2}$

Describing a Transformation In Exercises 29–32, compare the graph of $f(x) = 8/x^3$ with the graph of g.

29. $g(x) = f(x) + 5 = \frac{8}{x^3} + 5$

30. $g(x) = f(x - 3) = \frac{8}{(x - 3)^3}$

31. $g(x) = -f(x) = -\frac{8}{x^3}$

32. $g(x) = \frac{1}{4}f(x) = \frac{2}{x^3}$

Sketching the Graph of a Rational Function In Exercises 33–58, sketch the graph of the rational function. To aid in sketching the graphs, check for intercepts, symmetry, vertical asymptotes, and horizontal asymptotes. *See Examples 3, 4, and 5.*

33. $f(x) = \frac{1}{x + 3}$

34. $f(x) = \frac{1}{x - 3}$

35. $f(x) = \frac{-1}{x + 1}$

36. $f(x) = \frac{-2}{x - 3}$

37. $f(x) = \frac{x + 4}{x - 5}$

38. $f(x) = \frac{x - 2}{x - 3}$

39. $f(x) = \frac{2 + x}{1 - x}$

40. $f(x) = \frac{3 - x}{2 - x}$

41. $f(t) = \frac{3t + 1}{t}$

42. $f(t) = \frac{1 - 2t}{t}$

43. $C(x) = \frac{5 + 2x}{1 + x}$

44. $P(x) = \frac{1 - 3x}{1 - x}$

45. $g(x) = \frac{1}{x + 2} + 2$

46. $h(x) = \frac{1}{x - 3} + 1$

47. $f(x) = \frac{1}{x^2} + 2$

48. $f(x) = 2 - \frac{3}{x^2}$

49. $h(x) = \frac{x^2}{x^2 - 9}$

50. $h(t) = \frac{3t^2}{t^2 - 4}$

51. $g(s) = \frac{s}{s^2 + 1}$

52. $g(x) = \frac{x}{x^2 + 3}$

53. $f(x) = \frac{x}{x^2 - 3x - 4}$

54. $f(x) = \frac{-x}{x^2 + x - 6}$

55. $f(x) = \frac{3x}{x^2 - x - 2}$

56. $f(x) = \frac{2x}{x^2 + x - 2}$

57. $g(x) = \frac{4(x^2 - 4)}{x^2 - 16}$

58. $g(x) = \frac{3(x^2 - 16)}{x^2 - 9}$

A Rational Function with a Slant Asymptote In Exercises 59–64, sketch the graph of the rational function. To aid in sketching the graphs, check for intercepts, vertical asymptotes, and slant asymptotes. *See Example 6.*

59. $f(x) = \frac{2x^2 + 1}{x}$

60. $g(x) = \frac{1 - x^2}{x}$

61. $h(x) = \frac{x^2}{x - 1}$

62. $f(x) = \frac{x^3}{x^2 - 1}$

63. $g(x) = \frac{x^3}{2x^2 - 8}$

64. $f(x) = \frac{x^3}{x^2 + 4}$

Writing a Rational Function In Exercises 65–68, write a rational function f that has the specified characteristics. (There are many correct answers.)

65. Vertical asymptote: None
Horizontal asymptote: $y = 2$

66. Vertical asymptotes: $x = 0, x = \frac{5}{2}$
Horizontal asymptote: $y = -3$

67. Vertical asymptotes: $x = -2, x = 1$
Horizontal asymptote: None

68. Vertical asymptote: $x = 3$
Horizontal asymptote: x-axis



Graphical Analysis In Exercises 69–72, use a graphing utility to graph f and g in the same viewing window. (Notice that f has a common factor in the numerator and denominator.) Use the *trace* feature of the graphing utility to check the value of each function near any x-values excluded from its domain. Then, describe how the graphs of f and g are different.

69. $f(x) = \frac{(x - 3)(x + 3)}{x - 3}$, $g(x) = x + 3$

70. $f(x) = \frac{(x - 4)(x + 1)}{x - 4}$, $g(x) = x + 1$

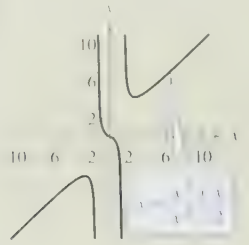
71. $f(x) = \frac{x - 2}{x^2 - 4}$, $g(x) = \frac{1}{x + 2}$

72. $f(x) = \frac{x - 5}{x^2 - 4x - 5}$, $g(x) = \frac{1}{x + 1}$

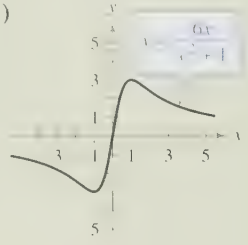
73. Think About It You divide out the common factor $(x - 7)$ from the numerator and denominator of a rational function. How do you specify the domain of the new function so that both functions are the same?

74. HOW DO YOU SEE IT? For each statement below, determine which graph(s) show that the statement is incorrect.

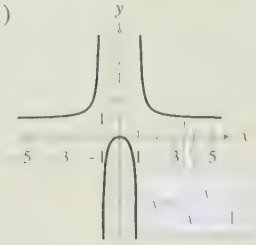
(i)



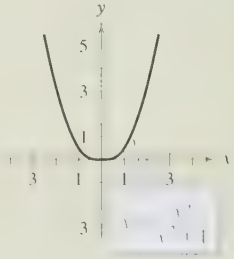
(ii)



(iii)



(iv)



(a) Every rational function has a vertical asymptote.

(b) Every rational function has at least one asymptote.

(c) A rational function can have at most one vertical asymptote.

(d) The graph of a rational function with a slant asymptote cannot cross the slant asymptote.

75. Think About It Is it possible for a rational function to have all three types of asymptotes (vertical, horizontal, and slant)? Why or why not?

76. Think About It Is it possible for a rational function to have more than one horizontal asymptote? Why or why not?

77. Federal Drug Enforcement The cost C (in millions of dollars) for the federal government to seize p percent of an illegal drug as it enters the country is

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the costs of seizing 25% of the drug, 50% of the drug, and 75% of the drug.

(b) According to this model, is it possible to seize 100% of the drug? Explain.

78. Population of Deer The Game Commission introduces 100 deer into newly acquired state game lands. The population N of the herd is given by $N = \frac{25(4 + 2t)}{1 + 0.02t}$, $t \ge 0$, where t is time (in years).

(a) Find the populations when t is 5, 10, and 25.

(b) What is the limiting size of the herd as time progresses?

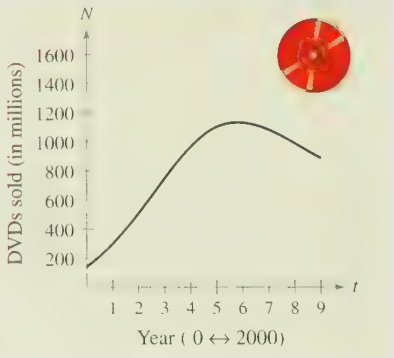
79. Retail Sales The table shows the numbers N (in millions) of DVDs sold in the years 2000 through 2009. The data can be modeled by

$$N = \frac{104.05t + 140.3}{0.026t^2 - 0.21t + 1}, \quad 0 \le t \le 9$$

where t is the year, with $t = 0$ corresponding to 2000 (see figure). *(Source: SNL Kagan)*

Year	DVDs, N
2000	146.6
2001	291.5
2002	507.5
2003	713.0
2004	976.6

Year	DVDs, N
2005	1072.4
2006	1069.3
2007	1063.4
2008	1007.5
2009	851.1



(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model represent the data?

(b) Use the model to predict the numbers of DVDs sold in the years 2012, 2015, and 2020. Are the predictions reasonable?

(c) Determine the horizontal asymptote and explain its meaning in the context of the problem.

80. Average Cost The cost C (in dollars) of producing x basketballs is

$$C = 375{,}000 + 4x.$$

The average cost $\overline{C}$ per basketball is

$$\overline{C} = \frac{C}{x} = \frac{375{,}000 + 4x}{x}, \quad x > 0.$$

(a) Sketch the graph of the average cost function.

(b) Find the average costs of producing 1000, 10,000, and 100,000 basketballs.

(c) Find the horizontal asymptote and explain its meaning in the context of the problem.

81. Human Memory Model Psychologists have developed mathematical models to predict memory performance as a function of the number of trials n of a certain task. Consider the learning curve modeled by

$$P = \frac{0.6 + 0.95(n - 1)}{1 + 0.95(n - 1)}, \quad n > 0$$

where P is the percent of correct responses (in decimal form) after n trials.

(a) Complete the table.

n	1	2	3	4	5	6	7	8	9	10
P										

(b) According to this model, what is the limiting percent of correct responses as n increases?

82. Human Memory Model Consider the learning curve modeled by

$$P = \frac{0.55 + 0.87(n - 1)}{1 + 0.87(n - 1)}, \quad n > 0$$

where P is the percent of correct responses (in decimal form) after n trials.

(a) Complete the table.

n	1	2	3	4	5	6	7	8	9	10
P										

(b) According to this model, what is the limiting percent of correct responses as n increases?

83. Average Recycling Cost The cost C (in dollars) of recycling a waste product is

$$C = 450{,}000 + 6x, \quad x > 0$$

where x is the number of pounds of waste. The average recycling cost $\overline{C}$ per pound is

$$\overline{C} = \frac{C}{x} = \frac{450{,}000 + 6x}{x}, \quad x > 0.$$

(a) Use a graphing utility to graph $\overline{C}$.

(b) Find the average costs of recycling 10,000, 100,000, 1,000,000, and 10,000,000 pounds of waste. What can you conclude?

84. Drug Concentration The concentration C of a medication in the bloodstream t minutes after sublingual (under the tongue) application is given by

$$C(t) = \frac{3t - 1}{2t^2 + 5}, \quad t > 0.$$

(a) Use a graphing utility to graph the function. Estimate when the concentration is greatest.

(b) Does this function have a horizontal asymptote? If so, discuss the meaning of the asymptote in terms of the concentration of the medication.

85. Domestic Sales The U.S. domestic sales S of electricity (in millions of kilowatt-hours) from 1990 through 2008 can be modeled by

$$S = 27{,}904t + 906{,}950, \quad 0 \le t \le 18$$

where t represents the year, with $t = 0$ corresponding to 1990. The population P (in millions) of the United States from 1990 through 2008 can be modeled by

$$P = 3.023t + 250.90, \quad 0 \le t \le 18$$

where t represents the year, with $t = 0$ corresponding to 1990. *(Source: Edison Electric Institute and the U.S. Census Bureau)*

(a) Construct a rational function E to represent the per capita domestic sales of electricity.

(b) Use a graphing utility to graph the rational function E.

(c) Use the model to estimate the per capita domestic sales (in kilowatt-hours) of electricity in 2008.

86. Health Care Spending The total health care spending H (in millions of dollars) in the United States from 1990 through 2008 can be modeled by

$$H = 3258.97t^2 + 31{,}416.1t + 754{,}111, \quad 0 \le t \le 18$$

where t represents the year, with t 0 corresponding to 1990. The population P (in millions) of the United States from 1990 through 2008 can be modeled by

$$P = 3.023t + 250.90, \quad 0 \le t \le 18$$

where t represents the year, with $t = 0$ corresponding to 1990. *(Sources: U.S. Centers for Medicare and Medicaid Services and the U.S. Census Bureau)*

(a) Construct a rational function S to represent the per capita health care spending.

(b) Use a graphing utility to graph the rational function S.

(c) Use the model to predict the per capita health care spending in 2012.

87. 3000-Meter Speed Skating The winning times for the women's 3000-meter speed skating race at the Olympics from 1960 through 2010 can be approximated by the quadratic model

$$y = 0.0237t^2 - 5.624t + 571.67, \quad 60 \le t \le 110$$

where y is the winning time (in seconds) and t represents the year, with $t = 60$ corresponding to 1960. *(Sources: DatabaseOlympics.com and Sports Reference LLC)*

(a) Use a graphing utility to graph the model.

(b) Use the model to predict the winning times in 2014 and 2018.

(c) Does this model have a horizontal asymptote? Do you think that a model for this type of data should have a horizontal asymptote?

ALGEBRA TUTOR

Factoring Expressions

Much of the algebra in Chapter 3 involves factoring. On this page, you can review some of the basic techniques for factoring polynomial expressions. The simplest type of factoring involves recognizing common factors and factoring them out of the expression. Other types of factoring involve special polynomial forms. Several of these are listed below.

Difference of Two Squares:	$u^2 - v^2 = (u + v)(u - v)$
Perfect Square Trinomial:	$u^2 + 2uv + v^2 = (u + v)^2$ $u^2 - 2uv + v^2 = (u - v)^2$
Sum or Difference of Two Cubes:	$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$ $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$

For trinomials of the form $ax^2 + bx + c$ that have binomial factors, you need to find a combination of factors of a and c such that the outer and inner products of the binomials add up to the middle term bx.

Example 1 Factoring Expressions

Completely factor the expression.

a. $6x^2 + 21x$ **b.** $36x^3 - 4x$

c. $4x^2 + 20x + 25$ **d.** $y^2 - 22y + 121$

e. $5z^3 + 40$ **f.** $x^2 - 12x + 27$

SOLUTION

a. $6x^2 + 21x = 3x(2x) + 3x(7)$ — $3x$ is a common factor.

$= 3x(2x + 7)$ — Factor completely.

b. $36x^3 - 4x = 4x(9x^2 - 1)$ — $4x$ is a common factor.

$= 4x[(3x)^2 - 1^2]$ — $9x^2 - 1$ is a difference of two squares.

$= 4x(3x + 1)(3x - 1)$ — Factor completely.

c. $4x^2 + 20x + 25 = (2x)^2 + 2(2x)(5) + 5^2$ — Perfect square trinomial

$= (2x + 5)^2$ — Factor completely.

d. $y^2 - 22y + 121 = y^2 - 2(y)(11) + 11^2$ — Perfect square trinomial

$= (y - 11)^2$ — Factor completely.

e. $5z^3 + 40 = 5(z^3 + 8)$ — 5 is a common factor.

$= 5(z^3 + 2^3)$ — $z^3 + 8$ is a sum of two cubes. Rewrite 8 as 2^3.

$= 5(z + 2)(z^2 - 2z + 4)$ — Factor completely.

f. For $x^2 - 12x + 27$, you have $a = 1$, $b = -12$, and $c = 27$. Because b is negative and c is positive, both factors of 27 must be negative. That is, $27 = (-1)(-27)$, or $27 = (-3)(-9)$. So, the possible factorizations of $x^2 - 12x + 27$ are $(x - 1)(x - 27)$ and $(x - 3)(x - 9)$. Testing the middle term, you find that

$$x^2 - 12x + 27 = (x - 3)(x - 9).$$

STUDY TIP

Be sure to remember that a polynomial is completely factored when each of its factors is prime.

Simplifying Radical Expressions

Performing operations with complex numbers involves several different algebraic skills. One such skill is simplifying expressions involving radicals. Here are some helpful simplification techniques.

1. To add or subtract radical expressions, the radicals must be like radicals (having the same index and the same radicand). The Distributive Property can then be used to combine like radicals.
2. To multiply or divide radical expressions, use the following properties.

   Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let n be a positive integer.

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \quad b \neq 0$$

Example 2 Simplifying Expressions Involving Radicals

Simplify the expression.

a. $\sqrt{48}$ **b.** $\sqrt{7} + \sqrt{28}$ **c.** $\sqrt{125} - \sqrt{45}$

d. $\sqrt{5} \cdot \sqrt{20}$ **e.** $\sqrt{2}(8 - \sqrt{3})$ **f.** $(3 + \sqrt{5})^2$

SOLUTION

a. $\sqrt{48} = \sqrt{16 \cdot 3}$ Find largest square factor.

$= \sqrt{4^2 \cdot 3}$ Rewrite.

$= 4\sqrt{3}$ Find root of perfect square.

b. $\sqrt{7} + \sqrt{28} = \sqrt{7} + \sqrt{4 \cdot 7}$ Find largest square factor.

$= \sqrt{7} + \sqrt{2^2 \cdot 7}$ Rewrite.

$= \sqrt{7} + 2\sqrt{7}$ Find root of perfect square.

$= (1 + 2)\sqrt{7}$ Distributive Property

$= 3\sqrt{7}$ Add.

c. $\sqrt{125} - \sqrt{45} = \sqrt{25 \cdot 5} - \sqrt{9 \cdot 5}$ Find largest square factors.

$= \sqrt{5^2 \cdot 5} - \sqrt{3^2 \cdot 5}$ Rewrite.

$= 5\sqrt{5} - 3\sqrt{5}$ Find roots of perfect squares.

$= (5 - 3)\sqrt{5}$ Distributive Property

$= 2\sqrt{5}$ Subtract.

d. $\sqrt{5} \cdot \sqrt{20} = \sqrt{5 \cdot 20}$ Rewrite.

$= \sqrt{100}$ Multiply.

$= 10$ Simplify.

e. $\sqrt{2}(8 - \sqrt{3}) = \sqrt{2} \cdot 8 - \sqrt{2} \cdot \sqrt{3}$ Distributive Property

$= 8\sqrt{2} - \sqrt{2 \cdot 3}$ Rewrite.

$= 8\sqrt{2} - \sqrt{6}$ Multiply.

f. $(3 + \sqrt{5})^2 = 9 + 3\sqrt{5} + 3\sqrt{5} + 5$ Distributive Property

$= 14 + 6\sqrt{5}$ Combine like terms.

STUDY TIP

Be sure to remember that one of the conditions for a radical expression to be in simplest form is that all possible factors have been removed from the radical.

SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 330. Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 3.1

	Review Exercises
■ Sketch the graph of a quadratic function.	*1–4*
■ Find the equation of a parabola given its vertex and a point on its graph.	*5, 6*
■ Construct and use a quadratic model to solve an application problem.	*7–12*

Section 3.2

■ Sketch a transformation of a monomial function.	*13–20*
■ Apply the Leading Coefficient Test to determine right-hand and left-hand behavior of graphs of polynomial functions.	*21–24*

When n is odd and the leading coefficient is positive,

$f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

When n is odd and the leading coefficient is negative,

$f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.

When n is even and the leading coefficient is positive,

$f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

When n is even and the leading coefficient is negative,

$f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.

■ Find the real zeros and sketch the graph of a polynomial function.	*25–28*

Section 3.3

■ Divide one polynomial by a second polynomial using long division.	*29, 30*
■ Use synthetic division to divide two polynomials.	*31, 32, 39, 40*
■ Use the Remainder Theorem and synthetic division to evaluate a polynomial.	*33, 34*
■ Use the Factor Theorem to factor a polynomial.	*35, 36*
■ Simplify a rational expression using long division.	*37, 38*

Section 3.4

■ Find all possible rational zeros of a function using the Rational Zero Test.	*41, 42*
■ Find all the real zeros of a function.	*43–50*
■ Approximate the real zeros of a polynomial function using the Intermediate Value Theorem.	*51, 52*
■ Approximate the real zeros of a polynomial function using a graphing utility.	*53, 54*
■ Apply techniques for approximating real zeros to solve an application problem.	*55, 56*

* A wide range of valuable study aids are available to help you master the material in this chapter. The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *www.cengagebrain.com* offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

Section 3.5

	Review Exercises
■ Find the complex conjugate of a complex number.	*57–60*
■ Perform operations with complex numbers and write the results in standard form.	*61–74*

$(a + bi) + (c + di) = (a + c) + (b + d)i$

$(a + bi) - (c + di) = (a - c) + (b - d)i$

$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

■ Solve a polynomial equation that has complex solutions.	*75–78*
■ Plot a complex number in the complex plane.	*79, 80*

Section 3.6

■ Use the Fundamental Theorem of Algebra and the Linear Factorization Theorem to write a polynomial as the product of linear factors.	*81–86*
■ Find a polynomial with real coefficients whose zeros are given.	*87, 88*
■ Factor a polynomial over the rational, real, and complex numbers, and find all real and complex zeros of a polynomial function.	*89–94*

Section 3.7

■ Find the domain of a rational function.	*95–98*
■ Find the vertical and horizontal asymptotes of the graph of a rational function.	*95–98*

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \quad a_n \neq 0, b_m \neq 0.$

Assume that $p(x)$ and $q(x)$ are polynomials with no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $q(x)$.
2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $p(x)$ and $q(x)$.
   - **a.** When $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   - **b.** When $n = m$, the graph of f has the line $y = a_n/b_m$ (ratio of the leading coefficients) as a horizontal asymptote.
   - **c.** When $n > m$, the graph of f has no horizontal asymptote.

■ Sketch the graph of a rational function, including graphs with slant asymptotes.	*99–104*
■ Use a rational function model to solve an application problem.	*105–108*

Study Strategies

■ **Use a Graphing Utility** A graphing calculator or graphing software for a computer can help you in this course in two important ways. As an *exploratory device*, a graphing utility helps you to learn concepts by comparing graphs of functions. For instance, sketching the graphs of $f(x) = x^3$ and $f(x) = -x^3$ shows that the negative coefficient has the effect of reflecting the graph in the x-axis. As a *problem-solving tool*, a graphing utility frees you from sketching complicated graphs by hand. The time you save can be spent using mathematics to solve real-life problems.

■ **Problem-Solving Strategies** When you get stuck trying to solve a real-life problem, consider the strategies below.

1. *Draw a Diagram.* Draw a diagram representing the problem. Label all known values and unknown values on the diagram.
2. *Solve a Simpler Problem.* Simplify the problem, or write several simple examples of the problem. For instance, if you are asked to find the dimensions that will produce a maximum area, try calculating the areas of several examples.
3. *Rewrite the Problem in Your Own Words.* Rewriting a problem can help you understand it better.
4. *Guess and Check.* Try guessing the answer, then check your guess in the statement of the original problem. By refining your guesses, you may be able to think of a general strategy for solving the problem.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Using Standard Form to Graph a Parabola In Exercises 1–4, write the quadratic function in standard form (if necessary) and sketch its graph. Identify the vertex and intercepts.

1. $f(x) = (x + 3)^2 - 5$
2. $g(x) = -(x - 1)^2 + 3$
3. $h(x) = 3x^2 - 12x + 11$
4. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$

Finding an Equation of a Parabola In Exercises 5 and 6, find an equation of the parabola that has the indicated vertex and whose graph passes through the given point.

5. Vertex: $(-5, -1)$; point: $(-2, 6)$
6. Vertex: $(2, 5)$; point: $(4, 7)$

7. **Optimal Area** The perimeter of a rectangular archaeological dig site is 500 feet. Let x represent the width of the dig site. Write a quadratic function for the area of the rectangle in terms of its width. Of all possible dig sites with perimeters of 500 feet, what are the measurements of the one with the greatest area?
8. **Optimal Revenue** Find the number of units that produces a maximum revenue R (in dollars) for

   $R = 900x - 0.015x^2$

   where x is the number of units produced.
9. **Optimal Cost** A manufacturer of retinal imaging systems has daily production costs C (in dollars per unit) of

   $C = 25{,}000 - 50x + 0.065x^2$

   where x is the number of units produced.
   (a) Use a graphing utility to graph the cost function.
   (b) Graphically estimate the number of units that should be produced to yield a minimum cost per unit.
   (c) Explain how to confirm the result of part (b) algebraically.
10. **Optimal Profit** The profit P (in dollars) for an electronics company is given by

    $P = -0.00015x^2 + 155x - 450{,}000$

    where x is the number of units produced.
    (a) Use a graphing utility to graph the profit function.
    (b) Graphically estimate the number of units that should be produced to yield a maximum profit.
    (c) Explain how to confirm the result of part (b) algebraically.
11. **Batting Distance** In a study, baseballs were batted at angles of x degrees above the horizontal, each with an initial speed of 40 meters per second and a backspin of 30 revolutions per second. The distances $d(x)$ (in meters) traveled by the balls are shown in the table. *(Source: The Physics of Sports)*

x	10	15	30	36	42	43
$d(x)$	61.2	83.0	130.4	139.4	143.2	143.3

x	44	45	48	54	60
$d(x)$	142.8	142.7	140.7	132.8	119.7

   (a) Use a graphing utility to create a scatter plot of the data.
   (b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.
   (c) Use the graphing utility to graph the model from part (b) in the same viewing window as the scatter plot of the data.
   (d) Find the vertex of the graph of the model from part (c). Interpret its meaning in the context of the problem.
12. **Dormitory Room Costs** The average costs C (in dollars) of a college dormitory room for the years 2003 through 2010 are shown in the table. *(Source: Digest of Education Statistics)*

Year	2003	2004	2005	2006
Cost, C	3179	3359	3569	3804

Year	2007	2008	2009	2010
Cost, C	4019	4214	4446	4657

   (a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 3$ corresponding to 2003.
   (b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.
   (c) Use the graphing utility to graph the model from part (b) in the same viewing window as the scatter plot of the data.
   (d) Use the graph of the model from part (c) to predict the average cost of a dormitory room in 2011.

Sketching a Transformation of a Monomial Function In Exercises 13–16, use the graph of $y = x^5$ to sketch the graph of the function.

13. $f(x) = (x + 4)^5$ **14.** $f(x) = x^5 + 1$

15. $f(x) = 3 - \frac{1}{2}x^5$ **16.** $f(x) = 2(x + 3)^5$

Sketching a Transformation of a Monomial Function In Exercises 17–20, use the graph of $y = x^6$ to sketch the graph of the function.

17. $f(x) = x^6 - 2$ **18.** $f(x) = -\frac{1}{4}x^6$

19. $f(x) = -\frac{1}{2}x^6 - 5$ **20.** $f(x) = -(x + 7)^6 + 2$

Applying the Leading Coefficient Test In Exercises 21–24, describe the right-hand and left-hand behavior of the graph of the polynomial function.

21. $f(x) = \frac{1}{2}x^3 + 2x$ **22.** $f(x) = 5 + 4x^3 - x^5$

23. $f(x) = -x^6 + 3x^4 - x^2 + 6$

24. $f(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

Finding Zeros of a Polynomial Function In Exercises 25–28, find all real zeros of the function algebraically and use the results to sketch the graph *by hand*. Then use a graphing utility to confirm your sketch.

25. $f(x) = 16 - x^2$ **26.** $f(x) = x^4 - 6x^2 + 8$

27. $f(x) = x^3 - 7x^2 + 10x$

28. $f(x) = x^3 - 6x^2 - 3x + 18$

Long Division of Polynomials In Exercises 29 and 30, use long division to divide.

29. $(2x^3 - 5x^2 - x) \div (2x + 1)$

30. $(x^4 - 5x^3 + 10x^2 - 12) \div (x^2 - 2x + 4)$

Using Synthetic Division In Exercises 31 and 32, use synthetic division to divide.

31. $(x^3 - 6x + 9) \div (x + 3)$

32. $(x^5 - x^4 + x^3 - 13x^2 + x + 6) \div (x - 2)$

Using Synthetic Division In Exercises 33 and 34, use synthetic division to find each function value.

33. $f(x) = 6 + 2x^2 - 3x^3$ (a) $f(2)$ (b) $f(-1)$

34. $f(x) = 2x^4 + 3x^3 + 6$ (a) $f\left(\frac{1}{2}\right)$ (b) $f(-1)$

Factoring a Polynomial In Exercises 35 and 36, (a) verify the given factors of $f(x)$, (b) find the remaining factors of $f(x)$, (c) use your results to write the complete factorization of $f(x)$, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.

Function	*Factors*
35. $f(x) = x^3 - 4x^2 - 11x + 30$	$(x - 5), (x + 3)$
36. $f(x) = 3x^3 + 23x^2 + 37x - 15$	$(3x - 1), (x + 5)$

Simplifying a Rational Expression In Exercises 37 and 38, simplify the rational expression.

37. $\dfrac{x^3 + 9x^2 + 2x - 48}{x - 2}$

38. $\dfrac{x^4 + 5x^3 - 20x - 16}{x^2 - 4}$

39. Profit The profit P (in dollars) from selling a motorcycle is given by

$$P = -42x^3 + 3000x^2 - 6000, \quad 0 \le x \le 65$$

where x is the advertising expense (in tens of thousands of dollars). For this motorcycle, the advertising expense was \$600,000 ($x = 60$) and the profit was \$1,722,000.

(a) Use a graphing utility to graph the function and use the result to find another advertising expense that would have produced the same profit.

(b) Use synthetic division to confirm the result of part (a) algebraically.

40. Profit The profit P (in dollars) from selling a novel is given by

$$P = -150x^3 + 7500x^2 - 450{,}000, \quad 0 \le x \le 45$$

where x is the advertising expense (in tens of thousands of dollars). For this novel, the advertising expense was \$400,000 ($x = 40$), and the profit was \$1,950,000.

(a) Use a graphing utility to graph the function and use the result to find another advertising expense that would have produced the same profit.

(b) Use synthetic division to confirm the result of part (a) algebraically.

Using the Rational Zero Test In Exercises 41 and 42, use the Rational Zero Test to list all possible rational zeros of f. Then use a graphing utility to graph the function. Use the graph to help determine which of the possible rational zeros are actual zeros of the function.

41. $f(x) = -4x^3 + 8x^2 - 3x + 15$

42. $f(x) = 3x^4 + 4x^3 - 5x^2 + 10x - 8$

Using the Rational Zero Test In Exercises 43–50, find all the real zeros of the function.

43. $f(x) = x^3 + 2x^2 - 5x - 6$

44. $g(x) = 2x^3 - 15x^2 + 24x + 16$

45. $h(x) = 3x^4 - 27x^2 + 60$

46. $f(x) = x^5 - 4x^3 + 3x$

47. $B(x) = 6x^3 - 19x^2 + 11x + 6$

48. $C(x) = 3x^4 + 3x^3 - 7x^2 - x + 2$

49. $p(x) = x^4 - x^3 - 2x - 4$

50. $q(x) = x^5 - 2x^4 + 2x^3 - 4x^2 - 3x + 6$

Using the Intermediate Value Theorem In Exercises 51 and 52, use the Intermediate Value Theorem to approximate the zero of f in the interval $[a, b]$. Give your approximation to the nearest tenth.

51. $f(x) = x^3 - 4x + 3, \quad [-3, -2]$

52. $f(x) = x^5 + 5x^2 + x - 1, \quad [0, 1]$



Approximating the Zeros of a Polynomial Function In Exercises 53 and 54, use the *zero*, *root*, or *zoom* and *trace* features of a graphing utility to approximate the real zeros of f. Give your approximations to the nearest thousandth.

53. $f(x) = 5x^3 - 11x - 3$

54. $f(x) = 2x^4 - 9x^3 - 5x^2 + 10x + 12$

55. Wholesale Revenue The revenues R (in millions of dollars) for Costco Wholesale for the years 2001 through 2010 are shown in the table. *(Source: Costco Wholesale)*

Year	Revenue, R
2001	34,797
2002	38,762
2003	42,546
2004	48,107
2005	52,935
2006	60,151
2007	64,909
2008	72,483
2009	71,449
2010	77,946

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the *regression* feature of the graphing utility to find a linear model, a quadratic model, and a quartic model for the data.

(c) Use the graphing utility to graph each model with the data. How well does each model fit the data?

(d) Use each model to predict the year in which the revenue will be about \$87 billion. Explain any differences in the predictions.

56. Home Furnishing Sales The sales S (in millions of dollars) for Bed Bath & Beyond for the years 2000 through 2009 are shown in the table. *(Source: Bed Bath & Beyond)*

Year	Sales, S
2000	2396.7
2001	2928.0
2002	3665.2
2003	4478.0
2004	5147.7

Year	Sales, S
2005	5809.6
2006	6617.4
2007	7048.9
2008	7208.3
2009	7828.8

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a linear model, a quadratic model, and a quartic model for the data.

(c) Use the graphing utility to graph each model with the data. How well does each model fit the data?

(d) Use each model to predict the year in which the sales will be about \$8.8 billion. Explain any differences in the predictions.

Writing Complex Numbers in Standard Form In Exercises 57–60, write the complex number in standard form and find its complex conjugate.

57. $\sqrt{-32}$ **58.** 12

59. $-3 + \sqrt{-16}$ **60.** $2 - \sqrt{-18}$

Adding, Subtracting, or Multiplying Complex Numbers In Exercises 61–70, perform the indicated operation and write the result in standard form.

61. $(7 - 3i) + (-2 + 5i)$ **62.** $(14 + 8i) - (-1 - 2i)$

63. $(5 + 8i)(5 - 8i)$ **64.** $(\frac{1}{2} + \frac{3}{4}i)(\frac{1}{2} - \frac{3}{4}i)$

65. $-2i(4 - 5i)$ **66.** $-3(-2 + 4i)$

67. $(3 + 4i)^2$ **68.** $(2 - 5i)^2$

69. $(3 + 2i)^2 + (3 - 2i)^2$ **70.** $(1 + i)^2 - (1 - i)^2$

Writing Quotients in Standard Form In Exercises 71–74, write the quotient in standard form.

71. $\dfrac{8 - i}{2 + i}$ **72.** $\dfrac{3 - 4i}{1 - 5i}$

73. $\dfrac{4 - 3i}{i}$ **74.** $\dfrac{2}{(1 + i)^2}$

Complex Solutions of a Quadratic Equation In Exercises 75–78, solve the quadratic equation.

75. $2x^2 - x + 3 = 0$ **76.** $3x^2 + 6x + 11 = 0$

77. $4x^2 + 11x + 3 = 0$ **78.** $9x^2 - 2x + 5 = 0$

Plotting Complex Numbers in the Complex Plane In Exercises 79 and 80, plot the complex number in the complex plane.

79. $-3 + 2i$ **80.** $-1 - 4i$

Finding the Zeros of a Polynomial Function In Exercises 81–86, find all the zeros of the function and write the polynomial as the product of linear factors.

81. $f(x) = x^4 - 81$ **82.** $h(x) = x^4 + 17x^2 + 16$

83. $f(t) = t^3 + 5t^2 + 3t + 15$

84. $h(x) = 2x^3 - 5x^2 + 4x - 10$

85. $g(x) = 4x^3 - 8x^2 + 9x - 18$

86. $f(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$

Finding a Polynomial Function with Given Zeros In Exercises 87 and 88, find a polynomial function with real coefficients that has the given zeros.

87. $1, 3i, -3i$ **88.** $1, -2, 1 - 3i, 1 + 3i$

Factoring a Polynomial In Exercises 89 and 90, write the polynomial (a) as the product of factors that are irreducible over the rationals, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

89. $x^4 + 5x^2 - 24$

90. $x^4 - 2x^3 - 2x^2 - 14x - 63$
(*Hint:* One factor is $x^2 + 7$.)

Finding the Zeros of a Polynomial Function In Exercises 91–94, use the given zero of f to find all the zeros of f.

91. $f(x) = 4x^3 - x^2 + 64x - 16, \quad -4i$

92. $f(x) = 50 - 75x + 2x^2 - 3x^3, \quad 5i$

93. $f(x) = x^4 + 7x^3 + 24x^2 + 58x + 40, \quad -1 + 3i$

94. $f(x) = x^4 + 4x^3 + 8x^2 + 4x + 7, \quad -2 - \sqrt{3}\,i$

Finding the Domain and Asymptotes of a Rational Function In Exercises 95–98, find the domain of the function and identify any horizontal or vertical asymptotes.

95. $f(x) = \dfrac{-3}{x + 2}$ **96.** $f(x) = \dfrac{3x^2 + 7x - 5}{x^2 + 1}$

97. $f(x) = \dfrac{2x^2}{x^2 - 9}$ **98.** $f(x) = \dfrac{3x}{x^2 + x - 6}$

Sketching the Graph of a Rational Function In Exercises 99–102, sketch the graph of the rational function. To aid in sketching the graphs, check for intercepts, symmetry, vertical asymptotes, and horizontal asymptotes.

99. $P(x) = \dfrac{3 - x}{x + 2}$ **100.** $f(x) = \dfrac{4}{(x - 1)^2}$

101. $g(x) = \dfrac{1}{x^2 - 4} + 2$ **102.** $h(x) = \dfrac{-3x}{2x^2 + 3x - 5}$

Sketching the Graph of a Rational Function In Exercises 103 and 104, find all asymptotes (vertical, horizontal, and/or slant) of the given function. Sketch the graph of f.

103. $f(x) = \dfrac{x^2 - 16}{x - 4}$ **104.** $f(x) = \dfrac{x^3}{x^2 - 5}$

105. Population of Fish The Wildlife Commission introduces 60,000 game fish into a large lake. The population N (in thousands) of the fish is

$$N = \frac{20(3 + 5t)}{1 + 0.06t}, \quad t \geq 0$$

where t is time (in years).

(a) Find the populations when $t = 5$, 10, and 25.

(b) What is the limiting number of fish in the lake as time progresses?

106. Average Recycling Cost The cost C (in dollars) of recycling a waste product is $C = 325{,}000 + 8.5x$, for $x > 0$, where x is the number of pounds of waste. The average recycling cost $\overline{C}$ per pound is

$$\overline{C} = \frac{C}{x} = \frac{325{,}000 + 8.5x}{x}, \quad x > 0.$$

(a) Sketch the graph of $\overline{C}$.

(b) Find the average costs of recycling 1000, 10,000, 100,000, and 1,000,000 pounds of waste. What can you conclude?

107. Smokestack Emissions The cost C (in dollars) of removing p percent of the air pollutants in the stack emissions of a utility company that burns coal to generate electricity is

$$C = \frac{105{,}000p}{100 - p}, \quad 0 \leq p < 100.$$

(a) Find the cost of removing 25% of the pollutants.

(b) Find the cost of removing 60% of the pollutants.

108. Human Memory Model Consider the learning curve modeled by

$$P = \frac{0.7 + 0.65(n - 1)}{1 + 0.65(n - 1)}, \quad n \geq 0$$

where P is the percent of correct responses (in decimal form) after n trials.

(a) Complete the table.

n	1	2	3	4	5	6	7	8	9	10
P										

(b) According to this model, what is the limiting percent of correct responses as n increases?

TEST YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

1. Sketch the graph of the quadratic function given by

$$f(x) = -\tfrac{1}{2}(x - 1)^2 - 5.$$

Identify the vertex and intercept(s).

2. Describe the right-hand and left-hand behavior of the graph of f.

(a) $f(x) = 12x^3 - 5x^2 - 49x + 15$

(b) $f(x) = 5x^4 - 3x^3 + 2x^2 + 11x + 12$

3. Simplify $\dfrac{x^4 + 4x^3 - 19x^2 - 106x - 120}{x^2 - 3x - 10}$.

4. List all possible rational zeros of

$$f(x) = 4x^4 - 16x^3 + 3x^2 + 36x - 27.$$

Use synthetic division to show that $x = -\frac{3}{2}$ and $x = \frac{3}{2}$ are zeros of f. Using these results, completely factor the polynomial.

5. The sales per share S (in dollars) for Hormel Foods for the years 2001 through 2010 are shown in the table at the left. *(Source: Hormel Foods Corporation)*

Year	Sales per share, S
2001	14.87
2002	14.13
2003	15.15
2004	17.33
2005	19.64
2006	20.92
2007	22.82
2008	25.11
2009	24.45
2010	27.15

Table for 5



(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the *regression* feature of the graphing utility to find a linear model, a quadratic model, and a quartic model for the data.

(c) Use the graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the sales per share will be about \$35. Then discuss the appropriateness of each model for predicting future values.

In Exercises 6–9, perform the indicated operation and write the result in standard form.

6. $(12 + 3i) + (4 - 5i)$

7. $(10 - 2i) - (3 + 8i)$

8. $\left(5 + \sqrt{-12}\right)\left(3 - \sqrt{-12}\right)$

9. $(4 + 3i)(2 - 5i)$

10. Write the quotient in standard form: $\dfrac{1 + i}{1 - i}$.

In Exercises 11 and 12, solve the quadratic equation.

11. $x^2 + 5x + 7 = 0$

12. $2x^2 - 5x + 11 = 0$

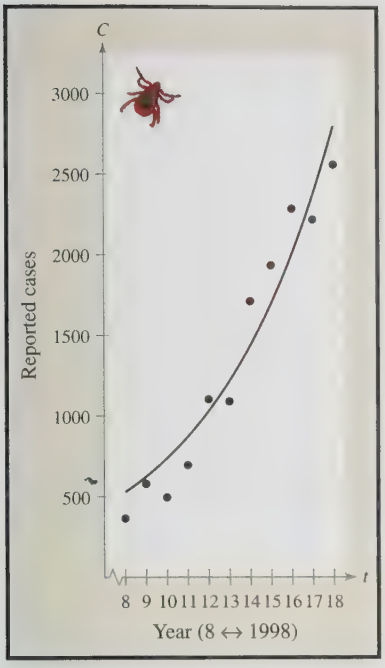
13. Find a polynomial function with real coefficients that has 2, 5, $3i$, and $-3i$ as zeros.

14. Find all the zeros of $f(x) = x^3 + 2x^2 + 5x + 10$, given that $\sqrt{5}i$ is a zero.

15. Sketch the graph of

$$f(x) = \frac{3x}{x - 2}.$$

Label any intercepts and asymptotes. What is the domain of f?



Example 1 on page 388 shows how an exponential growth function can be used to model the number of reported cases of Rocky Mountain spotted fever.

4 Exponential and Logarithmic Functions

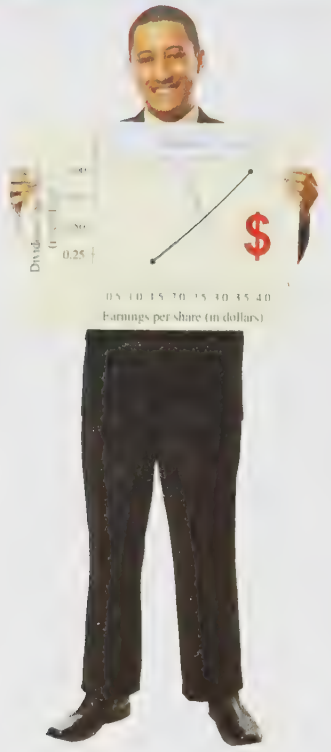
4.1 Inverse Functions
4.2 Exponential Functions
4.3 Logarithmic Functions
4.4 Properties of Logarithms
4.5 Solving Exponential and Logarithmic Equations
4.6 Exponential and Logarithmic Models



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4.1 Inverse Functions

- Determine whether a function has an inverse function.
- Find the inverse function of a function.
- Graph a function and its inverse function.



In Exercise 75 on page 345, you will use an inverse function to approximate the earnings per share in 2008 for Wal-Mart Stores.

Inverse Functions

Recall from Section 2.4 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as

$$f(x) = x + 4\text{: } \{(1, 5), (2, 6), (3, 7), (4, 8)\}.$$

By interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A and can be written as

$$f^{-1}(x) = x - 4\text{: } \{(5, 1), (6, 2), (7, 3), (8, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1} and vice versa, as shown in Figure 4.1. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function, as shown.

$$f(f^{-1}(x)) = f(x - 4)$$
$$= (x - 4) + 4 = x$$
$$f^{-1}(f(x)) = f^{-1}(x + 4)$$
$$= (x + 4) - 4 = x$$

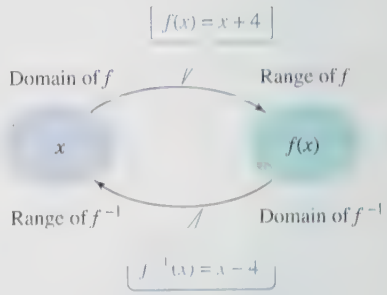


FIGURE 4.1

Example 1 Finding Inverse Functions Informally

Find the inverse function of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

SOLUTION The function $f(x) = 4x$ *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as shown.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x \qquad f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

✓Checkpoint 1

Find the inverse function of $f(x) = \frac{x}{6}$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function. ■

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Example 2 **Finding Inverse Functions Informally**

Find the inverse function of

$$f(x) = x - 6.$$

Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

SOLUTION The function $f(x) = x - 6$ *subtracts* 6 from each input. To "undo" this function, you need to *add* 6 to each input. So, the inverse function of $f(x) = x - 6$ is

$$f^{-1}(x) = x + 6.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as shown.

$$\begin{aligned} f(f^{-1}(x)) &= f(x + 6) && \text{Substitute } x + 6 \text{ for } f^{-1}(x). \\ &= (x + 6) - 6 && \text{Substitute } x + 6 \text{ into } f(x). \\ &= x && \text{Identity function} \\ f^{-1}(f(x)) &= f^{-1}(x - 6) && \text{Substitute } x - 6 \text{ for } f(x). \\ &= (x - 6) + 6 && \text{Substitute } x - 6 \text{ into } f^{-1}(x). \\ &= x && \text{Identity function} \end{aligned}$$

Checkpoint 2

Find the inverse function of $f(x) = x + 10$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

The formal definition of inverse function is shown below.

Definition of Inverse Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f.$$

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

Do not be confused by the use of -1 to denote the inverse function f^{-1}. In this text, f^{-1} *always* refers to the inverse function of the function f and *not* to the reciprocal of $f(x)$. That is,

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

If the function g is the inverse function of the function f, then it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other.*

Example 3 Verifying Inverse Functions

Show that the following functions are inverse functions.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

SOLUTION

$$f(g(x)) = f\left(\sqrt[3]{\frac{x+1}{2}}\right) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$
$$= 2\left(\frac{x+1}{2}\right) - 1$$
$$= x + 1 - 1$$
$$= x$$

$$g(f(x)) = g(2x^3 - 1) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}}$$
$$= \sqrt[3]{\frac{2x^3}{2}}$$
$$= \sqrt[3]{x^3}$$
$$= x$$

Checkpoint 3

Show that the following functions are inverse functions.

$$f(x) = x^3 + 6 \quad \text{and} \quad g(x) = \sqrt[3]{x - 6}$$

Example 4 Verifying Inverse Functions

Which of the functions given by

$$g(x) = \frac{x-2}{5} \quad \text{and} \quad h(x) = \frac{5}{x} + 2$$

is the inverse function of $f(x) = \dfrac{5}{x-2}$?

SOLUTION By forming the composition of f with g, you can see that

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{[(x-2)/5] - 2} = \frac{25}{x - 12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g is *not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{[(5/x) + 2] - 2} = \frac{5}{5/x} = x.$$

So, it appears that h is the inverse function of f. You can confirm this result by showing that the composition of h with f is also equal to the identity function.

Checkpoint 4

Which of the functions given by $g(x) = \dfrac{x+4}{3}$ and $h(x) = \dfrac{x}{3} + 4$ is the inverse function of $f(x) = 3x - 4$?

Finding Inverse Functions

For simple functions (such as the ones in Examples 1 and 2), you can find inverse functions by inspection. For more complicated functions, it is best to use the following guidelines. The key step in these guidelines is switching the roles of x and y. This step corresponds to the fact that inverse functions have ordered pairs with the coordinates reversed.

STUDY TIP

Note in Step 3 of the guidelines for finding inverse functions that it is possible for a function to have no inverse function. For instance, the function given by $f(x) = x^2$ has no inverse function.

Guidelines for Finding Inverse Functions

1. In the equation for $f(x)$, replace $f(x)$ with y.
2. Interchange the roles of x and y.
3. Solve the new equation for y. If the new equation does not represent y as a function of x, then the function f does not have an inverse function. If the new equation does represent y as a function of x, then continue to Step 4.
4. Replace y with $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and
$$f(f^{-1}(x)) = x = f^{-1}(f(x)).$$

Example 5 Finding Inverse Functions

Find the inverse function of

$$f(x) = \frac{5 - 3x}{2}.$$

SOLUTION



$$f(x) = \frac{5 - 3x}{2} \quad \text{Write original function.}$$

$$y = \frac{5 - 3x}{2} \quad \text{Replace } f(x) \text{ with } y.$$

$$x = \frac{5 - 3y}{2} \quad \text{Interchange } x \text{ and } y.$$

$$2x = 5 - 3y \quad \text{Multiply each side by 2.}$$

$$3y = 5 - 2x \quad \text{Isolate the } y\text{-term.}$$

$$y = \frac{5 - 2x}{3} \quad \text{Solve for } y.$$

$$f^{-1}(x) = \frac{5 - 2x}{3} \quad \text{Replace } y \text{ with } f^{-1}(x).$$

Note that both f and f^{-1} have domains and ranges that consist of the entire set of real numbers. Check that

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

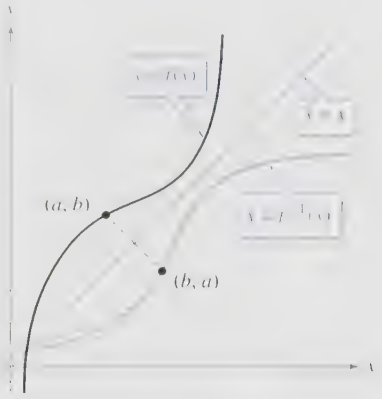
Checkpoint 5

Find the inverse function of $f(x) = 4x + 5$.

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The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a *reflection* of the graph of f in the line $y = x$, as shown in Figure 4.2.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

FIGURE 4.2

TECH TUTOR

A graphing utility can help you check that the graphs of f and f^{-1} are reflections of each other in the line $y = x$. To do this, graph $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ in the same viewing window, using a square setting.

Example 6 The Graphs of f and f^{-1}

Sketch the graphs of the inverse functions given by

$$f(x) = 2x - 3$$

and

$$f^{-1}(x) = \frac{1}{2}(x + 3)$$

in the same coordinate plane, and show that the graphs are reflections of each other in the line $y = x$.

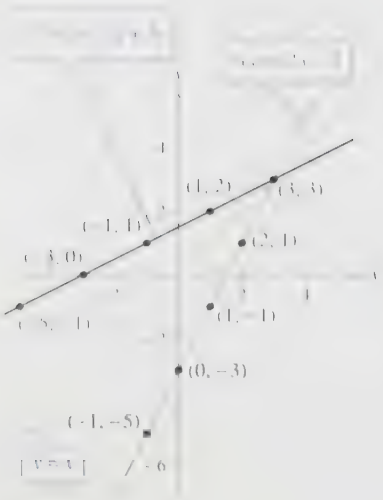


FIGURE 4.3

SOLUTION The graphs of f and f^{-1} are shown in Figure 4.3. Visually, it appears that the graphs are reflections of each other in the line $y = x$. You can further verify this reflective property by testing a few points on each graph. Note in the following list that if the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.

Graph of $f(x) = 2x - 3$	*Graph of* $f^{-1}(x) = \frac{1}{2}(x + 3)$
$(0, -3)$	$(-3, 0)$
$(1, -1)$	$(-1, 1)$
$(2, 1)$	$(1, 2)$
$(3, 3)$	$(3, 3)$

Checkpoint 6

Sketch the graphs of the inverse functions given by

$$f(x) = \frac{2}{5}x + 2 \quad \text{and} \quad f^{-1}(x) = \frac{5}{2}x - 5$$

in the same coordinate plane, and show that the graphs are reflections of each other in the line $y = x$. ■

The Study Tip on page 339 mentioned that the function given by $f(x) = x^2$ has no inverse function. This means, *assuming the domain of f is the entire real line*, that the function given by $f(x) = x^2$ has no inverse function. If the domain of f is restricted to the nonnegative real numbers, however, then f does have an inverse function, as demonstrated in Example 7.

Example 7 The Graphs of f and f^{-1}

Sketch the graphs of the inverse functions given by

$$f(x) = x^2, \quad x \geq 0, \quad \text{and} \quad f^{-1}(x) = \sqrt{x}$$

in the same coordinate plane, and show that the graphs are reflections of each other in the line $y = x$.

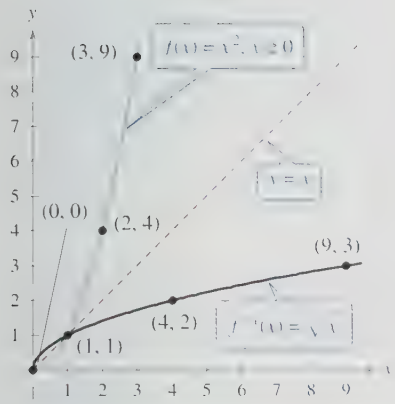


FIGURE 4.4

SOLUTION The graphs of f and f^{-1} are shown in Figure 4.4. Visually, it appears that the graphs are reflections of each other in the line $y = x$. You can further verify this reflective property by testing a few points on each graph. Note in the following list that if the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.

Graph of $f(x) = x^2, \quad x \geq 0$	*Graph of* $f^{-1}(x) = \sqrt{x}$
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(1, 1)$
$(2, 4)$	$(4, 2)$
$(3, 9)$	$(9, 3)$

You can verify algebraically that the functions are inverse functions of each other by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ as shown.

$$\begin{aligned} f(f^{-1}(x)) &= f(\sqrt{x}) \\ &= (\sqrt{x})^2 \\ &= x, \quad \text{if } x \geq 0 \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(x^2) \\ &= \sqrt{x^2} \\ &= x, \quad \text{if } x \geq 0 \end{aligned}$$

Checkpoint 7

Sketch the graphs of the inverse functions given by

$$f(x) = x^2 + 3, \quad x \geq 0, \quad \text{and} \quad f^{-1}(x) = \sqrt{x - 3}$$

in the same coordinate plane, and show that the graphs are reflections of each other in the line $y = x$.

The guidelines for finding the inverse function of a function include an *algebraic* test for determining whether a function has an inverse function. The reflective property of the graphs of inverse functions gives you a *geometric* test for determining whether a function has an inverse function. This test is called the **Horizontal Line Test** for inverse functions.

Horizontal Line Test for Inverse Functions

A function f has an inverse function if and only if no *horizontal* line intersects the graph of f at more than one point.

Example 8 **Applying the Horizontal Line Test**

Use the graph of f to determine whether the function has an inverse function.

a. $f(x) = x^3 - 1$ **b.** $f(x) = x^2 - 1$

SOLUTION

a. The graph of the function given by

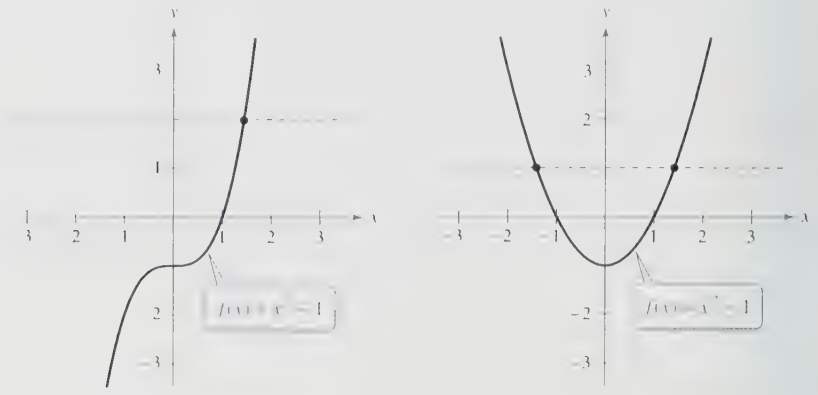
$$f(x) = x^3 - 1$$

is shown in Figure 4.5(a). Because no horizontal line intersects the graph of f at more than one point, you can conclude that f does have an inverse function.

b. The graph of the function given by

$$f(x) = x^2 - 1$$

is shown in Figure 4.5(b). Because it is possible to find a horizontal line that intersects the graph of f at more than one point, you can conclude that f does not have an inverse function.



(a) (b)

FIGURE 4.5

Checkpoint 8

Use the graph of f to determine whether the function has an inverse function.

a. $f(x) = |x|$ **b.** $f(x) = \sqrt{x}$



SUMMARIZE (Section 4.1)

1. State the definition of an inverse function *(page 337)*. For examples of finding inverse functions informally and verifying inverse functions, see Examples 1, 2, 3, and 4.
2. Describe how to find an inverse function *(page 339)*. For an example of finding an inverse function, see Example 5.
3. Describe the relationship between the graph of a function and the graph of its inverse function *(page 340)*. For examples of sketching the graphs of functions and their inverse functions, see Examples 6 and 7.
4. Describe the Horizontal Line Test for inverse functions *(page 341)*. For an example of applying the Horizontal Line Test, see Example 8.

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SKILLS WARM UP 4.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.4, 1.2, and 2.4.

In Exercises 1–4, find the domain of the function.

1. $f(x) = \sqrt[3]{x+1}$

2. $f(x) = \sqrt{x+1}$

3. $g(x) = \dfrac{2}{x^2 - 2x}$

4. $h(x) = \dfrac{x}{3x+5}$

In Exercises 5–8, simplify the expression.

5. $2\left(\dfrac{x+5}{2}\right) - 5$

6. $7 - 10\left(\dfrac{7-x}{10}\right)$

7. $\sqrt[3]{2\left(\dfrac{x^3}{2} - 2\right) + 4}$

8. $\sqrt[5]{(x+2)^5} - 2$

In Exercises 9 and 10, solve for x in terms of y.

9. $y = \dfrac{2x-6}{3}$

10. $y = \sqrt[3]{2x-4}$

Exercises 4.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding Inverse Functions In Exercises 1–4, find the inverse function of the function f given by the set of ordered pairs.

1. $\{(1, 4), (2, 5), (3, 6), (4, 7)\}$

2. $\{(6, 2), (5, 3), (4, 4), (3, 5)\}$

3. $\{(-1, 1), (-2, 2), (-3, 3), (-4, 4)\}$

4. $\{(6, -2), (5, -3), (4, -4), (3, -5)\}$

Finding Inverse Functions Informally In Exercises 5–8, find the inverse function informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. *See Examples 1 and 2.*

5. $f(x) = 2x$

6. $f(x) = -\dfrac{x}{4}$

7. $f(x) = x - 5$

8. $f(x) = x + 7$

Verifying Inverse Functions In Exercises 9–16, show that f and g are inverse functions by using the definition of inverse functions. *See Examples 3 and 4.*

9. $f(x) = 5x + 1, \quad g(x) = \dfrac{x-1}{5}$

10. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3-x}{4}$

11. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$

12. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

13. $f(x) = \sqrt{x-4}, \quad g(x) = x^2 + 4, \quad x \geq 0$

14. $f(x) = 9 - x^2, \quad x \geq 0, \quad g(x) = \sqrt{9-x}, \quad x \leq 9$

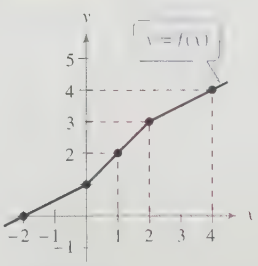
15. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1-x}$

16. $f(x) = \dfrac{1}{1+x}, \quad x \geq 0, \quad g(x) = \dfrac{1-x}{x}, \quad 0 < x \leq 1$

Sketching Inverse Functions In Exercises 17 and 18, use the graph of f to complete the table and to sketch the graph of f^{-1}.

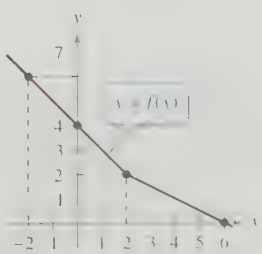
17.

x	0	1	2	3	4
$f^{-1}(x)$					



18.

x	0	2	4	6
$f^{-1}(x)$				



Finding Inverse Functions In Exercises 19–34, determine whether the function has an inverse function. If it does, find its inverse function. *See Example 5.*

19. $f(x) = x^4$
20. $f(x) = \dfrac{1}{x^2}$
21. $g(x) = \dfrac{x}{8}$
22. $f(x) = 3x + 5$
23. $p(x) = -4$
24. $f(x) = \dfrac{3x + 4}{5}$
25. $f(x) = (x + 3)^2,\ x \geq -3$
26. $q(x) = (x - 5)^2$
27. $h(x) = \dfrac{1}{x}$
28. $f(x) = |x - 2|,\ x \leq 2$
29. $f(x) = \sqrt{2x + 3}$
30. $f(x) = \sqrt{x - 2}$
31. $g(x) = x^2 - x^4$
32. $f(x) = \dfrac{x^2}{x^2 + 1}$
33. $f(x) = 25 - x^2,\ x \leq 0$
34. $f(x) = 36 + x^2,\ x \leq 0$

The Graphs of f and f^{-1} In Exercises 35–40, sketch the graphs of the inverse functions in the same coordinate plane and show that the graphs are reflections of each other in the line $y = x$. *See Examples 6 and 7.*

35. $f(x) = \dfrac{x}{3}$, $f^{-1}(x) = 3x$
36. $f(x) = x + 8$, $f^{-1}(x) = x - 8$
37. $f(x) = 7x + 1$, $f^{-1}(x) = \dfrac{x - 1}{7}$
38. $f(x) = \dfrac{4}{x}$, $f^{-1}(x) = \dfrac{4}{x}$
39. $f(x) = x^5 - 2$, $f^{-1}(x) = \sqrt[5]{x + 2}$
40. $f(x) = \sqrt{4 - x^2},\ 0 \leq x \leq 2$, $f^{-1}(x) = \sqrt{4 - x^2},\ 0 \leq x \leq 2$

Error Analysis In Exercises 41 and 42, describe and correct the error in finding the inverse function.

41. Find the inverse function f^{-1} of $f(x) = \sqrt{2x - 5}$.

$f(x) = \sqrt{2x - 5}$, so $f^{-1}(x) = \dfrac{1}{\sqrt{2x - 5}}$.

42. Find the inverse function f^{-1} of $f(x) = \frac{3}{5}x + \frac{1}{3}$.

$f(x) = \frac{3}{5}x + \frac{1}{3}$, so $f^{-1}(x) = \frac{5}{3}x - 3$.

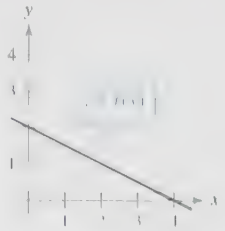


Graphing Inverse Functions In Exercises 43–52, find the inverse function f^{-1} of the function f. Then, using a graphing utility, graph both f and f^{-1} in the same viewing window.

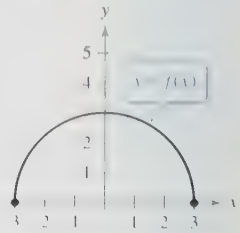
43. $f(x) = 2x - 3$
44. $f(x) = 5x + 2$
45. $f(x) = x^3$
46. $f(x) = x^3 + 1$
47. $f(x) = \sqrt{x}$
48. $f(x) = x^2,\ x \geq 0$
49. $f(x) = \sqrt{16 - x^2},\ 0 \leq x \leq 4$
50. $f(x) = \dfrac{3}{x + 1}$
51. $f(x) = \sqrt[3]{x + 2}$
52. $f(x) = x^{3/5} - 2$

Applying the Horizontal Line Test In Exercises 53–56, use the graph of f to determine whether the function has an inverse function. *See Example 8.*

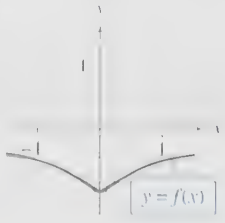
53.



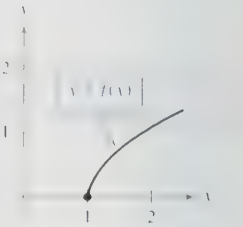
54.



55.



56.



Applying the Horizontal Test In Exercises 57–62, graph the function and use the Horizontal Line Test to determine whether the function has an inverse function.

57. $g(x) = \dfrac{5 - 2x}{3}$
58. $f(x) = 10$
59. $h(x) = |x - 5|$
60. $g(x) = (x - 3)^2$
61. $f(x) = -\sqrt{9 - x^2}$
62. $f(x) = (x - 1)^3$

Finding the Composition of Functions In Exercises 63–66, use the functions given by

$f(x) = x + 4$ and $g(x) = 2x - 6$

to find the composition of functions.

63. $g^{-1} \circ f^{-1}$
64. $f^{-1} \circ g^{-1}$
65. $(f \circ g)^{-1}$
66. $(g \circ f)^{-1}$

Evaluating Functions In Exercises 67–70, use the functions given by

$f(x) = \dfrac{1}{8}x - 3$ and $g(x) = x^2$

to find the value.

67. $(f^{-1} \circ g^{-1})(1)$
68. $(g^{-1} \circ f^{-1})(-3)$
69. $(f^{-1} \circ f^{-1})(6)$
70. $(g^{-1} \circ g^{-1})(-4)$

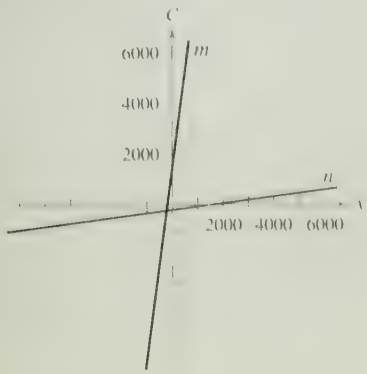
71. Profit A company's profit P for producing x units is given by $P(x) = 47x - 5736$. Find the inverse function $P^{-1}(x)$ and explain what it represents. Describe the domains of $P(x)$ and $P^{-1}(x)$.

72. HOW DO YOU SEE IT? The cost C for a business to make personalized T-shirts is given by

$$C(x) = 7.50x + 1500$$

where x represents the number of T-shirts.

(a) The graphs of $C(x)$ and $C^{-1}(x)$ are shown below. Match each function with its graph.



(b) Explain what $C(x)$ and $C^{-1}(x)$ represent in the context of the problem.

73. Movie Theaters The average prices of admission y (in dollars) to movie theaters for the years 2003 through 2010 are shown in the table. *(Source: National Association of Theater Owners)*

Year	2003	2004	2005	2006
Admission price, y	6.03	6.21	6.41	6.55

Year	2007	2008	2009	2010
Admission price, y	6.88	7.18	7.50	7.89

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Algebraically find the inverse function of the model in part (b). Explain what this inverse function represents in a real-life context.

(d) Use the inverse function you found in part (c) to predict the year in which the average admission price to a movie theater will reach \$9.00.

74. Lead Exposure A project is conducted to study the amount of lead accumulated in the bones of humans. The concentration L (in micrograms per gram of bone mineral) of lead found in the tibia of a man is measured every five years. The results are shown in the table.

Age	15	20	25	30	35	40
Lead, L	3.2	5.4	9.2	12.2	13.8	16.0

(a) Use a graphing utility to create a scatter plot of the data. Let x represent the age (in years) of the man.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

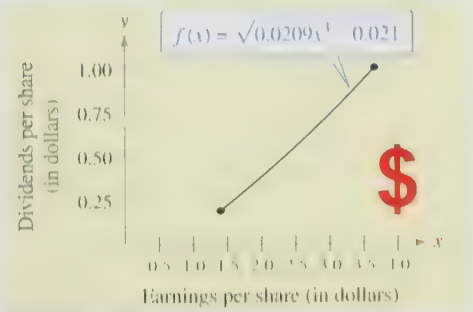
(c) Algebraically find the inverse function of the model in part (b). Explain what this inverse function represents in a real-life context.

(d) Use the inverse function you found in part (c) to estimate the age of the man when the concentration of lead in his tibia reaches 25 micrograms per gram of bone mineral.

75. Earnings-Dividend Ratio From 2000 through 2009, the earnings per share for Wal-Mart Stores were approximately related to the dividends per share by the function

$$f(x) = \sqrt{0.0209x^3 - 0.021}, \quad 1.40 \le x \le 3.66$$

where f represents the dividends per share (in dollars) and x represents the earnings per share (in dollars). (See figure.)



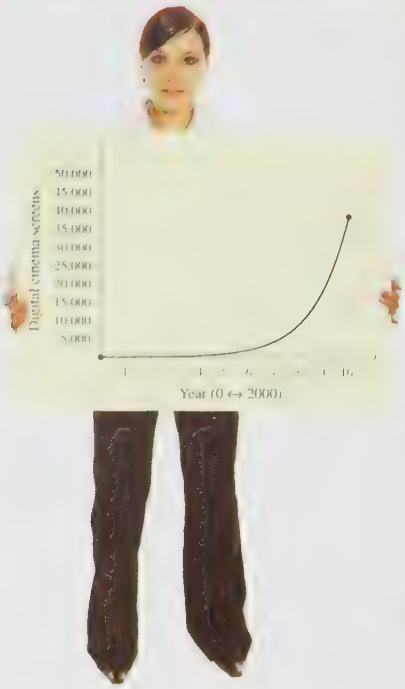
In 2008, Wal-Mart paid dividends of \$0.93 per share. Find the inverse function of f and use the inverse function to approximate the earnings per share in 2008. *(Source: Wal-Mart Stores, Inc.)*

76. Diesel Mechanics The function given by

$$y = 0.03x^2 + 245.5, \quad 0 < x < 100$$

approximates the exhaust temperature y for a diesel engine in degrees Fahrenheit, where x is the percent load for the diesel engine. Solve the equation for x in terms of y and use the result to find the percent load for a diesel engine when the exhaust temperature is 410°F.

4.2 Exponential Functions



In Exercise 73 on page 356, you will use a natural exponential function to determine the number of digital cinema screens in the world.

- Evaluate an exponential expression.
- Sketch the graph of an exponential function.
- Evaluate and sketch the graph of the natural exponential function.
- Use the compound interest formulas.
- Use an exponential model to solve an application problem.

Exponential Functions

So far, this text has dealt only with **algebraic functions,** which include polynomial functions and rational functions. In this chapter, you will study two types of nonalgebraic functions—*exponential* functions and *logarithmic* functions. These functions are examples of **transcendental functions.**

Definition of Exponential Function

The **exponential function** f **with base** a is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

The base $a = 1$ is excluded because it yields

$$f(x) = 1^x = 1.$$

This is a constant function, not an exponential function.

You already know how to evaluate a^x for integer and rational values of x. For example, you know that

$$4^3 = 64 \quad \text{and} \quad 4^{1/2} = 2.$$

However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents. For the purposes of this text, it is sufficient to think of

$$a^{\sqrt{2}} \qquad (\text{where } \sqrt{2} \approx 1.414214)$$

as that value having the successively closer approximations

$$a^{1.4},\ a^{1.41},\ a^{1.414},\ a^{1.4142},\ a^{1.41421},\ a^{1.414214},\ \ldots.$$

Example 1 Evaluating an Exponential Expression

Scientific Calculator

Number	*Keystrokes*	*Display*
$2^{-\pi}$	2 y^2 π +/− =	0.113314732

Graphing Calculator

Number	*Keystrokes*	*Display*
$2^{-\pi}$	2 ^ (−) π ENTER	.1133147323

Checkpoint 1

Use a calculator to evaluate $2.2^{\sqrt{3}}$. Round your result to three decimal places. ■

Artpose-Adam Borkowski/Shutterstock.com

Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 4.

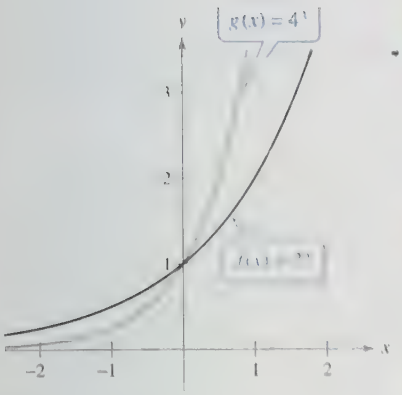


FIGURE 4.6

Example 2 Graphs of $y = a^x$

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$

b. $g(x) = 4^x$

SOLUTION The table below lists some values for each function, and Figure 4.6 shows their graphs. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$
$g(x) = 4^x$	$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$	$4^{-1} = \frac{1}{4^1} = \frac{1}{4}$	$4^0 = 1$	$4^1 = 4$	$4^2 = 16$	$4^3 = 64$

Checkpoint 2

Sketch the graph of $f(x) = 5^x$.

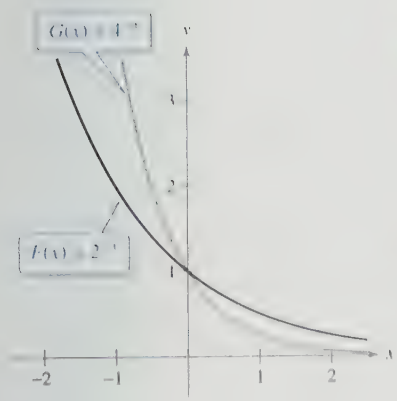


FIGURE 4.7

Example 3 Graphs of $y = a^{-x}$

In the same coordinate plane, sketch the graph of each function.

a. $F(x) = 2^{-x}$

b. $G(x) = 4^{-x}$

SOLUTION The table below lists some values for each function, and Figure 4.7 shows their graphs. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

x	-3	-2	-1	0	1	2
$F(x) = 2^{-x}$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$G(x) = 4^{-x}$	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

Checkpoint 3

Sketch the graph of $F(x) = 5^{-x}$.

The tables in Examples 2 and 3 were evaluated by hand. You could use the *table* feature of a graphing utility to construct tables with more values.

In Example 3, note that the functions given by $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$ can be rewritten with positive exponents.

$$F(x) = 2^{-x} = \left(\frac{2}{1}\right)^{-x} = \left(\frac{1}{2}\right)^{x} \quad \text{and} \quad G(x) = 4^{-x} = \left(\frac{4}{1}\right)^{-x} = \left(\frac{1}{4}\right)^{x}$$

ALGEBRA TUTOR

For help with evaluating expressions involving exponents, see the *Chapter 4 Algebra Tutor* on page 400.

Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \quad \text{and} \quad G(x) = 4^{-x} = g(-x).$$

Consequently, the graph of F is a reflection (in the y-axis) of the graph of f. The graphs of G and g have the same relationship.

The graphs in Figures 4.6 and 4.7 are typical of the exponential functions $y = a^x$ and $y = a^{-x}$. They have one y-intercept and one horizontal asymptote (the x-axis), and they are continuous. The basic characteristics of these exponential functions are summarized in Figures 4.8 and 4.9.



Characteristics of Exponential Functions

Graph of $y = a^x$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Increasing
- x-axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$)
- Continuous

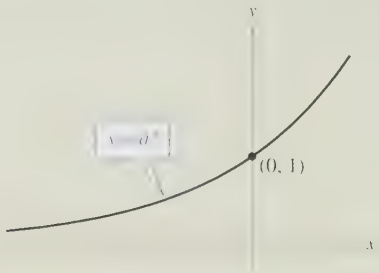


FIGURE 4.8

Graph of $y = a^{-x}$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Decreasing
- x-axis is a horizontal asymptote ($a^{-x} \to 0$ as $x \to \infty$)
- Continuous
- Reflection of graph of $y = a^x$ in y-axis

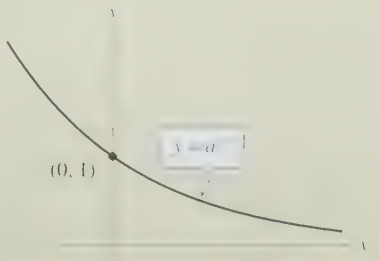


FIGURE 4.9

Dmitriy Pochutalin/Shutterstock.com

In the following example, notice how the graph of $y = a^x$ is used to sketch the graphs of functions of the form

$$f(x) = b \pm a^{x+c}.$$

Example 4 **Transformations of Graphs of Exponential Functions**

Each of the following graphs is a transformation of the graph of $f(x) = 3^x$, as shown in Figure 4.10.

a. Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *left*.

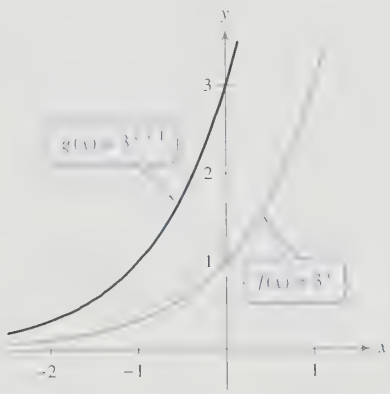
b. Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f *downward* two units.

c. Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by *reflecting* the graph of f in the x-axis.

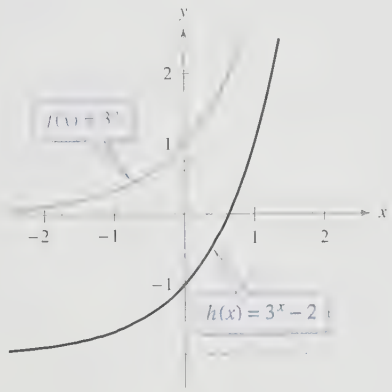
d. Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by *reflecting* the graph of f in the y-axis.

STUDY TIP

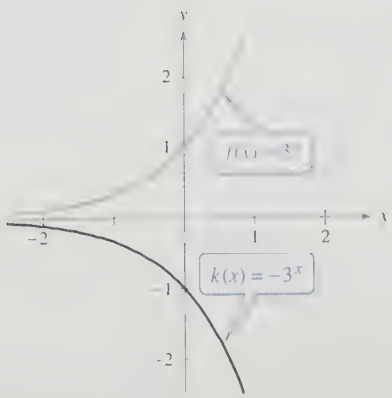
Notice in Example 4(b) that shifting the graph downward two units also shifts the horizontal asymptote of $f(x) = 3^x$ from the x-axis ($y = 0$) to the line $y = -2$.



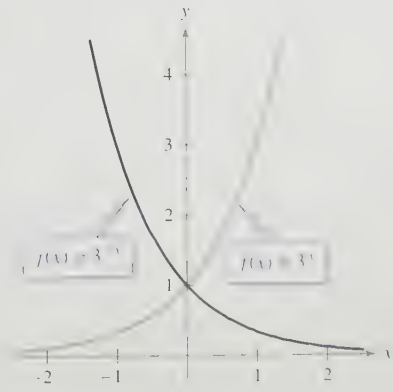
(a)



(b)



(c)



(d)

FIGURE 4.10

Checkpoint 4

Sketch the graph of

$$f(x) = 2^{x-1}.$$

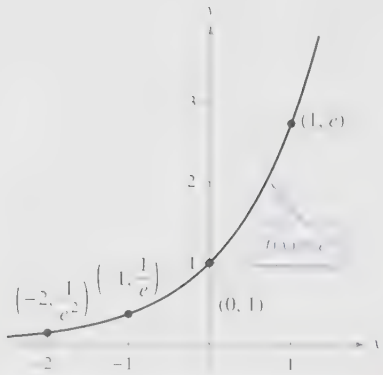


FIGURE 4.11

The Natural Base e

In many applications, the most convenient choice for a base is the irrational number

$$e = 2.718281828\ldots$$

called the **natural base.** The function given by

$$f(x) = e^x$$

is called the **natural exponential function.** Its graph is shown in Figure 4.11. The graph of the natural exponential function has the same basic characteristics as the graph of the exponential function given by $f(x) = a^x$ (see page 348). Be sure you see that for the exponential function given by $f(x) = e^x$, e is the constant 2.718281828 . . . , whereas x is the variable.

Example 5 Evaluating the Natural Exponential Function

Use a calculator to evaluate the function given by $f(x) = e^x$ when $x = 2$ and $x = -1$.

SOLUTION

Scientific Calculator

Number	*Keystrokes*	*Display*
e^2	2 [2nd] [e^x]	7.389056099
e^{-1}	1 [+/−] [2nd] [e^x]	0.367879441

Graphing Calculator

Number	*Keystrokes*	*Display*
e^2	[2nd] [e^x] 2 [)] [ENTER]	7.389056099
e^{-1}	[2nd] [e^x] [(−)] 1 [)] [ENTER]	.3678794412

Checkpoint 5

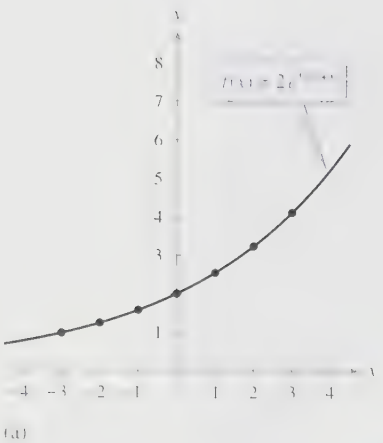
Use a calculator to evaluate $f(x) = e^x$ when $x = 6$ and $x = -3.5$. ■

Example 6 Graphing Natural Exponential Functions

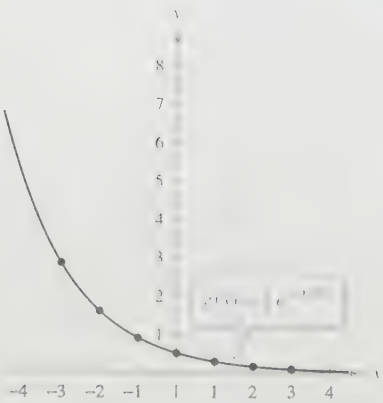
Sketch the graph of each natural exponential function.

a. $f(x) = 2e^{0.24x}$ **b.** $g(x) = \frac{1}{2}e^{-0.58x}$

SOLUTION Use a calculator to find several points on each graph, as shown in the table. Then, plot the points and connect them with smooth curves, as shown in Figure 4.12. Note that the graph in part (a) is increasing, whereas the graph in part (b) is decreasing.



(a)



(b)

FIGURE 4.12

x	−3	−2	−1	0	1	2	3
$f(x) = 2e^{0.24x}$	0.974	1.238	1.573	2	2.542	3.232	4.109
$g(x) = \frac{1}{2}e^{-0.58x}$	2.849	1.595	0.893	0.5	0.280	0.157	0.088

Checkpoint 6

Sketch the graph of $f(x) = e^{0.5x}$. ■

Compound Interest

One of the most familiar examples of exponential growth is that of an investment earning **continuously compounded interest.** The formula for the balance in an account that is compounded n times per year is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A is the balance in the account, P is the initial deposit, r is the annual interest rate (in decimal form), and t is the number of years. Using exponential functions, you will *develop* this formula and show how it leads to continuous compounding.

Suppose a principal P is invested at an annual interest rate r, compounded once a year. The principal at the end of the first year, P_1, is equal to the initial deposit P plus the interest earned, Pr. So,

$$P_1 = P + Pr.$$

This can be rewritten by factoring out P from each term as shown.

$$\begin{aligned} P_1 &= P + Pr \\ &= P(1 + r) \end{aligned}$$

This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown below.

Year	*Balance After Each Compounding*
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
3	$P_3 = P_2(1 + r) = P(1 + r)^2(1 + r) = P(1 + r)^3$
	$\vdots$
t	$P_t = P(1 + r)^t$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. Then the rate per compounding is r/n and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$ Amount (balance) with n compoundings per year

When you let the number of compoundings n increase without bound, the process approaches what is called **continuous compounding.** In the formula for n compoundings per year, let $m = n/r$. This produces

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Amount with } n \text{ compoundings per year} \\ &= P\left(1 + \frac{1}{m}\right)^{mrt} && \text{Substitute } mr \text{ for } n \text{ and simplify.} \\ &= P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}. && \text{Property of exponents} \end{aligned}$$

As m increases without bound, it can be shown that

$$\left[1 + \left(\frac{1}{m}\right)\right]^{m}$$

approaches e. From this, you can conclude that the formula for continuous compounding is

$$A = Pe^{rt}.$$ Continuous compounding

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. For continuous compounding: $A = Pe^{rt}$

Be sure that the annual interest rate is written in decimal form. For instance, 6% should be written as 0.06 when using compound interest formulas.

Example 7 Compound Interest

You invest \$12,000 at an annual rate of 3%. Find the balance after 5 years when the interest is compounded (a) quarterly, (b) monthly, and (c) continuously. Which type of compounding earns the most money?

SOLUTION

a. For quarterly compounding, you have $n = 4$. So, in 5 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

$$= 12{,}000\left(1 + \frac{0.03}{4}\right)^{4(5)}$$ Substitute for P, r, n, and t.

$$\approx \$13{,}934.21.$$ Use a calculator.

b. For monthly compounding, you have $n = 12$. So, in 5 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

$$= 12{,}000\left(1 + \frac{0.03}{12}\right)^{12(5)}$$ Substitute for P, r, n, and t.

$$\approx \$13{,}939.40.$$ Use a calculator.

c. For continuous compounding, the balance is

$$A = Pe^{rt}$$ Formula for continuous compounding

$$= 12{,}000e^{0.03(5)}$$ Substitute for P, r, and t.

$$\approx \$13{,}942.01.$$ Use a calculator.

Continuous compounding yields more than quarterly and monthly compounding. This is typical of the two types of compounding. That is, for a given principal, interest rate, and time, continuous compounding will always yield a larger balance than compounding n times a year.

Checkpoint 7

You invest \$6000 at an annual rate of 4%. Find the balance after 7 years when the interest is compounded

a. quarterly.

b. monthly.

c. continuously.

Another Application

Example 8 Radioactive Decay

In 1986, a nuclear reactor accident occurred in Chernobyl in what was then the Soviet Union. The explosion spread highly toxic radioactive chemicals, such as plutonium, over hundreds of square miles, and the government evacuated the city and the surrounding area. Consider the model

$$P = 10\left(\frac{1}{2}\right)^{t/24{,}100}$$

which represents the amount of plutonium P that remains (from an initial amount of 10 pounds) after t years. Sketch the graph of this function over the interval from $t = 0$ to $t = 100{,}000$, where $t = 0$ represents 1986. How much of the 10 pounds of plutonium will remain in the year 2012? How much of the 10 pounds will remain after 100,000 years? Why is this city uninhabited?

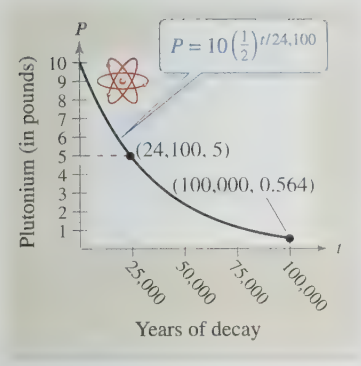


FIGURE 4.13

SOLUTION The graph of this function is shown in Figure 4.13. Plutonium has a *half-life* of about 24,100 years. That is, after 24,100 years, *half* of the original amount of plutonium will remain. After another 24,100 years, one-quarter of the original amount will remain, and so on. In the year 2012 ($t = 26$), there will still be

$$P = 10\left(\frac{1}{2}\right)^{26/24{,}100} \qquad \text{Substitute 26 for } t.$$

$$\approx 10\left(\frac{1}{2}\right)^{0.0010788} \qquad \text{Approximate exponent.}$$

$$\approx 9.993 \text{ pounds} \qquad \text{Use a calculator.}$$

of plutonium remaining. After 100,000 years, there will still be

$$P = 10\left(\frac{1}{2}\right)^{100{,}000/24{,}100} \qquad \text{Substitute 100,000 for } t.$$

$$\approx 10\left(\frac{1}{2}\right)^{4.149} \qquad \text{Approximate exponent.}$$

$$\approx 0.564 \text{ pound} \qquad \text{Use a calculator.}$$

of plutonium remaining. This city is uninhabited because much of the original amount of radioactive plutonium still remains.



Checkpoint 8

In Example 8, how much of the initial 10 pounds of plutonium will remain in the year 2086?

SUMMARIZE (Section 4.2)

1. State the definition of an exponential function f with base a *(page 346)*. For an example of evaluating an exponential expression, see Example 1.
2. Describe the relationship between the graphs of the functions $f(x) = a^x$ and $g(x) = a^{-x}$, where $a > 1$ *(pages 347 and 348)*. For examples of graphing exponential functions, see Examples 2, 3, and 4.
3. State the definitions of the natural base and the natural exponential function *(page 350)*. For examples of evaluating and graphing natural exponential functions, see Examples 5 and 6.

David Gilder/Shutterstock.com

SKILLS WARM UP 4.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.4.

In Exercises 1–12, use the properties of exponents to simplify the expression.

1. $5^{2x}(5^{-x})$
2. $3^{-x}(3^{3x})$
3. $\dfrac{4^{5x}}{4^{2x}}$
4. $\dfrac{10^{2x}}{10^{x}}$
5. $(4^{x})^{2}$
6. $(4^{2x})^{5}$
7. $\left(\dfrac{2^{x}}{3^{x}}\right)^{-1}$
8. $(4^{6x})^{1/2}$
9. $(2^{3x})^{-1/3}$
10. $\left(\dfrac{3^{4x}}{5^{4x}}\right)^{1/4}$
11. $(16^{x})^{1/4}$
12. $(27^{x})^{1/3}$

Exercises 4.2

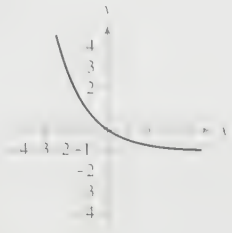
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Evaluating an Exponential Expression In Exercises 1–6, use a calculator to evaluate the expression. Round your result to three decimal places. *See Example 1.*

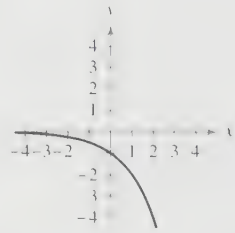
1. $(2.6)^{1.3}$
2. $(1.12)^{20.5}$
3. $100(1.03)^{-9/5}$
4. $1500(2^{-5/2})$
5. $6^{-\sqrt{2}}$
6. $1.3^{-\sqrt{5}}$

Matching In Exercises 7–14, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

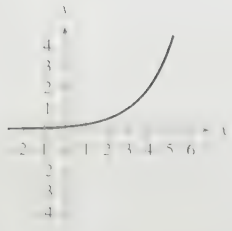
(a)



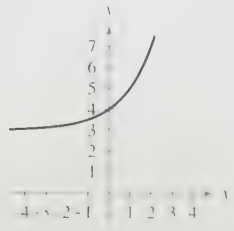
(b)



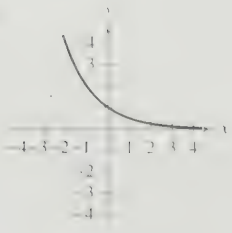
(c)



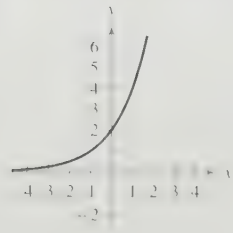
(d)



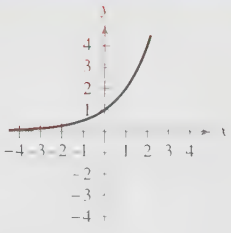
(e)



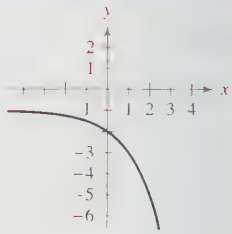
(f)



(g)



(h)



7. $f(x) = 2^{x}$
8. $f(x) = 2^{-x}$
9. $f(x) = -2^{x}$
10. $f(x) = -2^{x} - 1$
11. $f(x) = 2^{x} + 3$
12. $f(x) = 2^{-x} - 1$
13. $f(x) = 2^{x+1}$
14. $f(x) = 2^{x-3}$

Graphing an Exponential Function In Exercises 15–26, sketch the graph of the function. *See Examples 2, 3, and 4.*

15. $g(x) = 4^{x}$
16. $f(x) = \left(\frac{3}{2}\right)^{x}$
17. $f(x) = 4^{-x}$
18. $h(x) = \left(\frac{3}{2}\right)^{-x}$
19. $h(x) = 4^{x-3}$
20. $g(x) = \left(\frac{3}{2}\right)^{x+2}$
21. $g(x) = 4^{x} - 2$
22. $f(x) = \left(\frac{3}{2}\right)^{-x} + 2$
23. $h(x) = -4^{x}$
24. $g(x) = -\left(\frac{3}{2}\right)^{x}$
25. $y = 2^{-x^{2}}$
26. $y = 3^{-x^{2}}$

Evaluating the Natural Exponential Function In Exercises 27–30, use a calculator to evaluate the function $f(x) = e^{x}$ for the given value of x. Round your result to three decimal places. *See Example 5.*

27. $x = 4$
28. $x = -5$
29. $x = -2.7$
30. $x = 0.3$

Graphing a Natural Exponential Function In Exercises 31–38, sketch the graph of the function. *See Example 6.*

31. $y = e^{-0.1x}$ **32.** $y = e^{0.2x}$

33. $f(x) = 2e^{0.12x}$ **34.** $f(x) = 3e^{-0.2x}$

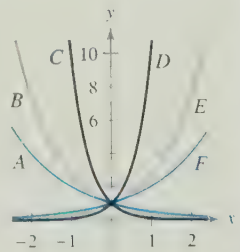
35. $f(x) = e^{x+3}$ **36.** $h(x) = e^{x-2}$

37. $g(x) = 1 + e^{-x}$ **38.** $N(t) = 2 - e^{t}$

39. Think About It Is the natural exponential function also an exponential function with base a? Explain.

40. HOW DO YOU SEE IT?

The figure shows the graphs of $y = 2^x$, $y = e^x$, $y = 10^x$, $y = 2^{-x}$, $y = e^{-x}$, and $y = 10^{-x}$. Match each function with its graph. (The graphs are labeled A through F.) Explain your reasoning.



Compound Interest In Exercises 41–44, complete the table to find the balance A for P dollars invested at rate r for t years, compounded n times per year. *See Example 7.*

n	1	2	4	12	365	Continuous
A						

41. $P = \$5000$, $r = 8\%$, $t = 5$ years

42. $P = \$1000$, $r = 10\%$, $t = 10$ years

43. $P = \$2500$, $r = 5\%$, $t = 20$ years

44. $P = \$1000$, $r = 10\%$, $t = 40$ years

45. Compound Interest A bank offers two types of interest-bearing accounts. The first account pays 5% interest compounded quarterly. The second account pays 3% interest compounded continuously. Which account earns more money? Why?

46. Compound Interest A bank offers two types of interest-bearing accounts. The first account pays 6% interest compounded monthly. The second account pays 5% interest compounded continuously. Which account earns more money? Why?

Present Value The present value of money is the principal P you need to invest today so that it will grow to an amount A at the end of a specified time. The present value formula

$$P = A\left(1 + \frac{r}{n}\right)^{-nt}$$

is obtained by solving the compound interest formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

for P. Recall that t is the number of years, r is the interest rate per year, and n is the number of compoundings per year. In Exercises 47–50, find the present value of amount A invested at rate r for t years, compounded n times per year.

47. $A = \$10{,}000$, $r = 6\%$, $t = 5$ years, $n = 4$

48. $A = \$50{,}000$, $r = 7\%$, $t = 10$ years, $n = 12$

49. $A = \$20{,}000$, $r = 8\%$, $t = 6$ years, $n = 4$

50. $A = \$1{,}000{,}000$, $r = 8\%$, $t = 20$ years, $n = 2$

Compound Interest In Exercises 51–54, complete the table to find the amount P that must be invested at rate r to obtain a balance of $A = \$100{,}000$ in t years.

t	1	10	20	30	40	50
P						

51. $r = 10\%$, compounded monthly

52. $r = 7\%$, compounded daily

53. $r = 9\%$, compounded continuously

54. $r = 5\%$, compounded continuously

Compound Interest In Exercises 55–58, you deposit a lump sum P in a trust fund on the day your child is born. The fund earns 6.5% interest compounded continuously. Find the amount P that will yield the given balance A on your child's 25th birthday.

55. $A = \$100{,}000$ **56.** $A = \$500{,}000$

57. $A = \$750{,}000$ **58.** $A = \$1{,}000{,}000$

Compound Interest In Exercises 59–62, you deposit a lump sum P in a trust fund on the day your grandchild is born. The fund earns 7.5% interest compounded continuously. Find the amount P that will yield the given balance A on your grandchild's 21st birthday.

59. $A = \$100{,}000$ **60.** $A = \$500{,}000$

61. $A = \$750{,}000$ **62.** $A = \$1{,}000{,}000$

63. Demand Function The demand function for a limited edition comic book is given by

$$p = 3000\left(1 - \frac{5}{5 + e^{-0.015x}}\right).$$

(a) Find the price p for a demand of $x = 75$ units.

(b) Find the price p for a demand of $x = 200$ units.

(c) Use a graphing utility to graph the demand function.

(d) Use the graph from part (c) to approximate the demand when the price is \$100.

64. Demand Function The demand function for a home theater system is given by

$$p = 7500\left(1 - \frac{7}{7 + e^{-0.003x}}\right).$$

(a) Find the price p for a demand of $x = 200$ units.

(b) Find the price p for a demand of $x = 900$ units.

(c) Use a graphing utility to graph the demand function.

(d) Use the graph from part (c) to approximate the demand when the price is \$400.

65. Bacteria Growth The number of a certain type of bacteria increases according to the model

$$P(t) = 100e^{0.01896t}$$

where t is time (in hours).

(a) Find $P(0)$. (b) Find $P(5)$.

(c) Find $P(10)$. (d) Find $P(24)$.

66. Bacteria Growth As a result of a medical treatment, the number of a certain type of bacteria decreases according to the model

$$P(t) = 100e^{-0.685t}$$

where t is time (in hours).

(a) Find $P(0)$. (b) Find $P(5)$.

(c) Find $P(10)$. (d) Find $P(24)$.

67. Population Growth The population P of a town increases according to the model

$$P(t) = 4500e^{0.0272t}$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to predict the population in each year.

(a) 2010 (b) 2012

(c) 2015 (d) 2020

68. Population Growth The population P of a small city increases according to the model

$$P(t) = 36{,}000e^{0.0156t}$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to predict the population in each year.

(a) 2009 (b) 2011

(c) 2015 (d) 2018

69. Radioactive Decay Strontium-90 has a half-life of 29.1 years. The amount S of 100 kilograms of strontium-90 present after t years is given by

$$S = 100e^{-0.0238t}.$$

How much of the 100 kilograms will remain after 50 years?

70. Radioactive Decay Neptunium-237 has a half-life of 2.1 million years. The amount N of 200 kilograms of neptunium-237 present after t years is given by

$$N = 200e^{-0.000000033007t}.$$

How much of the 200 kilograms will remain after 20,000 years?

71. Radioactive Decay Five pounds of the element polonium (^{210}Po) is released in a nuclear accident. The amount of polonium P that is present after t months is given by $P = 5e^{-0.1507t}$.

(a) Use a graphing utility to graph this function over the interval from $t = 0$ to $t = 10$.

(b) How much of the 5 pounds of polonium will remain after 10 months?

(c) Use the graph to estimate the half-life of ^{210}Po. Explain your reasoning.

72. Radioactive Decay One hundred grams of radium (^{226}Ra) is stored in a container. The amount of radium R present after t years is given by $R = 100e^{-0.0004335t}$.

(a) Use a graphing utility to graph this function over the interval from $t = 0$ to $t = 10{,}000$.

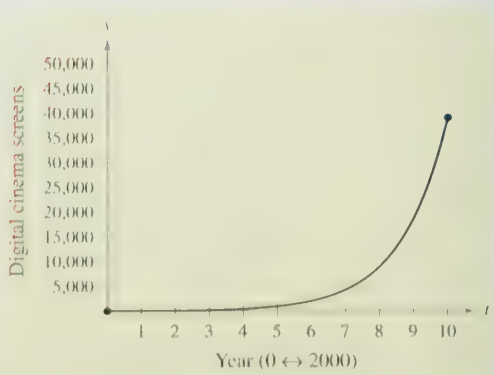
(b) How much of the 100 grams of radium will remain after 10,000 years?

(c) Use the graph to estimate the half-life of ^{226}Ra. Explain your reasoning.

73. Digital Cinema Screens The numbers y of digital cinema screens in the world from 2000 through 2010 can be modeled by

$$y = 25.87e^{0.7323t}, \quad 0 \le t \le 10$$

where t represents the year, with $t = 0$ corresponding to 2000 (see figure). *(Source: IHS Screen Digest)*



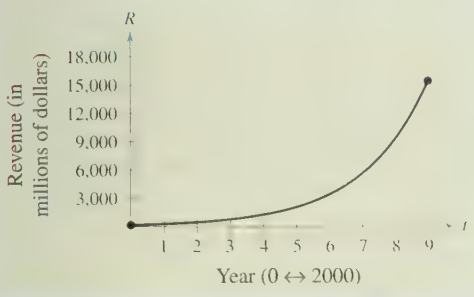
(a) Use the graph to estimate *graphically* the numbers of digital cinema screens in 2008 and 2010.

(b) Use the model to confirm *algebraically* the estimates obtained in part (a).

74. Revenue The revenue R (in millions of dollars) for Research in Motion each year from 2000 through 2009 can be modeled by

$$R = 162.53e^{0.5061t}, \quad 0 \le t \le 9$$

where t represents the year, with $t = 0$ corresponding to 2000 (see figure). *(Source: Research in Motion Limited)*



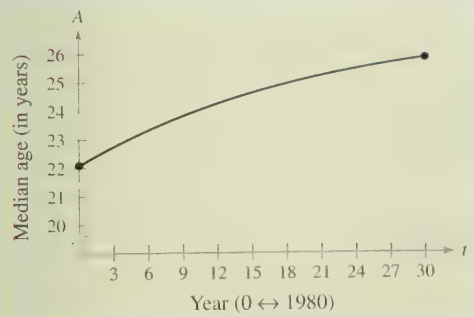
(a) Use the graph to estimate *graphically* the revenues for Research in Motion in 2003, 2007, and 2009.

(b) Use the model to confirm *algebraically* the estimates obtained in part (a).

75. Age at First Marriage For each year from 1980 through 2010, the median age A of an American woman at her first marriage can be modeled by

$$A = -45.10 + \frac{72.02}{1 + e^{-0.0523t - 2.6314}}, \quad 0 \le t \le 30$$

where t represents the year, with $t = 0$ corresponding to 1980 (see figure). *(Source: U.S. Census Bureau)*



(a) Use the graph to estimate *graphically* the median age of an American woman at her first marriage in each of the years 1980, 1990, 2000, and 2010.

(b) Use the model to confirm *algebraically* the estimates obtained in part (a).

76. Prescriptions The numbers of mail-order prescriptions P (in millions) filled in the United States from 2002 through 2009 can be modeled by

$$P = 151 + \frac{89.24}{1 + e^{-0.0895t + 2.8739}}, \quad 2 \le t \le 9$$

where t represents the year, with $t = 2$ corresponding to 2002. *(Source: National Association of Chain Drug Stores)*

(a) Use a graphing utility to graph P for the years 2002 through 2009.

(b) Use the graph from part (a) to estimate the numbers of mail-order prescriptions filled in 2002, 2006, and 2009.

77. Gross Domestic Product The U.S. annual gross domestic product G (in billions of dollars) from 1985 through 2009 can be modeled by

$$G = 3355.4(1.0541)^t, \quad 5 \le t \le 29$$

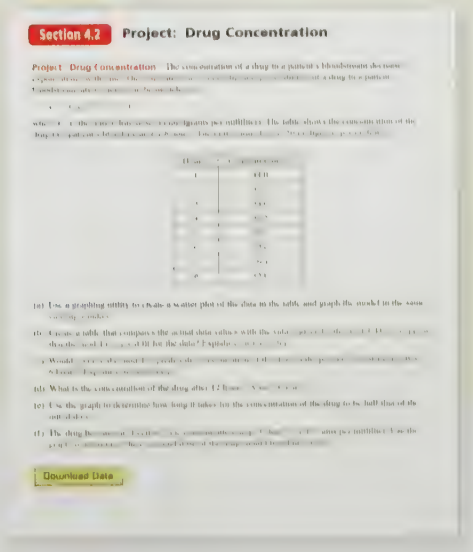
where t represents the year, with $t = 5$ corresponding to 1985. *(Source: U.S. Bureau of Economic Analysis)*

(a) Use a graphing utility to graph G for the years 1985 through 2009.

(b) Use the graph from part (a) to estimate the gross domestic product in 1990, 2000, and 2009.

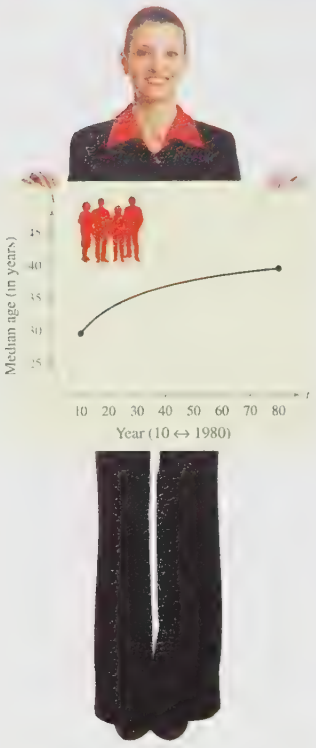
78. Writing Determine whether $e = \frac{271,801}{99,990}$. Justify your answer.

79. Project: Drug Concentration For a project involving the concentration of a drug in a patient's bloodstream, visit this text's website at *www.cengagebrain.com.*



4.3 Logarithmic Functions

- Recognize and evaluate a logarithmic function with base a.
- Sketch the graph of a logarithmic function.
- Recognize and evaluate the natural logarithmic function.
- Use a logarithmic model to solve an application problem.



In Exercise 100 on page 367, you will use a natural logarithmic model to approximate the median age of the United States population.

Logarithmic Functions

In Section 4.1, you learned that if a function has the property that no horizontal line intersects the graph of the function more than once, then the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Section 4.2, you will see that every function of the form

$$f(x) = a^x \text{ (where } a > 0 \text{ and } a \neq 1)$$

passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base a.**

Definition of Logarithmic Function

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function given by

$$f(x) = \log_a x$$

is called the **logarithmic function with base a.**

The equations $y = \log_a x$ and $x = a^y$ are equivalent. The first equation is in logarithmic form and the second is in exponential form.

When evaluating logarithms, remember that *a logarithm is an exponent.* This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance,

$$\log_2 8 = 3$$

because 2 must be raised to the third power to obtain 8.

STUDY TIP

By the definition of a logarithmic function,

$$3^4 = 81$$

(a = 3, y = 4, x = 81)

can be written as

$$\log_3 81 = 4.$$

(a = 3, x = 81, y = 4)

Example 1 Evaluating Logarithmic Expressions

a. $\log_2 32 = 5$ because $2^5 = 32$.

b. $\log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.

c. $\log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

d. $\log_3 1 = 0$ because $3^0 = 1$.

Checkpoint 1

Evaluate each expression.

a. $\log_9 81$ **b.** $\log_7 \frac{1}{49}$

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The logarithmic function with base 10 is called the **common logarithmic function.** On most calculators, this function is denoted by LOG.

STUDY TIP

Because $y = \log_a x$ is the inverse function of $y = a^x$, it follows that the domain of $y = \log_a x$ is the range of $y = a^x$, $(0, \infty)$. In other words, $y = \log_a x$ is defined only if x is positive.

Example 2 Evaluating Logarithmic Expressions on a Calculator

Scientific Calculator

	Number	*Keystrokes*	*Display*
a.	$\log_{10} 10$	10 LOG	1
b.	$2 \log_{10} 2.5$	2.5 LOG × 2 =	0.795880017
c.	$\log_{10}(-2)$	2 +/− LOG	ERROR

Graphing Calculator

	Number	*Keystrokes*	*Display*
a.	$\log_{10} 10$	LOG 10) ENTER	1
b.	$2 \log_{10} 2.5$	2 LOG 2.5) ENTER	.7958800173
c.	$\log_{10}(-2)$	LOG (−) 2) ENTER	ERROR

Many calculators display an error message (or a complex number) when you try to evaluate $\log_{10}(-2)$. This is because the domain of every logarithmic function is the set of *positive real numbers*. In other words, there is no real number power to which 10 can be raised to obtain -2.

Checkpoint 2

Use a calculator to evaluate the expression $\log_{10} 200$. Round your result to three decimal places.

The following properties follow directly from the definition of the logarithmic function with base a.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ Inverse Properties
4. If $\log_a x = \log_a y$, then $x = y$. One-to-One Property

Example 3 Using Properties of Logarithms

a. Solve the equation $\log_2 x = \log_2 3$ for x.

b. Solve the equation $\log_5 x = 1$ for x.

SOLUTION

a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.

b. Using Property 2, you can conclude that $x = 5$.

Checkpoint 3

Solve the equation $\log_4 1 = x$ for x.

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Example 4 Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

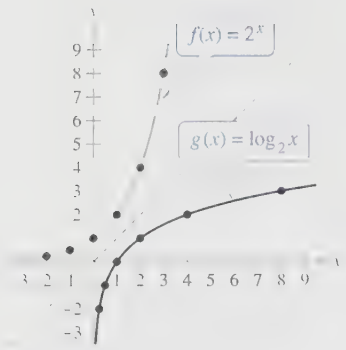
SOLUTION

a. For $f(x) = 2^x$, construct a table of values.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

By plotting these points and connecting them with a smooth curve, you obtain the graph of f shown in Figure 4.14.

b. Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 4.14.



Inverse Functions

FIGURE 4.14

Checkpoint 4

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 4^x$

b. $g(x) = \log_4 x$

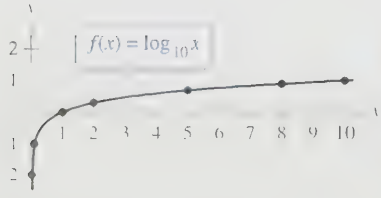
Before you can confirm the result of Example 4 with a graphing utility, you need to know how to enter $\log_2 x$. You will learn how to do this using the *change-of-base formula* discussed in Section 4.4.

Example 5 Sketching the Graph of a Logarithmic Function

Sketch the graph of the common logarithmic function given by $f(x) = \log_{10} x$.

SOLUTION Begin by constructing a table of values. Note that some of the values can be obtained without a calculator by using the properties of logarithms and the definition of a logarithmic function. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 4.15.

	Without Calculator				With Calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log_{10} x$	-2	-1	0	1	0.301	0.699	0.903



Common Logarithmic Function

FIGURE 4.15

Checkpoint 5

Sketch the graph of the function given by $f(x) = 2 \log_{10} x$.

The nature of the graph in Figure 4.15 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$. The basic characteristics of logarithmic graphs are summarized below and shown in Figure 4.16. Note that the vertical asymptote occurs at $x = 0$, where $\log_a x$ is *undefined*.

Characteristics of Logarithmic Functions

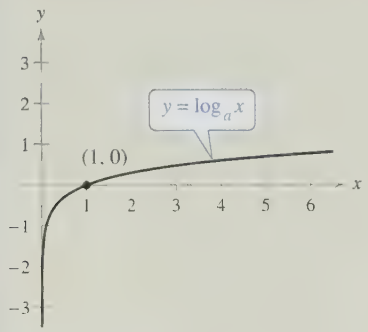


FIGURE 4.16

Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Intercept: $(1, 0)$
- Increasing
- Passes the Horizontal Line Test; therefore has an inverse function
- y-axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$)
- Continuous
- Reflection of graph of $y = a^x$ about the line $y = x$

Remember that the domain of $y = \log_a x$ is the range of $y = a^x$ and the range of $y = \log_a x$ is the domain of $y = a^x$.

Example 6 Sketching the Graphs of Logarithmic Functions

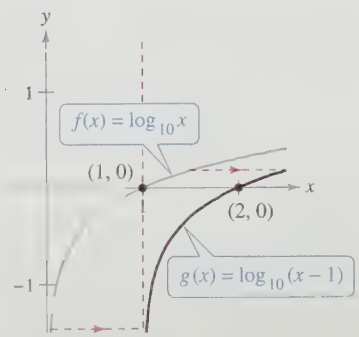
The graph of each function below is similar to the graph of $f(x) = \log_{10} x$.

a. Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *right*. The domain of g is $(1, \infty)$ because $\log_{10}(x - 1)$ is defined only when $x - 1 > 0$. See Figure 4.17(a).

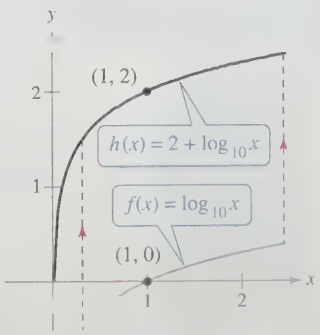
b. Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units *upward*. The domain of h is $(0, \infty)$ because $2 + \log_{10} x$ is defined only when $x > 0$. See Figure 4.17(b).

STUDY TIP

Notice in Example 6(a) that shifting the graph of $f(x)$ one unit to the right shifts the vertical asymptote from the y-axis ($x = 0$) to the line $x = 1$. It also shifts the x-intercept from $(1, 0)$ to $(2, 0)$.



(a) Right shift of one unit
Domain of g: $(1, \infty)$



(b) Upward shift of two units
Domain of h: $(0, \infty)$

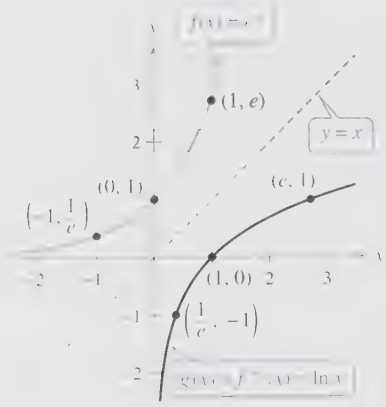
FIGURE 4.17

Checkpoint 6

Sketch the graph of $f(x) = \log_{10}(x + 3)$.

The Natural Logarithmic Function

By looking back at the graph of the natural exponential function introduced in Section 4.2, you will see that the graph of $f(x) = e^x$ passes the Horizontal Line Test. So, it has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as "el en of x."



Inverse Functions
FIGURE 4.18

The Natural Logarithmic Function

The function defined by

$$f(x) = \log_e x = \ln x, \quad x > 0$$

is called the **natural logarithmic function.**

Because the functions given by

$f(x) = e^x$ and $g(x) = \ln x$ Inverse functions

are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 4.18. The four properties of logarithms listed on page 359 are also valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$ Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. One-to-One Property

Example 7 Using Properties of Natural Logarithms

Evaluate each logarithmic expression.

a. $\ln \frac{1}{e}$ **b.** $e^{\ln 5}$ **c.** $\frac{\ln 1}{3}$ **d.** $2 \ln e$

SOLUTION

a. $\ln \frac{1}{e} = \ln e^{-1} = -1$ Inverse Property

b. $e^{\ln 5} = 5$ Inverse Property

c. $\frac{\ln 1}{3} = \frac{0}{3} = 0$ Property 1

d. $2 \ln e = 2(1) = 2$ Property 2

Checkpoint 7

Evaluate each logarithmic expression.

a. $\ln e^7$

b. $5 \ln 1$

c. $\frac{3}{4} \ln e$



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On most calculators, the natural logarithm is denoted by LN, as illustrated in Example 8.

Example 8 **Evaluating the Natural Logarithmic Function**

Use a calculator to evaluate each expression.

a. $\ln 2$ **b.** $\ln 0.3$ **c.** $\ln e^2$ **d.** $\ln(-1)$

SOLUTION

Scientific Calculator

	Number	*Keystrokes*	*Display*
a.	$\ln 2$	2 LN	0.69314718
b.	$\ln 0.3$	.3 LN	−1.203972804
c.	$\ln e^2$	2 2nd [e^x] LN	2
d.	$\ln(-1)$	1 +/− LN	ERROR

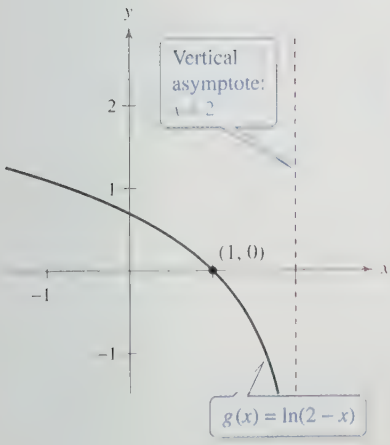
Graphing Calculator

	Number	*Keystrokes*	*Display*
a.	$\ln 2$	LN 2) ENTER	.6931471806
b.	$\ln 0.3$	LN .3) ENTER	−1.203972804
c.	$\ln e^2$	LN 2nd [e^x] 2)) ENTER	2
d.	$\ln(-1)$	LN (−) 1) ENTER	ERROR

Checkpoint 8

Use a calculator to evaluate the expression $\ln 0.1$. Round your result to three decimal places.

In Example 8, note that $\ln(-1)$ gives an error message on most calculators. This occurs because the domain of $\ln x$ is the set of *positive real numbers* (see Figure 4.18). So, $\ln(-1)$ is undefined.



Domain of g: $(-\infty, 2)$

FIGURE 4.19

Example 9 **Finding the Domains of Natural Logarithmic Functions**

Find the domain of each function.

a. $f(x) = \ln(x - 2)$ **b.** $g(x) = \ln(2 - x)$ **c.** $h(x) = \ln x^2$

SOLUTION

a. Because $\ln(x - 2)$ is defined only when $x - 2 > 0$, it follows that the domain of f is $(2, \infty)$.

b. Because $\ln(2 - x)$ is defined only when $2 - x > 0$, it follows that the domain of g is $(-\infty, 2)$. The graph of g is shown in Figure 4.19.

c. Because $\ln x^2$ is defined only when $x^2 > 0$, it follows that the domain of h is all real numbers except $x = 0$.

Checkpoint 9

Find the domain of the function given by $f(x) = \ln(x + 5)$.

Application

Example 10 Human Memory Model

A group of students participating in a psychological experiment attended several lectures on a subject. Every month for a year after that, the students were tested to see how much of the material they remembered. The average scores $f(t)$ for the group are given by the *human memory model*

$$f(t) = 75 - 6\ln(t + 1), \quad 0 \le t \le 12$$

where t is the time (in months). Based on the results of the experiment, how many months can a student wait before retaking the exam and still expect to score 60 or better? (Do not count portions of months.)

SOLUTION To determine how many months a student can wait before retaking the exam and still expect to score 60 or better, use the model to create a table of values showing the scores for several months.

Month, t	0	1	2	3	4	5	6
Score, $f(t)$	75	70.84	68.41	66.68	65.34	64.25	63.32

Month, t	7	8	9	10	11	12
Score, $f(t)$	62.52	61.82	61.18	60.61	60.09	59.61

From the table, you can see that a student would need to retake the exam by the 11th month in order to score 60 or better. The graph of f is shown in Figure 4.20.

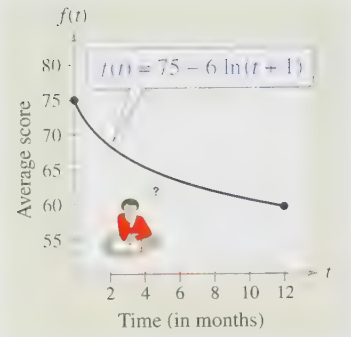


FIGURE 4.20

Checkpoint 10

Biologists have found that an alligator's length l (in inches) can be approximated by the model

$$l = 27.1 \ln w - 32.8$$

where w is the weight (in pounds) of the alligator. Find the lengths of alligators for which $w = 150$, 225, 380, 450, and 625 pounds. Round your results to the nearest tenth of an inch.



SUMMARIZE (Section 4.3)

1. State the definition of a logarithmic function with base a *(page 358)*. For examples of evaluating logarithmic expressions, see Examples 1 and 2.
2. Make a list of the properties of logarithms *(page 359)*. For an example that uses properties of logarithms, see Example 3.
3. Describe the relationship between the graphs of the functions $f(x) = a^x$ and $g(x) = \log_a x$, where $a > 1$ *(pages 360 and 361)*. For examples of graphing exponential and logarithmic functions, see Examples 4, 5, and 6.
4. State the definition of the natural logarithmic function and make a list of the properties of natural logarithms *(page 362)*. For examples of evaluating natural logarithms and using the properties of natural logarithms, see Examples 7 and 8.

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SKILLS WARM UP 4.3 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.6 and 4.2.

In Exercises 1–4, determine the value of x.

1. $2^x = 8$ **2.** $4^x = 1$ **3.** $10^x = 0.1$ **4.** $e^x = e$

In Exercises 5 and 6, evaluate the expression. (Round the result to three decimal places.)

5. e^2 **6.** e^{-1}

In Exercises 7–10, describe how the graph of g is related to the graph of f.

7. $g(x) = f(x + 2)$ **8.** $g(x) = -f(x)$ **9.** $g(x) = -1 + f(x)$ **10.** $g(x) = f(-x)$

Exercises 4.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Logarithmic Equations in Exponential Form In Exercises 1–6, match the logarithmic equation with its exponential form. [The exponential forms are labeled (a), (b), (c), (d), (e), and (f).]

1. $\log_4 16 = 2$ (a) $4^{1/2} = 2$
2. $\log_2 16 = 4$ (b) $2^{-4} = \frac{1}{16}$
3. $\log_2 \frac{1}{16} = -4$ (c) $4^2 = 16$
4. $\log_4 \frac{1}{16} = -2$ (d) $4^{-2} = \frac{1}{16}$
5. $\log_4 2 = \frac{1}{2}$ (e) $16^{1/2} = 4$
6. $\log_{16} 4 = \frac{1}{2}$ (f) $2^4 = 16$

Writing Logarithmic Equations In Exercises 7–12, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.

7. $4^4 = 256$ **8.** $7^3 = 343$
9. $81^{1/4} = 3$ **10.** $9^{3/2} = 27$
11. $6^{-2} = \frac{1}{36}$ **12.** $10^{-3} = 0.001$

Writing Exponential Equations In Exercises 13–20, write the logarithmic equation in exponential form. For example, the exponential form of $\log_5 125 = 3$ is $5^3 = 125$.

13. $\log_4 16 = 2$ **14.** $\log_{10} 1000 = 3$
15. $\log_2 \frac{1}{2} = -1$ **16.** $\log_3 \frac{1}{9} = -2$
17. $\log_5 0.2 = -1$ **18.** $\log_{10} 0.1 = -1$
19. $\log_{27} 3 = \frac{1}{3}$ **20.** $\log_8 2 = \frac{1}{3}$

Evaluating Logarithmic Expressions In Exercises 21–30, evaluate the expression without using a calculator. *See Example 1.*

21. $\log_3 9$ **22.** $\log_5 125$
23. $\log_2 \frac{1}{16}$ **24.** $\log_6 \frac{1}{36}$
25. $\log_8 2$ **26.** $\log_{64} 4$
27. $\log_7 7$ **28.** $\log_{12} 1$
29. $\log_{10} 0.0001$ **30.** $\log_{10} 100$

Evaluating Logarithmic Expressions on a Calculator In Exercises 31–36, use a calculator to evaluate the logarithm. Round your result to three decimal places. *See Example 2.*

31. $\log_{10} 345$ **32.** $\log_{10} 163$
33. $\log_{10} \frac{4}{5}$ **34.** $\log_{10} \frac{3}{4}$
35. $\log_{10} \sqrt{8}$ **36.** $\log_{10} \sqrt{3}$

Using Properties of Logarithms In Exercises 37–44, solve the equation for x. *See Example 3.*

37. $\log_5 5 = x$ **38.** $\log_6 6^2 = x$
39. $\log_2 2^{-1} = x$ **40.** $\log_4 4^3 = x$
41. $\log_2(x + 1) = \log_2 4$ **42.** $\log_2(x - 3) = \log_2 9$
43. $\log(2x + 1) = \log 15$ **44.** $\log(5x + 3) = \log 12$

Graphs of Exponential and Logarithmic Functions In Exercises 45–48, sketch the graphs of f and g in the same coordinate plane. *See Example 4.*

45. $f(x) = 7^x$, $g(x) = \log_7 x$
46. $f(x) = 5^x$, $g(x) = \log_5 x$
47. $f(x) = 6^x$, $g(x) = \log_6 x$
48. $f(x) = 10^x$, $g(x) = \log_{10} x$

Sketching the Graphs of Logarithmic Functions In Exercises 49–54, find the domain, vertical asymptote, and x-intercept of the logarithmic function. Then sketch its graph. *See Examples 5 and 6.*

49. $f(x) = \log_2 x$ **50.** $g(x) = \log_4 x$
51. $h(x) = \log_2(x + 4)$ **52.** $f(x) = \log_4(x - 3)$
53. $f(x) = -\log_2 x$ **54.** $h(x) = -\log_4(x - 1)$

Writing Logarithmic Equations In Exercises 55–58, write the exponential equation in logarithmic form.

55. $e^1 = e$
56. $e^4 = 54.5981\ldots$
57. $e^x = 4$
58. $e^{-x} = 2$

Writing Exponential Equations In Exercises 59–62, write the logarithmic equation in exponential form.

59. $\ln e = 1$
60. $\ln \frac{1}{e} = -1$
61. $\ln 1 = 0$
62. $\ln \frac{1}{2} = -0.693\ldots$

Using Properties of Natural Logarithms In Exercises 63–66, evaluate the expression without using a calculator. *See Example 7.*

63. $\ln e$
64. $\ln e^{10}$
65. $\ln e^{-4}$
66. $\ln \frac{1}{e^3}$

Evaluating the Natural Logarithmic Function In Exercises 67–72, use a calculator to evaluate the logarithm. Round your result to three decimal places. *See Example 8.*

67. $\ln 7$
68. $2 \ln 9$
69. $\ln 18.42$
70. $\ln 36.7$
71. $\ln \sqrt{6}$
72. $\ln \sqrt{10}$

Using Properties of Logarithms In Exercises 73–76, use the One-to-One Property to solve the equation for x.

73. $\ln(x + 2) = \ln 6$
74. $\ln(x - 4) = \ln 2$
75. $\ln(x^2 - 2) = \ln 23$
76. $\ln(x^2 - x) = \ln 6$

Graphs of Exponential and Logarithmic Functions In Exercises 77 and 78, sketch the graphs of f and g in the same coordinate plane.

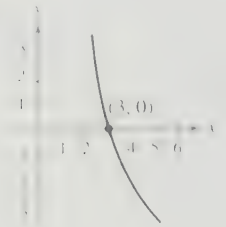
77. $f(x) = e^x,\ g(x) = \ln x$
78. $f(x) = e^{2x},\ g(x) = \frac{1}{2} \ln x$

Finding the Domains of Natural Logarithmic Functions In Exercises 79–82, find the domain, vertical asymptote, and x-intercept of the logarithmic function. Then sketch its graph. *See Example 9.*

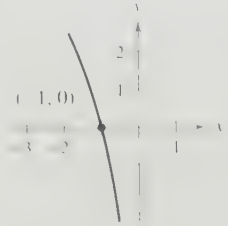
79. $g(x) = \ln(-x)$
80. $f(x) = \ln(3 - x)$
81. $h(x) = \ln(x + 1)$
82. $f(x) = 3 + \ln x$

Matching In Exercises 83–86, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

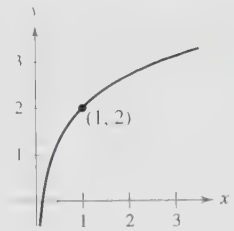
(a)



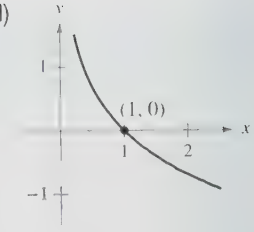
(b)



(c)



(d)



83. $f(x) = \ln x + 2$
84. $f(x) = -\ln x$
85. $f(x) = -3 \ln(x - 2)$
86. $f(x) = 4 \ln(-x)$

Graphing Logarithmic Functions In Exercises 87–92, use a graphing utility to graph the function. Be sure to use an appropriate viewing window.

87. $f(x) = \log(x + 1)$
88. $f(x) = \log(x - 1)$
89. $f(x) = \ln(x - 1)$
90. $f(x) = \ln(x + 2)$
91. $f(x) = \ln x + 1$
92. $f(x) = 3 \ln x - 1$

93. **Population Growth** The population of a town will double in

$$t = \frac{8 \ln 3}{\ln 63 - \ln 45} \text{ years.}$$

Find t.

94. **Work** The work W (in foot-pounds) done in compressing a volume of 9 cubic feet at a pressure of 15 pounds per square inch to a volume of 3 cubic feet is $W = 19{,}440(\ln 9 - \ln 3)$. Find W.

Skill Retention Model In Exercises 95 and 96, participants in an industrial psychology study were taught a simple mechanical task and tested monthly on this mechanical task for a period of 1 year. The average scores for the participants are given by the model $f(t) = 98 - 14 \log_{10}(t + 1)$, for $0 \le t \le 12$, where t is the time (in months).

95. Use a graphing utility to graph the function. Use the graph to discuss the domain and range of the function.
96. **Think About It** Based on the graph of f, do you think the study's participants practiced the simple mechanical task very often? Cite the behavior of the graph to justify your answer.

97. **Human Memory Model** Students in a seventh-grade class were given an exam. During the next 2 years, the same students were retested several times. The average score g can be approximated by the model

$$g(t) = 87 - 16 \log_{10}(t + 1), \quad 0 \le t \le 24$$

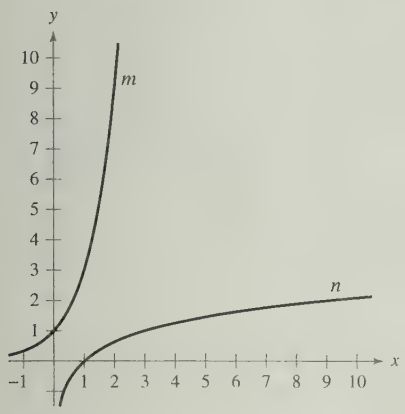
where t is the time (in months).

(a) What was the average score on the original exam?

(b) What was the average score after 6 months?

(c) When did the average score drop below 70?

98. HOW DO YOU SEE IT? The figure shows the graphs of $f(x) = 3^x$ and $g(x) = \log_3 x$. [The graphs are labeled m and n.]



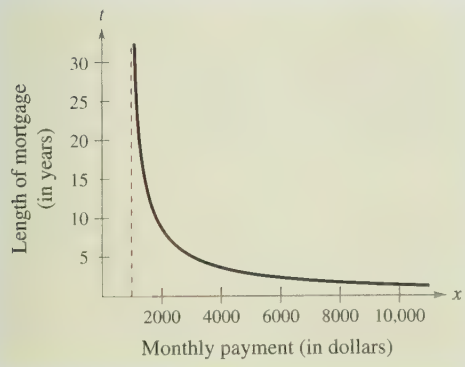
(a) Match each function with its graph.

(b) Given that $f(a) = b$, what is $g(b)$? Explain.

99. Monthly Payment The length t (in years) of a home mortgage of \$150,000 at 8% interest can be approximated by the model

$$t = 12.542 \ln\left(\frac{x}{x - 1000}\right), \quad x > 1000$$

where x represents the monthly payment (in dollars) (see figure).



(a) Use the model to approximate the length of a \$150,000 mortgage at 8% interest when the monthly payment is \$1100.65 and when the monthly payment is \$1254.68.

(b) Approximate the total amount paid over the term of the mortgage with a monthly payment of \$1100.65 and with a monthly payment of \$1254.68.

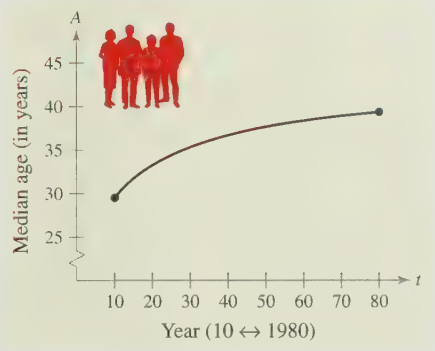
(c) Approximate the total interest charge for a monthly payment of \$1100.65 and for a monthly payment of \$1254.68.

(d) What is the vertical asymptote of the model? Interpret its meaning in the context of the problem.

100. Median Age of U.S. Population The median age A of the U.S. population for each year from 1980 to 2050 can be approximated by the model

$$A = 16.15 - 0.036t + 5.956 \ln t, \quad 10 \le t \le 80$$

where t represents the year, with $t = 10$ corresponding to 1980 (see figure). *(Source: U.S. Census Bureau)*



(a) Use the model to estimate the median age of the U.S. population in 1980.

(b) Use the model to estimate the median age of the U.S. population in 1990.

(c) Use the model to estimate the change in the median age of the U.S. population from 1980 to 2050.

(d) Use the model to project the change in the median age of the U.S. population from 1980 to 2050.

101. Investment Time A principal P, invested at 5.25% interest compounded continuously, increases to an amount that is K times the principal after t years, where t is given by

$$t = \frac{\ln K}{0.0525}.$$

(a) Complete the table.

K	1	2	4	6	8	10	12
t							

(b) Use the table in part (a) to graph the function.

102. Think About It The table of values was obtained by evaluating a function. Determine which of the statements may be true and which must be false.

x	1	2	8
y	0	1	3

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

QUIZ YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, determine whether the function has an inverse function. If it does, find its inverse function and graph f and f^{-1} in the same coordinate plane.

1. $f(x) = 3x^2$

2. $f(x) = \sqrt[3]{x + 1}$

In Exercises 3 and 4, (a) find f^{-1}, (b) sketch the graphs of f and f^{-1} in the same coordinate plane, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

3. $f(x) = \frac{1}{2}x - 3$

4. $f(x) = \sqrt{x + 1}$

In Exercises 5–8, use the graph of $f(x) = 3^x$ to sketch the graph of the function.

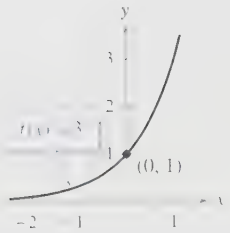


Figure for 5–8

5. $g(x) = 3^x - 2$

6. $h(x) = 3^{-x}$

7. $k(x) = \log_3 x$

8. $j(x) = \log_3(x - 1)$

9. For $P = \$10{,}000$, $r = 5.15\%$, and $t = 8$ years, find the balance in an account when interest is compounded (a) monthly and (b) continuously.

10. The numbers of children C (in millions) participating in the Federal School Breakfast Program from 2000 through 2009 can be approximated by the model

$$C = 7.50(1.04)^t, \quad 0 \le t \le 9$$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: U.S. Department of Agriculture)*

(a) Use the model to estimate the numbers of children participating in the Federal School Breakfast Program in 2006 and 2008.

(b) Use the model to predict the numbers of children who will participate in the Federal School Breakfast Program in 2014 and 2015.

11. The size of a bacteria population is modeled by

$$P(t) = 100e^{0.2154t}$$

where t is the time in hours. Find (a) $P(0)$, (b) $P(6)$, and (c) $P(12)$.

12. Use the demand function

$$p = 4000\left(1 - \frac{8}{8 + e^{-0.003x}}\right)$$

to find the price for a demand of $x = 500$ MP3 players.

In Exercises 13–16, evaluate the expression without using a calculator.

13. $\log_{10} 100$

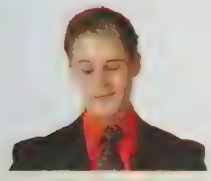
14. $\ln e^4$

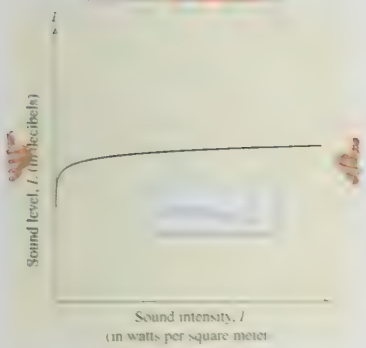
15. $\log_4 \frac{1}{16}$

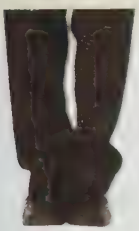
16. $\ln 1$

17. Sketch the graphs of $f(x) = 3^x$ and $g(x) = \log_3 x$ in the same coordinate plane. Identify the domains of f and g. Discuss the special relationship between f and g that is shown by their graphs.

4.4 Properties of Logarithms







In Exercise 107 on page 376, you will use properties of logarithms to determine the difference in loudness between two sounds.

- Evaluate a logarithm using the change-of-base formula.
- Use properties of logarithms to evaluate or rewrite a logarithmic expression.
- Use properties of logarithms to expand or condense a logarithmic expression.
- Use logarithmic functions to model and solve real-life applications.

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). In some situations, you must use logarithms with other bases. To evaluate logarithms with other bases, you can use the **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as shown.

Base b	*Base 10*	*Base e*
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms to base a are simply *constant multiples* of logarithms to base b. The constant multiplier is

$$\frac{1}{\log_b a}.$$

Example 1 Changing Bases Using Common Logarithms

a. $\log_4 30 = \dfrac{\log_{10} 30}{\log_{10} 4} \approx \dfrac{1.47712}{0.60206} \approx 2.4534$

b. $\log_2 14 = \dfrac{\log_{10} 14}{\log_{10} 2} \approx \dfrac{1.14613}{0.30103} \approx 3.8074$

Example 2 Changing Bases Using Natural Logarithms

a. $\log_4 30 = \dfrac{\ln 30}{\ln 4} \approx \dfrac{3.40120}{1.386294} \approx 2.4534$

b. $\log_2 14 = \dfrac{\ln 14}{\ln 2} \approx \dfrac{2.63906}{0.693147} \approx 3.8074$

✓ Checkpoints 1 and 2

Evaluate $\log_8 56$ using common logarithms and natural logarithms. Round your results to three decimal places.

Notice in Examples 1 and 2 that the result is the same whether common logarithms or natural logarithms are used in the change-of-base formula.

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Properties of Logarithms

You know from the preceding section that the logarithmic function with base a is the *inverse function* of the exponential function with base a. So, it makes sense that the properties of exponents should have corresponding properties involving logarithms. For instance, the exponential property $a^0 = 1$ has the corresponding logarithmic property $\log_a 1 = 0$.

STUDY TIP

There is no general property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(x + y)$ is *not* equal to $\log_a x + \log_a y$.

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

Logarithm with Base a	*Natural Logarithm*	
$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$	Product Property
$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$	Quotient Property
$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$	Power Property

Example 3 Using Properties of Logarithms

Write each logarithm in terms of ln 2 and ln 3.

a. $\ln 6$ **b.** $\ln \frac{2}{27}$

SOLUTION

a. $\ln 6 = \ln(2 \cdot 3)$ Rewrite 6 as $2 \cdot 3$.

$= \ln 2 + \ln 3$ Product Property

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$ Quotient Property

$= \ln 2 - \ln 3^3$ Rewrite 27 as 3^3.

$= \ln 2 - 3 \ln 3$ Power Property

Checkpoint 3

Write $\log_{10} \frac{25}{3}$ in terms of $\log_{10} 3$ and $\log_{10} 5$.

Example 4 Using Properties of Logarithms

Use the properties of logarithms to verify that $-\log_{10} \frac{1}{100} = \log_{10} 100$.

SOLUTION

$-\log_{10} \frac{1}{100} = -\log_{10}(100^{-1})$ Definition of negative exponent

$= -(-1) \log_{10} 100$ Power Property

$= \log_{10} 100$ Simplify.

Checkpoint 4

Use the properties of logarithms to verify that $-\ln \frac{2}{e} = 1 - \ln 2$.

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because these properties convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

Example 5 Expanding Logarithmic Expressions

Expand each logarithmic expression.

a. $\log_4 5x^3y$ **b.** $\ln \dfrac{\sqrt{3x-5}}{7}$

SOLUTION

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ Product Property

$= \log_4 5 + 3\log_4 x + \log_4 y$ Power Property

b. $\ln \dfrac{\sqrt{3x-5}}{7} = \ln \dfrac{(3x-5)^{1/2}}{7}$ Rewrite using rational exponent.

$= \ln(3x-5)^{1/2} - \ln 7$ Quotient Property

$= \dfrac{1}{2}\ln(3x-5) - \ln 7$ Power Property

STUDY TIP

Be careful when rewriting the logarithmic expression $\log ax^n$. It should be rewritten as

$$\log a + n \log x$$

instead of $n \log ax$.

✓Checkpoint 5

Expand the expression $\ln 2mn^2$.

Example 6 Condensing Logarithmic Expressions

Condense each logarithmic expression.

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1)$ **b.** $2\ln(x+2) - \ln x$ **c.** $\frac{1}{3}[\log_2 x + \log_2(x+1)]$

SOLUTION

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1) = \log_{10} x^{1/2} + \log_{10}(x+1)^3$ Power Property

$= \log_{10}\left[\sqrt{x}(x+1)^3\right]$ Product Property

b. $2\ln(x+2) - \ln x = \ln(x+2)^2 - \ln x$ Power Property

$= \ln \dfrac{(x+2)^2}{x}$ Quotient Property

c. $\frac{1}{3}[\log_2 x + \log_2(x+1)] = \frac{1}{3}\{\log_2[x(x+1)]\}$ Product Property

$= \log_2[x(x+1)]^{1/3}$ Power Property

$= \log_2 \sqrt[3]{x(x+1)}$ Rewrite with a radical.

STUDY TIP

Be careful when rewriting the logarithmic expression $\log x - \log y$. It should be rewritten as

$$\log \frac{x}{y}$$

instead of

$$\frac{\log x}{\log y}.$$

✓Checkpoint 6

Condense the expression $2\log_{10}(x+1) - 3\log_{10}(x-1)$.

When applying the properties of logarithms to a logarithmic function, be careful to check the domain of the function. For example, the domain of $f(x) = \ln x^2$ is all real $x \neq 0$, whereas the domain of $g(x) = 2\ln x$ is all real $x > 0$.

Applications

One method of determining how the x- and y-values of a set of nonlinear data are related begins by taking the natural logarithm of each of the x- and y-values. If you graph the points $(\ln x, \ln y)$ and they fall in a straight line, then you can determine that the x- and y-values are related by the equation

$$\ln y = m \ln x$$

where m is the slope of the straight line.

Example 7 Finding a Mathematical Model

The table shows the mean distance from the sun x and the period (the time it takes a planet to orbit the sun) y for each of the six planets that are closest to the sun. In the table, the mean distance is given in astronomical units (where Earth's mean distance is defined as 1.0), and the period is given in years. Find an equation that relates y and x.

Planet	Mean distance, x	Period, y
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.863
Saturn	9.537	29.447

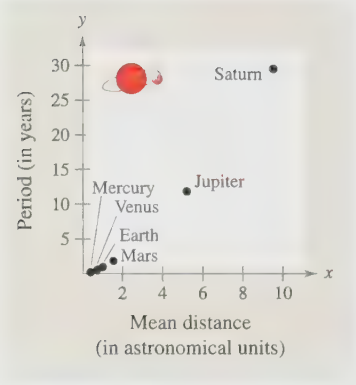


FIGURE 4.21

SOLUTION The points in the table are plotted in Figure 4.21. From this figure, it is not clear how to find an equation that relates y and x. To solve this problem, take the natural logarithm of each of the x- and y-values in the table. This produces the following results.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
$\ln x$	−0.949	−0.324	0.000	0.421	1.649	2.255
$\ln y$	−1.423	−0.486	0.000	0.632	2.473	3.383

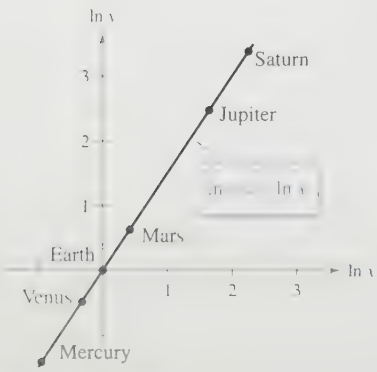


FIGURE 4.22

Now, by plotting the points in the second table, you can see that all six of the points appear to lie in a line (see Figure 4.22). Choose any two points to determine the slope of the line. Using the two points (0.421, 0.632) and (0, 0), you can determine that the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

By the point-slope form, the equation of the line is $Y = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. You can therefore conclude that $\ln y = \frac{3}{2} \ln x$.

Checkpoint 7

Find a logarithmic equation that relates y and x.

x	1	2	3	4
y	1	1.414	1.732	2

Example 8 Sound Intensity

The level of sound L (in decibels) with an intensity of I (in watts per square meter) is given by

$$L = 10 \log_{10} \frac{I}{I_0}$$

where I_0 represents the faintest sound that can be heard by the human ear, and is approximately equal to 10^{-12} watt per square meter. You and your roommate are playing your stereos at the same time and at the same intensity. How much louder is the music when both stereos are playing than when just one stereo is playing?

SOLUTION Let L_1 represent the level of sound when one stereo is playing and let L_2 represent the level of sound when both stereos are playing. Using the formula for level of sound, you can express L_1 as

$$L_1 = 10 \log_{10} \frac{I}{10^{-12}}.$$

For L_2, multiply I by 2 as shown below

$$L_2 = 10 \log_{10} \frac{2I}{10^{-12}}$$

because L_2 represents the level of sound when *two* stereos are playing at the same intensity I. To determine the increase in loudness, subtract L_1 from L_2 as shown.

$$\begin{aligned} L_2 - L_1 &= 10 \log_{10} \frac{2I}{10^{-12}} - 10 \log_{10} \frac{I}{10^{-12}} \\ &= 10\left(\log_{10} \frac{2I}{10^{-12}} - \log_{10} \frac{I}{10^{-12}}\right) \\ &= 10\left(\log_{10} 2 + \log_{10} \frac{I}{10^{-12}} - \log_{10} \frac{I}{10^{-12}}\right) \\ &= 10 \log_{10} 2 \approx 3 \end{aligned}$$

So, the music is about 3 decibels louder. Notice that the variable I drops out of the equation when it is simplified. This means that the loudness increases by 3 decibels when both stereos are playing at the same intensity, regardless of the individual intensities of the stereos.



Checkpoint 8

Two sounds have intensities of $I_1 = 10^{-6}$ watt per square meter and $I_2 = 10^{-9}$ watt per square meter. Use the formula for the level of sound in Example 8 to find the difference in loudness between the two sounds.

SUMMARIZE (Section 4.4)

1. State the change-of-base formula *(page 369)*. For examples that use the change-of-base formula, see Examples 1 and 2.
2. Make a list of the properties of logarithms *(page 370)*. For examples that use properties of logarithms, see Examples 3 and 4.
3. Describe how the properties of logarithms are useful for rewriting logarithmic expressions *(page 371)*. For examples of expanding and condensing logarithmic expressions, see Examples 5 and 6.

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SKILLS WARM UP 4.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, 0.4, and 4.3.

In Exercises 1–4, evaluate the expression without using a calculator.

1. $\log_7 49$ **2.** $\log_2 \frac{1}{32}$ **3.** $\ln \frac{1}{e^2}$ **4.** $\log_{10} 0.001$

In Exercises 5–8, simplify the expression.

5. e^2e^3 **6.** $\frac{e^2}{e^3}$ **7.** $(e^2)^3$ **8.** $(e^2)^0$

In Exercises 9–12, rewrite the equation in exponential form.

9. $y = \frac{1}{x^2}$ **10.** $y = \sqrt{x}$ **11.** $\log_4 64 = 3$ **12.** $\log_{16} 4 = \frac{1}{2}$

Exercises 4.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Changing the Base Using Common Logarithms In Exercises 1–4, evaluate the logarithm using common logarithms. Round your result to three decimal places. *See Example 1.*

1. $\log_5 8$ **2.** $\log_7 12$
3. $\log_4 30$ **4.** $\log_9 20$

Changing the Base Using Natural Logarithms In Exercises 5–8, evaluate the logarithm using natural logarithms. Round your result to three decimal places. *See Example 2.*

5. $\log_5 8$
6. $\log_7 12$
7. $\log_6 14$
8. $\log_8 18$

Changing the Base In Exercises 9–18, evaluate the logarithm. Round your result to three decimal places. *See Examples 1 and 2.*

9. $\log_2 6$ **10.** $\log_8 3$
11. $\log_{27} 35$ **12.** $\log_{19} 42$
13. $\log_{15} 1250$ **14.** $\log_{20} 1575$
15. $\log_5 \frac{1}{3}$ **16.** $\log_9 \frac{3}{5}$
17. $\log_{1/4} 10$
18. $\log_{1/3} 5$

Using Properties of Logarithms In Exercises 19–24, write the logarithm in terms of ln 2 and ln 5. *See Example 3.*

19. $\ln 10$ **20.** $\ln \frac{2}{5}$
21. $\ln \frac{5}{2}$ **22.** $\ln 20$
23. $\ln 100$ **24.** $\ln \frac{125}{16}$

Using Properties of Logarithms In Exercises 25–28, use the properties of logarithms to verify the statement. *See Example 4.*

25. $-\log_{10} \frac{11}{8} = \log_{10} \frac{8}{11}$
26. $-\ln \frac{3}{4} = \ln \frac{4}{3}$
27. $-2 \ln \frac{e}{7} = 2 \ln 7 - 2$
28. $-3 \log_{10} \frac{8}{100} = 6 - 3 \log_{10} 8$

Using Properties of Logarithms In Exercises 29–42, approximate the logarithm using the properties of logarithms, given $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.

29. $\log_b 10$ **30.** $\log_b 15$
31. $\log_b \frac{2}{3}$ **32.** $\log_b \frac{3}{5}$
33. $\log_b 8$ **34.** $\log_b 81$
35. $\log_b \sqrt{2}$ **36.** $\log_b \sqrt{5}$
37. $\log_b 40$ **38.** $\log_b 45$
39. $\log_b (2b)^{-2}$ **40.** $\log_b (3b^2)$
41. $\log_b \sqrt[3]{4b}$ **42.** $\log_b \sqrt[3]{3b}$

Using Properties of Logarithms In Exercises 43–50, use the properties of logarithms to simplify the logarithmic expression.

43. $\log_9 \frac{1}{18}$ **44.** $\log_5 \frac{1}{15}$
45. $\log_7 \sqrt{70}$ **46.** $\log_5 \sqrt{75}$
47. $\log_5 \frac{1}{250}$
48. $\log_{10} \frac{9}{300}$
49. $\ln(5e^6)$
50. $\ln \frac{6}{e^2}$

Error Analysis In Exercises 51–54, describe and correct the error.

51. $\log_2 5 = \frac{\ln 2}{\ln 5}$

52. $\ln 8x = \ln 8 \cdot \ln x$

53. $\ln 8 - \ln 2 = \frac{\ln 8}{\ln 2}$

54. $\log_{10} 5x^4 = 4 \log_{10} 5x$

Expanding a Logarithmic Expression In Exercises 55–76, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms. (Assume all variables are positive.) *See Example 5.*

55. $\log_2(4^3 \cdot 3^5)$

56. $\log_3(3^2 \cdot 4^2)$

57. $\log_3 4n$

58. $\log_6 6x$

59. $\log_5 \frac{x}{25}$

60. $\log_{10} \frac{y}{2}$

61. $\log_2 x^4$

62. $\log_2 z^{-3}$

63. $\ln \sqrt{z}$

64. $\ln \sqrt[3]{t}$

65. $\ln xyz$

66. $\ln \frac{xy}{z}$

67. $\ln \sqrt{a-1}, \quad a > 1$

68. $\ln \sqrt[3]{y-2}, \quad y > 2$

69. $\ln\left[\frac{(z-1)^2}{z}\right]$

70. $\ln\left(\frac{x}{\sqrt{x^2+1}}\right)$

71. $\ln \frac{yz}{\sqrt[3]{z+3}}$

72. $\log_9 \frac{\sqrt{y}}{7z^2}$

73. $\ln 5\sqrt[3]{\frac{x}{y}}$

74. $\ln 3\sqrt{\frac{x^2}{y^3}}$

75. $\ln \sqrt[4]{x^3(x^2+3)}$

76. $\ln \sqrt{x^2(x+2)}$

Condensing a Logarithmic Expression In Exercises 77–92, condense the expression to the logarithm of a single quantity. *See Example 6.*

77. $\log_3 x + \log_3 5$

78. $\log_5 y + \log_5 x$

79. $\log_4 8 - \log_4 x$

80. $\log_{10} 4 - \log_{10} z$

81. $2 \log_{10}(x+4)$

82. $-4 \log_{10} 2x$

83. $-\ln x - 3 \ln 6$

84. $2 \ln 8 + 5 \ln z$

85. $\frac{1}{3} \ln 5x - \ln(x+1)$

86. $\frac{3}{2} \ln(z-2) + \ln z$

87. $\log_8(x-2) - \log_8(x+2)$

88. $3 \log_7 x + 2 \log_7 y - 4 \log_7 z$

89. $2 \ln 4 + 2 \ln 3 - \frac{1}{2} \ln(x^2+1)$

90. $2 \ln(x^2-2) + \frac{3}{2} \ln t^6 - \frac{3}{4} \ln t^4$

91. $2 \ln x - [\ln(x+2) + \ln(x-2)]$

92. $4 \ln(x+1) + 2 \ln(x-1) - 3 \ln x$

93. Graphical Analysis Use a graphing utility to graph

$f(x) = \ln 5x$ and $g(x) = \ln 5 + \ln x$

in the same viewing window. How do the two graphs compare? What property of logarithms do the graphs demonstrate?

94. Think About It Consider the functions below.

$$f(x) = \ln \frac{x}{2}, \quad g(x) = \frac{\ln x}{\ln 2}, \quad h(x) = \ln x - \ln 2$$

Which two functions have identical graphs? Verify your answer by using a graphing utility to graph all three functions in the same viewing window.

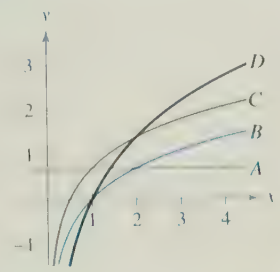
95. Graphical Analysis Use a graphing utility to graph

$f(x) = \log_{10} x^3$ and $g(x) = 3 \log_{10} x$

in the same viewing window. How do the two graphs compare? What property of logarithms is shown?

96. HOW DO YOU SEE IT?

The figure shows the graphs of $y = \ln x$, $y = \ln x^2$, $y = \ln 2x$, and $y = \ln 2$. Match each function with its graph. (The graphs are labeled *A* through *D*.) Explain your reasoning.



Using Inverse Properties In Exercises 97–102, find the exact value of the logarithmic expression without using a calculator.

97. $\log_4 \sqrt[3]{4}$

98. $\log_8 \sqrt[4]{8}$

99. $\ln \frac{1}{\sqrt{e}}$

100. $\ln \sqrt[4]{e^3}$

101. $\log_5 \frac{1}{125}$

102. $\log_7 \frac{49}{343}$

Finding a Mathematical Model In Exercises 103 and 104, find a logarithmic equation that relates y and x. Explain the steps used to find the equation. *See Example 7.*

103.

x	1	2	3	4	5	6
y	1	1.189	1.316	1.414	1.495	1.565

104.

x	1	2	3	4	5	6
y	1	1.587	2.080	2.520	2.924	3.302

105. Nail Length The approximate lengths and diameters (in inches) of common nails are shown in the table. Find a logarithmic equation that relates the diameter y of a common nail to its length x.

Length, x	Diameter, y	Length, x	Diameter, y
1	0.070	4	0.176
2	0.111	5	0.204
3	0.146	6	0.231

106. Galloping Speeds of Animals Four-legged animals run with two different types of motion: trotting and galloping. An animal that is trotting has at least one foot on the ground at all times, whereas an animal that is galloping has all four feet off the ground at some point in its stride. The number of strides per minute at which an animal breaks from a trot to a gallop depends on the weight of the animal. Use the table to find a logarithmic equation that relates an animal's lowest galloping speed y (in strides per minute) to its weight x (in pounds).

Weight, x	Galloping speed, y	Weight, x	Galloping speed, y
25	191.5	75	164.2
35	182.7	500	125.9
50	173.8	1000	114.2

107. Sound Intensity Use the equation for the level of sound in Example 8 to find the difference in loudness between an average office and a broadcast studio with the intensities given below.

Office: 1.26×10^{-7} watt per square meter

Broadcast studio: 3.16×10^{-10} watt per square meter

108. Sound Intensity Use the equation for the level of sound in Example 8 to find the difference in loudness between a bird singing and rustling leaves with the intensities given below.

Bird singing: 10^{-8} watt per square meter

Rustling leaves : 10^{-10} watt per square meter

109. Think About It For each property of logarithms given below, write a related property of exponents.

(a) $\log_a(uv) = \log_a u + \log_a v$

(b) $\log_a \frac{u}{v} = \log_a u - \log_a v$

(c) $\log_a u^n = n \log_a u$

110. Proof Complete the proof of the logarithmic property

$\log_a uv = \log_a u + \log_a v.$

Let $\log_a u = x$ and $\log_a v = y$.

$a^x =$ ____ and $a^y =$ ____ Rewrite in exponential form.

$u \cdot v =$ ____ $\cdot$ ____ $= a$ Multiply and substitute for u and v.

____ $= x + y$ Rewrite in logarithmic form.

$\log_a uv =$ ____ $+$ ____ Substitute for x and y.

111. Proof Complete the proof of the logarithmic property

$\log_a \frac{u}{v} = \log_a u - \log_a v.$

Let $\log_a u = x$ and $\log_a v = y$.

$a^x =$ ____ and $a^y =$ ____ Rewrite in exponential form.

$\frac{u}{v} =$ ____ $= a$ Divide and substitute for u and v.

____ $= x - y$ Rewrite in logarithmic form.

$\log_a \frac{u}{v} =$ ____ $-$ ____ Substitute for x and y.



Courtesy of Maggie's Place

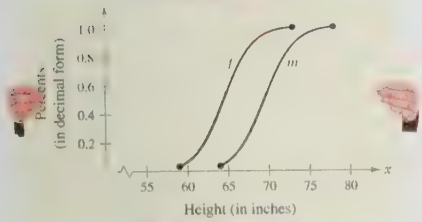
Business Capsule

Co-founded by five recent college graduates, Maggie's Place is a community of homes that provides hospitality for pregnant women who are alone or living on the streets. In homes shared with staff members, expectant mothers have their immediate needs met while being connected to community resources such as prenatal care, education programs, and low-cost housing. Maggie's Place opened its first home, the Magdalene House, on May 13, 2000 in Phoenix, Arizona. Today, there are five homes, and a resource and support center for women and children who have completed the Maggie's Place program.

112. Research Project Use your campus library, the Internet, or some other reference source to find information about a nonprofit group or company whose growth can be modeled by a logarithmic function. Write a brief report about the growth of the group or company.

4.5 Solving Exponential and Logarithmic Equations







In Exercise 110 on page 386, you will use exponential models to find the median heights of American males and females between the ages of 20 and 29 years old.

- Understand the strategies for solving exponential and logarithmic equations.
- Solve an exponential equation.
- Solve a logarithmic equation.
- Use an exponential or a logarithmic model to solve an application problem.

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving these exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties	*Inverse Properties*
$a^x = a^y$ if and only if $x = y$.	$a^{\log_a x} = x$
$\log_a x = \log_a y$ if and only if $x = y$.	$\log_a a^x = x$

Example 1 Solving Simple Equations

Original Equation	*Rewritten Equation*	*Solution*	*Property*
a. $2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b. $\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
c. $e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
d. $\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
e. $\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse

✓Checkpoint 1

Solve each equation for x.

a. $3^x = 81$

b. $\log_6 x = 3$

c. $\ln 5 - \ln x = 0$

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Property of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Edyta Pawlowska /Shutterstock.com

Solving Exponential Equations

Example 2 Solving Exponential Equations

Solve each equation and approximate the result to three decimal places.

a. $4^x = 72$ **b.** $3(2^x) = 42$

SOLUTION

a.
$$4^x = 72$$ Write original equation.

$$\log_4 4^x = \log_4 72$$ Take log (base 4) of each side.

$$x = \log_4 72$$ Inverse Property

$$x = \frac{\ln 72}{\ln 4} \approx 3.085$$ Change-of-base formula

The solution is $x = \log_4 72 \approx 3.085$. Check this in the original equation.

b.
$$3(2^x) = 42$$ Write original equation.

$$2^x = 14$$ Divide each side by 3 to isolate the exponential expression.

$$\log_2 2^x = \log_2 14$$ Take log (base 2) of each side.

$$x = \log_2 14$$ Inverse Property

$$x = \frac{\ln 14}{\ln 2} \approx 3.807$$ Change-of-base formula

The solution is $x = \log_2 14 \approx 3.807$. Check this in the original equation.

Checkpoint 2

Solve each equation and approximate the result to three decimal places.

a. $6^x = 84$ **b.** $62 - 10^x = 24$

In Example 2(a), the exact solution is $x = \log_4 72$ and the approximate solution is $x \approx 3.085$. An exact answer is preferred when the solution is an intermediate step in a larger problem. For a final answer, an approximate solution in decimal form is easier to comprehend.

Example 3 Solving an Exponential Equation

$$e^x + 5 = 60$$ Original equation

$$e^x = 55$$ Subtract 5 from each side to isolate the exponential expression.

$$\ln e^x = \ln 55$$ Take natural log of each side.

$$x = \ln 55$$ Inverse Property

$$x \approx 4.007$$ Use a calculator.

The solution is $x = \ln 55 \approx 4.007$. Check this in the original equation.

Checkpoint 3

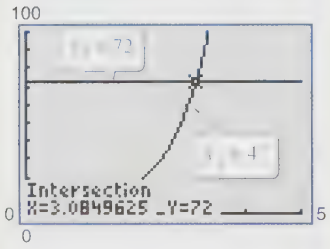
Solve $2e^x - 7 = 23$ and approximate the result to three decimal places.

TECH TUTOR

When solving an exponential or logarithmic equation, you can check your solution graphically by "graphing the left and right sides separately" and using the *intersect* feature of your graphing utility to determine the point of intersection. For instance, to check the solution of the equation in Example 2(a), graph

$$y_1 = 4^x \quad \text{and} \quad y_2 = 72$$

in the same viewing window, as shown below. Using the *intersect* feature, you can determine that the graphs intersect when $x \approx 3.085$, which confirms the solution found in Example 2(a).



For instructions on how to use the *intersect* feature, see Appendix A. Consult the user's guide for your graphing utility for more information.

Example 4 Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$ and approximate the result to three decimal places.

SOLUTION

$$2(3^{2t-5}) - 4 = 11$$ Write original equation

$$2(3^{2t-5}) = 15$$ Add 4 to each side

$$3^{2t-5} = \frac{15}{2}$$ Divide each side by 2

$$\log_3 3^{2t-5} = \log_3 \frac{15}{2}$$ Take log (base 3) of each side

$$2t - 5 = \log_3 \frac{15}{2}$$ Inverse Property

$$2t = 5 + \log_3 7.5$$ Add 5 to each side.

$$t = \frac{5}{2} + \frac{1}{2}\log_3 7.5$$ Divide each side by 2.

$$t \approx 3.417$$ Use a calculator.

The solution is $t = \frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.417$. Check this in the original equation.

STUDY TIP

Remember that to evaluate a logarithm such as $\log_3 7.5$, you need to use the change-of-base formula as shown.

$$\log_3 7.5 = \frac{\log_{10} 7.5}{\log_{10} 3} \approx 1.834$$

Checkpoint 4

Solve $4(4^{2t-7}) + 14 = 110$ and approximate the result to three decimal places.

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in Examples 2, 3, and 4. However, the algebra is a bit more complicated.

Example 5 Solving an Exponential Equation of Quadratic Type

$$e^{2x} - 3e^x + 2 = 0$$ Original equation

$$(e^x)^2 - 3e^x + 2 = 0$$ Write in quadratic form.

$$(e^x - 2)(e^x - 1) = 0$$ Factor

$$e^x - 2 = 0 \Rightarrow x = \ln 2$$ Set 1st factor equal to 0

$$e^x - 1 = 0 \Rightarrow x = 0$$ Set 2nd factor equal to 0.

The solutions are $x = \ln 2$ and $x = 0$. Check these in the original equation. Or, check by graphing $y = e^{2x} - 3e^x + 2$ using a graphing utility. The graph should have two x-intercepts: $x = \ln 2$ and $x = 0$, as shown in Figure 4.23.

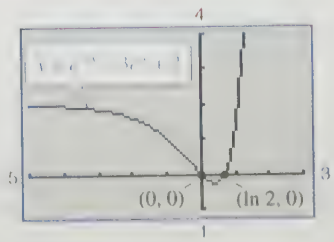


FIGURE 4.23

ALGEBRA TUTOR

For help with the algebra used to factor the left side of the equation in Example 5, see the *Chapter 4 Algebra Tutor* on page 401.

Checkpoint 5

Solve $e^{2x} - 7e^x + 12 = 0$.

Solving Logarithmic Equations

To solve a logarithmic equation such as

$\ln x = 3$ Logarithmic form

write the equation in exponential form as shown.

$e^{\ln x} = e^3$ Exponentiate each side.

$x = e^3$ Exponential form

This procedure is called **exponentiating** each side of an equation.



Example 6 Solving Logarithmic Equations

a. Solve $\ln(x + 5) = 1$. **b.** Solve $2 \log_5 3x = 4$.

SOLUTION

a. $\ln(x + 5) = 1$ Write original equation.

$e^{\ln(x+5)} = e^1$ Exponentiate each side.

$x + 5 = e$ Inverse Property

$x = e - 5$ Subtract 5 from each side.

The solution is $x = e - 5$. Check this in the original equation.

b. $2 \log_5 3x = 4$ Write original equation.

$\log_5 3x = 2$ Divide each side by 2.

$5^{\log_5 3x} = 5^2$ Exponentiate each side (base 5).

$3x = 25$ Inverse Property

$x = \frac{25}{3}$ Divide each side by 3.

The solution is $x = \frac{25}{3}$. Check this in the original equation.

✓Checkpoint 6

Solve each logarithmic equation.

a. $\ln(x - 4) = 0$ **b.** $\log_3 2x = 4$

Example 7 Solving a Logarithmic Equation

Solve $\log_3(5x - 1) = \log_3(x + 7)$.

SOLUTION

$\log_3(5x - 1) = \log_3(x + 7)$ Write original equation.

$5x - 1 = x + 7$ One-to-One Property

$4x = 8$ Add $-x$ and 1 to each side.

$x = 2$ Divide each side by 4.

The solution is $x = 2$. Check this in the original equation.

✓Checkpoint 7

Solve $\ln(3x + 2) = \ln(x + 8)$.

Ken Hurst/www shutterstock com

Example 8 Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$ and approximate the result to three decimal places.

SOLUTION

$$5 + 2 \ln x = 4$$ Write original equation.

$$2 \ln x = -1$$ Subtract 5 from each side.

$$\ln x = -\frac{1}{2}$$ Divide each side by 2.

$$e^{\ln x} = e^{-1/2}$$ Exponentiate each side.

$$x = e^{-1/2}$$ Inverse Property

$$x \approx 0.607$$ Use a calculator.

The solution is $x = e^{-1/2} \approx 0.607$. Check this in the original equation.

Checkpoint 8

Solve $4 + 3 \ln x = 16$ and approximate the result to three decimal places.

Because the domain of a logarithmic function generally does not include all real numbers, be sure to check for extraneous solutions of logarithmic equations.

Example 9 Checking for Extraneous Solutions

Solve $\log_{10} 5x + \log_{10}(x - 1) = 2$.

SOLUTION

$$\log_{10} 5x + \log_{10}(x - 1) = 2$$ Write original equation.

$$\log_{10}[5x(x - 1)] = 2$$ Product Property of logarithms

$$10^{\log_{10}(5x^2 - 5x)} = 10^2$$ Exponentiate each side (base 10).

$$5x^2 - 5x = 100$$ Inverse Property

$$x^2 - x - 20 = 0$$ Write in general form.

$$(x - 5)(x + 4) = 0$$ Factor.

$$x - 5 = 0$$ Set 1st factor equal to 0.

$$x = 5$$ Solution

$$x + 4 = 0$$ Set 2nd factor equal to 0.

$$x = -4$$ Solution

The solutions appear to be $x = 5$ and $x = -4$. However, when you check these in the original equation, you can see that $x = 5$ is the only solution.

Checkpoint 9

Solve $\log_6 x + \log_6(x + 5) = 2$.

In Example 9, the domain of $\log_{10} 5x$ is $x > 0$ and the domain of $\log_{10}(x - 1)$ is $x > 1$, so the domain of the original equation is $x > 1$. Because the domain is all real numbers greater than 1, the solution $x = -4$ is extraneous.

TECH TUTOR

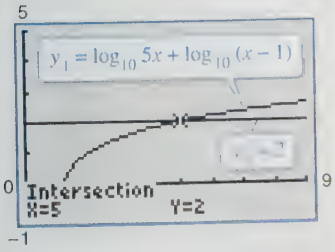
You can use a graphing utility to verify that the equation in Example 9 has $x = 5$ as its only solution. Graph

$$y_1 = \log_{10} 5x + \log_{10}(x - 1)$$

and

$$y_2 = 2$$

in the same viewing window, as shown below. It appears that the graphs intersect at one point. Use the *intersect* feature or the *zoom* and *trace* features to determine that $x = 5$ is the solution.



For instructions on how to use the *zoom* and *trace* features, see Appendix A. Consult the user's guide for your graphing utility for more information.

Applications

Example 10 Doubling and Tripling an Investment

You deposit \$500 in an account that pays 6.75% interest, compounded continuously.

a. How long will it take your money to double?

b. How long will it take your money to triple?

SOLUTION Using the formula for compound interest with continuous compounding, you can find that the balance in the account is given by

$$A = Pe^{rt} = 500e^{0.0675t}.$$

a. To find the time required for the balance to double, let $A = 1000$ and solve the resulting equation for t.

$$\begin{aligned} 500e^{0.0675t} &= 1000 && \text{Substitute 1000 for } A. \\ e^{0.0675t} &= 2 && \text{Divide each side by 500.} \\ \ln e^{0.0675t} &= \ln 2 && \text{Take natural log of each side.} \\ 0.0675t &= \ln 2 && \text{Inverse Property} \\ t &= \frac{1}{0.0675} \ln 2 && \text{Divide each side by 0.0675.} \\ t &\approx 10.27 && \text{Use a calculator.} \end{aligned}$$

The balance in the account will double after approximately 10.27 years.

b. To find the time required for the balance to triple, let $A = 1500$ and solve the resulting equation for t.

$$\begin{aligned} 500e^{0.0675t} &= 1500 && \text{Substitute 1500 for } A. \\ e^{0.0675t} &= 3 && \text{Divide each side by 500.} \\ \ln e^{0.0675t} &= \ln 3 && \text{Take natural log of each side.} \\ 0.0675t &= \ln 3 && \text{Inverse Property} \\ t &= \frac{1}{0.0675} \ln 3 && \text{Divide each side by 0.0675.} \\ t &\approx 16.28 && \text{Use a calculator.} \end{aligned}$$

The balance in the account will triple after approximately 16.28 years.

Notice that it took 10.27 years to earn the first \$500 and only 6.01 years to earn the second \$500. This result is graphically demonstrated in Figure 4.24.

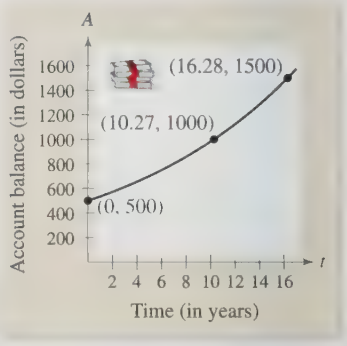


FIGURE 4.24

Checkpoint 10

In Example 10, how long will it take for the account balance to reach \$600? ■

Example 11 Hospital Expenses

From 1997 through 2008, the yearly expenses y (in billions of dollars) of hospitals in the United States can be approximated by

$$y = 208.05e^{0.067t}$$

where t represents the year, with $t = 7$ corresponding to 1997 (see Figure 4.25). Use the model to estimate the year in which the expenses of hospitals reached about \$610 billion. *(Source: Health Forum, An American Hospital Association Company)*

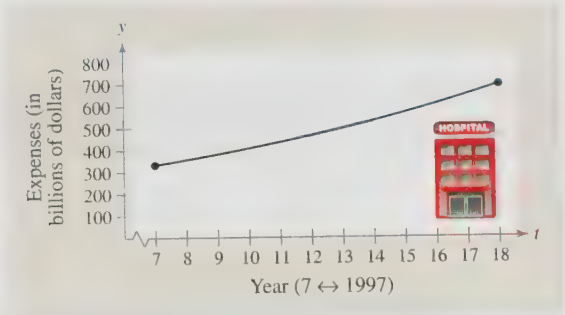


FIGURE 4.25

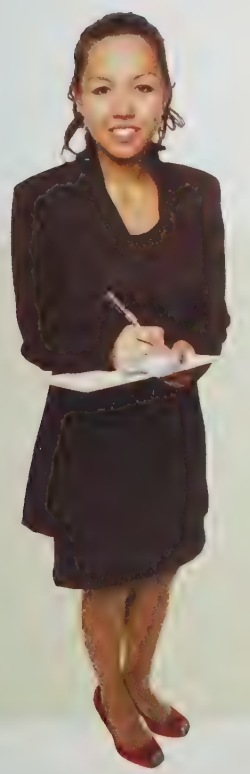
SOLUTION To solve this problem, let $y = 610$ and solve the resulting equation for t.

$208.05e^{0.067t} = 610$	Substitute 610 for y.
$e^{0.067t} \approx 2.932$	Divide each side by 208.05.
$\ln e^{0.067t} \approx \ln 2.932$	Take natural log of each side.
$0.067t \approx 1.076$	Inverse Property
$t \approx 16$	Divide each side by 0.067.

The solution is $t \approx 16$. Because $t = 7$ represents 1997, you can conclude that, according to the model, expenses were about \$610 billion in 2006.

✓Checkpoint 11

Use the model in Example 11 to estimate the year in which the expenses of hospitals reached about \$435 billion.



SUMMARIZE (Section 4.5)

1. State the One-to-One Properties and the Inverse Properties that can be used to solve simple exponential and logarithmic equations *(page 377)*. For an example of solving simple exponential and logarithmic equations, see Example 1.
2. Describe the strategies for solving exponential and logarithmic equations *(page 377)*. For examples of solving exponential equations, see Examples 2, 3, 4, and 5.
3. Describe what it means to exponentiate each side of an equation *(page 380)*. For examples of solving logarithmic equations, see Examples 6, 7, 8, and 9.
4. Describe a real-life example of how an exponential equation can be used to find the balance in an investment account *(page 382, Example 10)*.

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SKILLS WARM UP 4.5

The following warm up exercises involve skills that were covered in earlier sections. You will use those skills in the exercise set for this section. For additional help, review Sections 1.4 and 4.3.

In Exercises 1–6, solve for x.

1. $x \ln 2 = \ln 3$
2. $(x - 1) \ln 4 = 2$
3. $2xe^2 = e^3$
4. $4xe^{-1} = 8$
5. $x^2 - 4x + 5 = 0$
6. $2x^2 - 3x + 1 = 0$

In Exercises 7–10, simplify the expression.

7. $\log_{10} 10^x$
8. $\log_{10} 10^{2x}$
9. $\ln e^{2x}$
10. $\ln e^{-x^2}$

Exercises 4.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Solving Simple Equations In Exercises 1–10, solve for x. *See Example 1.*

1. $5^x = 25$
2. $2^x = 64$
3. $7^x = \frac{1}{49}$
4. $4^x = \frac{1}{256}$
5. $4^{x-1} = 64$
6. $3^{x-1} = 27$
7. $\log_4 x = 3$
8. $\log_{10} x = -2$
9. $\log_5 5x = \log_5 2$
10. $\ln 2x = \ln 6$

Using Inverse Properties In Exercises 11–22, apply the Inverse Property of logarithmic or exponential functions to simplify the expression.

11. $\ln e^{x^2}$
12. $\ln e^{2x-1}$
13. $\log_{10} 10^{x+1}$
14. $\log_{10} 10^{2x+3}$
15. $\log_5 5^{x^2} - 7$
16. $\log_8 8^{x^2} + 1$
17. $8 + e^{\ln x^3}$
18. $-1 + \ln e^{2x}$
19. $10^{\log_{10}(x+5)}$
20. $9^{\log_9(3x+7)}$
21. $2^{\log_2 x^2}$
22. $10^{\log_{10}(x^2+7x+10)}$

Solving Exponential Equations In Exercises 23–60, solve the exponential equation algebraically. Approximate the result to three decimal places. *See Examples 2, 3, 4, and 5.*

23. $3e^x = 9$
24. $5e^x = 20$
25. $2(3^x) = 16$
26. $3(4^x) = 81$
27. $e^x - 9 = 19$
28. $e^x + 4 = 18$
29. $3^{2x} = 80$
30. $6^{3x} = 3000$
31. $5^{-t/2} = 0.20$
32. $4^{-t/3} = 0.15$
33. $3^{x-1} = 28$
34. $2^{x-3} = 31$
35. $2^{3-x} = 565$
36. $8^{2-x} = 431$
37. $8(10^{3x}) = 12$
38. $5(10^{2x}) = 7$
39. $3(5^{x-1}) = 21$
40. $8(3^{6-x}) = 40$
41. $e^{3x} = 12$
42. $e^{2x} = 50$
43. $500e^{-x} = 300$
44. $1000e^{-4x} = 75$
45. $7 - 2e^{2x} = 6$
46. $14 + 3e^{2x} = 11$
47. $6(2^{3x-1}) - 7 = 9$
48. $8(4^{6-2x}) + 13 = 41$
49. $e^{2x} - 8e^x + 12 = 0$
50. $e^{2x} - 5e^x + 6 = 0$
51. $e^{2x} - 3e^x - 4 = 0$
52. $e^{2x} - 9e^x - 36 = 0$
53. $\frac{500}{100 - e^{x/2}} = 20$
54. $\frac{400}{1 + e^{-x}} = 350$
55. $\frac{3000}{2 + e^{2x}} = 2$
56. $\frac{119}{e^{6x} - 14} = 7$
57. $\left(1 + \frac{0.065}{365}\right)^{365t} = 4$
58. $\left(1 + \frac{0.075}{4}\right)^{4t} = 5$
59. $\left(1 + \frac{0.10}{12}\right)^{12t} = 2$
60. $\left(1 + \frac{0.0825}{26}\right)^{26t} = 9$

Solving Logarithmic Equations In Exercises 61–86, solve the logarithmic equation algebraically. Approximate the result to three decimal places. *See Examples 6, 7, 8, and 9.*

61. $\log_{10} x = 4$
62. $\log_{10} x = -5$
63. $\ln x = -3$
64. $\ln x = 6$
65. $\ln 2x = 2.4$
66. $\ln 4x = 1$
67. $\log_{10} 2x = 7$
68. $\log_{10} 3z = 2$
69. $4 \log_3(x + 1) = 12$
70. $5 \log_{10}(x - 2) = 10$
71. $3 \ln 5x = 10$
72. $2 \ln x = 7$
73. $\ln \sqrt{x + 2} = 1$
74. $\ln \sqrt{x - 8} = 5$
75. $7 + 3 \ln x = 5$
76. $2 - 6 \ln x = 10$
77. $\ln x - \ln(x + 1) = 2$
78. $\ln x - \ln(x + 2) = 3$
79. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$
80. $\ln(x + 1) - \ln(x - 2) = \ln x$
81. $\log_2(2x - 3) = \log_2(x + 4)$
82. $\log_3(x + 8) = \log_3(3x + 2)$
83. $\log_{10}(x + 4) - \log_{10} x = \log_{10}(x + 2)$
84. $\log_{10} x + \log_{10}(x + 1) = \log_{10}(x + 3)$
85. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$
86. $\log_3 x + \log_3(x - 8) = 2$

Rewriting Logarithmic Equations In Exercises 87–90, solve for y in terms of x.

87. $\ln y = \ln(2x + 1) + \ln 1$

88. $\ln y = 2 \ln x + \ln(x - 3)$

89. $\log_{10} y = 2 \log_{10}(x - 1) - \log_{10}(x + 2)$

90. $\log_{10}(y - 4) + \log_{10} x = 3 \log_{10} x$



Solving Equations Using a Graphing Utility In Exercises 91–94, use a graphing utility to solve the equation. Approximate the result to three decimal places. Verify your result algebraically.

91. $2^x - 7 = 0$

92. $500 - 1500e^{-x/2} = 0$

93. $3 - \ln x = 0$

94. $10 - 4 \ln(x - 2) = 0$

Doubling an Investment In Exercises 95 and 96, find the time required for a \$1000 investment to double at interest rate r, compounded continuously. *See Example 10.*

95. $r = 0.0725$

96. $r = 0.065$

Tripling an Investment In Exercises 97 and 98, find the time required for a \$1000 investment to triple at interest rate r, compounded continuously. *See Example 10.*

97. $r = 0.0825$

98. $r = 0.0775$

99. Suburban Wildlife The number V of varieties of suburban nondomesticated wildlife in a community is approximated by the model

$$V = 15 \cdot 10^{0.02x}, \quad 0 \le x \le 36$$

where x is the number of months since the development of the community was completed. Use this model to approximate the number of months since the development was completed when $V = 60$.

100. Native Prairie Grasses The number A of varieties of native prairie grasses per acre within a farming region is approximated by the model

$$A = 10.5 \cdot 10^{0.04x}, \quad 0 \le x \le 24$$

where x is the number of months since the farming region was plowed. Use this model to approximate the number of months since the region was plowed using a test acre for which $A = 80$.

101. Demand Function The demand function for a special limited edition coin set is given by

$$p = 1000\left(1 - \frac{5}{5 + e^{-0.001x}}\right).$$

(a) Find the demand x for a price of $p = \$139.50$.

(b) Find the demand x for a price of $p = \$99.99$.

(c) Use a graphing utility to confirm graphically the results found in parts (a) and (b).

102. Demand Function The demand function for a hot tub spa is given by

$$p = 105{,}000\left(1 - \frac{3}{3 + e^{-0.002x}}\right).$$

(a) Find the demand x for a price of $p = \$25{,}000$.

(b) Find the demand x for a price of $p = \$21{,}000$.

(c) Use a graphing utility to confirm graphically the results found in parts (a) and (b).

103. Forest Yield The yield V (in millions of cubic feet per acre) for a forest at age t years is given by



$$V = 6.7e^{-48.1/t}, \quad t > 0.$$

(a) Use a graphing utility to find the time necessary to obtain a yield of 1.3 million cubic feet per acre.

(b) Use the graphing utility to find the time necessary to obtain a yield of 2 million cubic feet per acre.

104. Human Memory Model In a group project on learning theory, a mathematical model for the percent P (in decimal form) of correct responses after n trials was found to be

$$P = \frac{0.98}{1 + e^{-0.3n}}, \quad n \ge 0.$$

(a) After how many trials will 80% of the responses be correct? (That is, for what value of n will $P = 0.8$?)

(b) Use a graphing utility to graph the memory model and confirm the result found in part (a).

(c) Write a paragraph describing the memory model.

105. Drug Prescriptions The numbers y (in millions) of mail-order drug prescriptions in the United States from 2002 through 2009 can be approximated by the model $y = 143.09 + 47.2 \ln t$, for $2 \le t \le 9$, where t represents the year, with $t = 2$ corresponding to 2002. *(Source: National Association of Chain Drug Stores)*



(a) Use a graphing utility to graph the model.

(b) Use the graphing utility to estimate the year when the number of mail-order drug prescriptions exceeded 200 million.

(c) Verify your answer to part (b) algebraically.

106. Air Transportation The numbers of employees y (in thousands) in air transportation in the United States from 2001 through 2009 can be approximated by the model $y = 608 - 64.2 \ln t$, for $1 \le t \le 9$, where t represents the year, with $t = 1$ corresponding to 2001. *(Source: U.S. Bureau of Labor Statistics)*



(a) Use a graphing utility to graph the model.

(b) Use the graphing utility to estimate the year in which the number of air transportation employees fell below 500,000.

(c) Verify your answer to part (b) algebraically.

107. Automobiles Automobiles are designed with crumple zones that help protect their occupants. The crumple zones allow the occupants to move short distances when the automobiles come to abrupt stops. The greater the distance moved, the fewer g's the crash victims experience. (One g is equal to the acceleration due to gravity. For very short periods of time, humans have withstood as much as 40 g's.) In crash tests with vehicles moving at 90 kilometers per hour, analysts measured the numbers of g's experienced during deceleration by crash dummies that were permitted to move x meters during impact. The data are shown in the table.

x	0.2	0.4	0.6	0.8	1.0
g's	158	80	53	40	32

A model for these data is given by

$$y = -3.00 + 11.88 \ln x + \frac{36.94}{x}$$

where y is the number of g's.

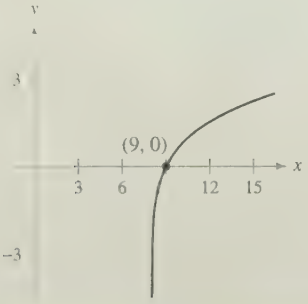
(a) Use the model to expand the table by adding a row of y-values that correspond to the given x-values.

(b) Use a graphing utility to graph the data points and the model in the same viewing window. How do they compare?

(c) Use the model to estimate the least distance traveled during impact for which the passenger does not experience more than 30 g's.

(d) Do you think it is practical to lower the number of g's experienced during impact to fewer than 23? Explain your reasoning.

108. HOW DO YOU SEE IT? Solving $\log_3 x + \log_3(x - 8) = 2$ algebraically, the solutions appear to be $x = 9$ and $x = -1$. Use the graph of $y = \log_3 x + \log_3(x - 8) - 2$ to determine whether each value is an actual solution of the equation. Explain your reasoning.



109. Think About It Is it possible for a logarithmic equation to have more than one extraneous solution? Explain.

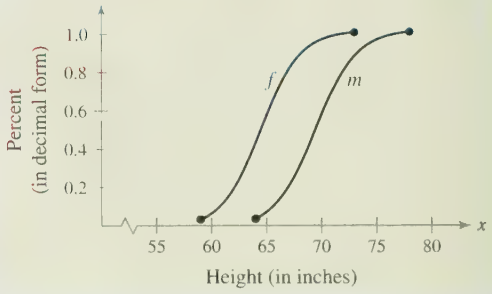
110. Average Heights The percent m of American males (between 20 and 29 years old) who are less than x inches tall is approximated by

$$m = -0.018 + \frac{1.041}{1 + e^{-0.5325(x - 69.44)}}, \quad 64 \le x \le 78$$

and the percent f of American females (between 20 and 29 years old) who are less than x inches tall is approximated by

$$f = -0.013 + \frac{1.031}{1 + e^{-0.5604(x - 64.48)}}, \quad 59 \le x \le 73$$

where m and f are the percents (in decimal form) and x is the height (in inches) (see figure). *(Source: U.S. National Center for Health Statistics)*



(a) What is the median height for each sex between 20 and 29 years old? (In other words, for what values of x are m and f equal to 0.5?)

(b) Write a paragraph describing each height model.

Writing Equations In Exercises 111–114, rewrite each verbal statement as an equation. Then decide whether the statement is true or false. Justify your answer.

111. The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

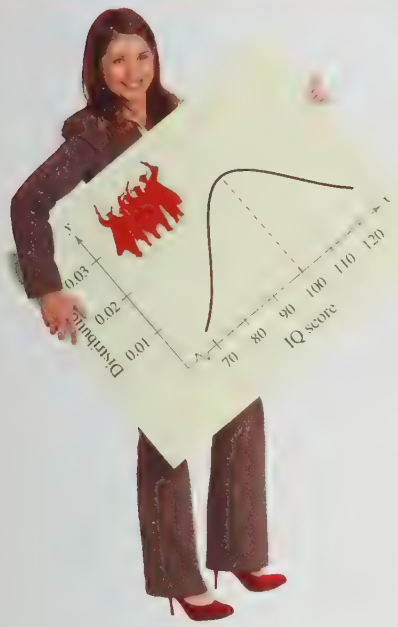
112. The logarithm of the sum of two numbers is equal to the product of the logarithms of the numbers.

113. The logarithm of the difference of two numbers is equal to the difference of the logarithms of the numbers.

114. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

115. Think About It Are the times required for the investments in Exercises 95 and 96 to quadruple twice as long as the times for them to double? Give a reason for your answer and verify your answer algebraically.

4.6 Exponential and Logarithmic Models



In Exercise 43 on page 397, you will use the graph of a Gaussian model to determine the average IQ score for the students at a college.

- Understand the shapes of the graphs of the most common types of mathematical models involving exponential and logarithmic functions.
- Construct and use a model for exponential growth or exponential decay.
- Use a Gaussian model to solve an application problem.
- Use a logistic growth model to solve an application problem.
- Use a logarithmic model to solve an application problem.
- Choose an appropriate model involving exponential or logarithmic functions for a real-life situation.

Introduction

The five most common types of mathematical models involving exponential functions and logarithmic functions are shown below.

1. Exponential growth model: $y = ae^{bx}, \quad b > 0$
2. Exponential decay model: $y = ae^{-bx}, \quad b > 0$
3. Gaussian model: $y = ae^{-(x-b)^2/c}$
4. Logistic growth model: $y = \dfrac{a}{1 + be^{-rx}}$
5. Logarithmic models: $y = a + b \ln x$

   $y = a + b \log_{10} x$

The basic shape of the graph of each of these models is shown in Figure 4.26.

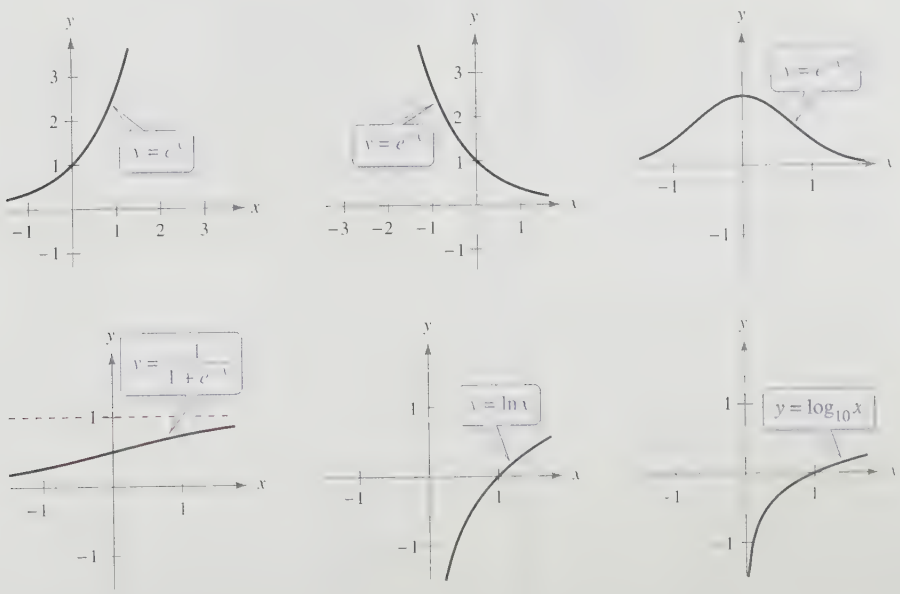


FIGURE 4.26

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes. Use the graphs in Figure 4.26 to identify the asymptote(s) of the graph of each function.

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Exponential Growth and Decay

Example 1 Rocky Mountain Spotted Fever

The number of reported cases of Rocky Mountain spotted fever for each year from 1998 through 2008 are shown in the table. *(Source: Centers for Disease Control and Prevention)*

Year	1998	1999	2000	2001	2002	2003
Reported cases	365	579	495	695	1104	1091

Year	2004	2005	2006	2007	2008
Reported cases	1713	1936	2288	2221	2563

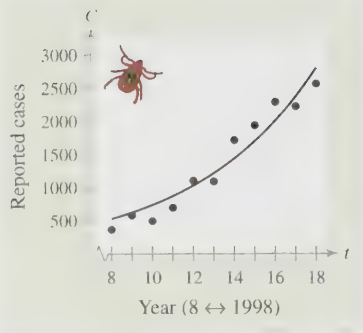


FIGURE 4.27

An exponential growth model that approximates these data is

$$C = 138.78e^{0.167t}, \quad 8 \le t \le 18$$

where C is the number of reported cases and t represents the year, with $t = 8$ corresponding to 1998. Compare the estimates given by the model with the values given by the Centers for Disease Control and Prevention. Use the model to predict the year in which the number of reported cases will reach 5500.

SOLUTION The following table compares the two sets of data. The graph of the model and the original data values are shown in Figure 4.27.

Year	1998	1999	2000	2001	2002	2003
Reported cases	365	579	495	695	1104	1091
Model	528	624	737	871	1030	1217

Year	2004	2005	2006	2007	2008
Reported cases	1713	1936	2288	2221	2563
Model	1438	1699	2008	2373	2804

To find the year in which the number of reported cases will reach 5500, let $C = 5500$ in the model and solve for t.

$138.78e^{0.167t} = C$	Write original model.
$138.78e^{0.167t} = 5500$	Substitute 5500 for C.
$e^{0.167t} \approx 39.631$	Divide each side by 138.78.
$\ln e^{0.167t} \approx \ln 39.631$	Take natural log of each side.
$0.167t \approx 3.680$	Inverse Property
$t \approx 22$	Divide each side by 0.167.

The solution is $t \approx 22$. Because $t = 8$ represents 1998, you can conclude that, according to the model, the number of reported cases will reach 5500 in 2012.

TECH TUTOR

Some graphing utilities have an *exponential regression* feature that can be used to find an exponential model that represents data. If you have such a graphing utility, try using it to find a model for the data given in Example 1. How does your model compare with the model given in Example 1? For instructions on how to use the *regression* feature, see Appendix A. Consult the user's guide for your graphing utility for more information.

Checkpoint 1

Use the model in Example 1 to predict the year in which the number of reported cases will reach 7500.

The exponential model in Example 1 increases by the same percent each year. What is the annual percent increase for this exponential model?

In Example 1, you are given the exponential growth model. How could you find such a model if you are given only a set of data? To find an exponential growth model that fits a set of data, choose any two of the points and substitute them in the general exponential growth model $y = ae^{bx}$. This technique is demonstrated in Example 2.

Example 2 Finding an Exponential Growth Model

Find an exponential growth model whose graph passes through the points (0, 4453) and (7, 5024), as shown in Figure 4.28(a).

SOLUTION The general form of the model is

$$y = ae^{bx}.$$

From the fact that the graph passes through the point (0, 4453), you know that $y = 4453$ when $x = 0$. By substituting these values into the general model, you have

$$4453 = ae^0 \quad \Rightarrow \quad a = 4453.$$

In a similar way, from the fact that the graph passes through the point (7, 5024), you know that $y = 5024$ when $x = 7$. Substitute these values into the model $y = ae^{bx}$ and solve for b.

$5024 = 4453e^{7b}$	Substitute for y, a, and x.
$1.12823 \approx e^{7b}$	Divide each side by 4453.
$\ln 1.12823 \approx \ln e^{7b}$	Take natural log of each side.
$0.12065 \approx 7b$	Inverse Property
$0.01724 \approx b$	Divide each side by 7.

So, the exponential growth model is

$$y = 4453e^{0.01724x}.$$

The graph of the model is shown in Figure 4.28(b).

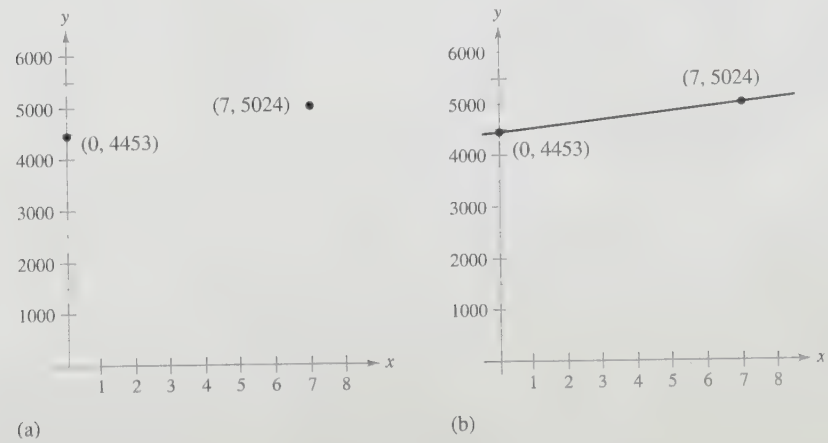


FIGURE 4.28

✓ Checkpoint 2

Find an exponential growth model whose graph passes through the points (0, 3) and (5, 8).

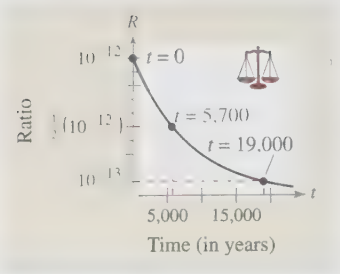


FIGURE 4.29

In living organic material, the ratio of the number of radioactive carbon isotopes (carbon 14) to the number of nonradioactive carbon isotopes (carbon 12) is about 1 to 10^{12}. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of about 5700 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio R of carbon 14 to carbon 12 present at any time t (in years).

$$R = \frac{1}{10^{12}} e^{-t/8223}$$ Carbon dating model

In Figure 4.29, note that R decreases as the time t increases. Any material that is composed of carbon, such as wood, bone, hair, pottery, paper, and water, can be dated.

Example 3 Carbon Dating

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{10^{13}}.$$

Estimate the age of the fossil.

SOLUTION In the carbon dating model, substitute for R and solve for t.

$$\frac{1}{10^{12}} e^{-t/8223} = R$$ Write original model.

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}}$$ Substitute $\frac{1}{10^{13}}$ for R.

$$e^{-t/8223} = \frac{1}{10}$$ Multiply each side by 10^{12}.

$$\ln e^{-t/8223} = \ln \frac{1}{10}$$ Take natural log of each side.

$$-\frac{t}{8223} \approx -2.3026$$ Inverse Property

$$t \approx 18,934$$ Multiply each side by -8223.

So, you can estimate the age of the fossil to be about 19,000 years.



In 1960, Willard Libby of the University of Chicago won the Nobel Prize for Chemistry for the carbon 14 method, a valuable tool for estimating the ages of ancient materials.

Checkpoint 3

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{9^{13}}.$$

Estimate the age of the fossil.

An exponential model can be used to determine the *decay* of radioactive isotopes. For instance, to find how much of an initial 10 grams of radioactive radium (^{226}Ra), with a half-life of 1599 years, is left after 500 years, use the exponential decay model, as shown.

$$y = ae^{-bt} \Rightarrow \frac{1}{2}(10) = 10e^{-b(1599)} \Rightarrow \ln\frac{1}{2} = -1599b \Rightarrow -\frac{\ln\frac{1}{2}}{1599} = b$$

Using the value of b found above, $a = 10$, and $t = 500$, the amount left is

$$y = 10e^{-[-\ln(1/2)/1599](500)} \approx 8.05 \text{ grams}.$$

Kutlayev Dmitry/www.shutterstock.com

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The graph of a Gaussian model is called a **bell-shaped curve.** Try to sketch the standard normal distribution curve with a graphing utility. Can you see why it is called a bell-shaped curve?

The **average value** of a population can be found from the bell-shaped curve by observing where the maximum y-value of the function occurs. The x-value corresponding to the maximum y-value of the function represents the average value of the independent variable, x.

Example 4 ACT Scores

In 2010, the ACT science scores for high school graduates in the United States roughly followed a normal distribution given by

$$y = 0.0782e^{-(x-20.9)^2/52.02}, \quad 1 \le x \le 36$$

where x is the ACT science score. Sketch the graph of the function. From the graph, estimate the average ACT science score. *(Source: ACT, Inc.)*

SOLUTION The graph of the function is shown in Figure 4.30. On this bell-shaped curve, the x-value corresponding to the maximum value of the curve represents the average score. From the graph, you can estimate that the average ACT science score for high school graduates in 2010 was about 21.

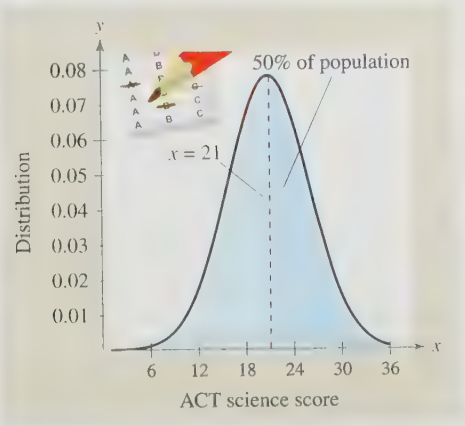


FIGURE 4.30

Checkpoint 4

In 2010, the SAT critical reading scores for high school graduates in the United States roughly followed a normal distribution given by

$$y = 0.0036e^{-(x-501)^2/25{,}088}, \quad 200 \le x \le 800$$

where x is the SAT score for critical reading. Sketch the graph of the function. From the graph, estimate the average SAT critical reading score. *(Source: The College Board)*



Logistic Curve
FIGURE 4.31

Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as shown by the graph in Figure 4.31. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions, followed by less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

Example 5 Spread of a Virus

On a college campus of 5000 students, one student returned from vacation with a contagious flu virus. The spread of the virus through the student population is given by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where y is the total number of students infected after t days.

a. How many students are infected after 5 days?

b. The college will cancel classes when 40% or more of the students become infected. After how many days will the college cancel classes?

SOLUTION

a. After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

b. Classes are cancelled when the number infected is $(0.40)(5000) = 2000$. So, substitute 2000 for y and solve for t as shown.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$2000(1 + 4999e^{-0.8t}) = 5000$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$4999e^{-0.8t} = 1.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

$$\ln e^{-0.8t} = \ln \frac{1.5}{4999}$$

$$-0.8t = \ln \frac{1.5}{4999}$$

$$t \approx 10.1$$

So, after about 10 days, 40% of the students will become infected, and the college will cancel classes. The graph of the function is shown in Figure 4.32.

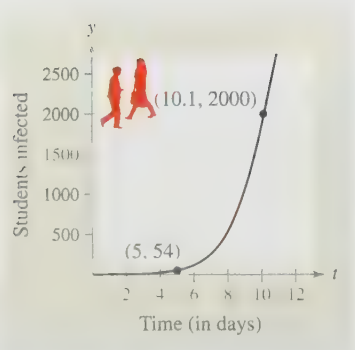


FIGURE 4.32

Checkpoint 5

In Example 5, how many days does it take for 25% of the students on campus to become infected?

Logarithmic Models



The severity of the destruction caused by an earthquake depends on its magnitude and duration. Earthquakes can destroy buildings, and can cause landslides and tsunamis.

Example 6 Magnitudes of Earthquakes

On the Richter scale, the magnitude R of an earthquake of intensity I per unit of area is given by $R = \log_{10}(I/I_0)$, where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity per unit of area for each earthquake. (Intensity is a measure of the wave energy of an earthquake.)

a. Chile, in 1960; $R = 9.5$

b. Southern Sumatra, Indonesia, in 2007; $R = 8.5$

SOLUTION

a. Because $I_0 = 1$ and $R = 9.5$,

$$9.5 = \log_{10} I$$

$$I = 10^{9.5} = 3{,}162{,}277{,}660.$$

b. For $R = 8.5$,

$$8.5 = \log_{10} I$$

$$I = 10^{8.5} = 316{,}227{,}766.$$

Note that an increase of 1 unit on the Richter scale (from 8.5 to 9.5) represents an intensity change by a factor of

$$\frac{3{,}162{,}277{,}660}{316{,}227{,}766} = 10.$$

In other words, the intensity of the earthquake in Chile in 1960 was 10 times greater than the intensity of the earthquake in Southern Sumatra, Indonesia in 2007.

Checkpoint 6

In June 2010, an earthquake measuring 7.5 on the Richter scale occurred near the Nicobar Islands in the Indian Ocean. Find the intensity I per unit of area for this earthquake. (Let $I_0 = 1$.) How many times greater is the intensity of the 1960 Chile earthquake?

Example 7 pH Levels

Acidity, or pH level, is a measure of the hydrogen ion concentration $[H^+]$ (measured in moles of hydrogen per liter) of a solution. Use the model given by

$$\text{pH} = -\log_{10}[H^+]$$

to determine the hydrogen ion concentration of milk of magnesia, which has a pH of 10.5.

SOLUTION

$$\text{pH} = -\log_{10}[H^+] \qquad \text{Write original model.}$$

$$10.5 = -\log_{10}[H^+] \qquad \text{Substitute 10.5 for pH.}$$

$$-10.5 = \log_{10}[H^+] \qquad \text{Multiply each side by } -1.$$

$$10^{-10.5} = 10^{\log_{10}[H^+]} \qquad \text{Exponentiate each side (base 10).}$$

$$3.16 \times 10^{-11} = [H^+] \qquad \text{Simplify.}$$

So, the hydrogen ion concentration is 3.16×10^{-11} mole of hydrogen per liter.

Checkpoint 7

Use the model in Example 7 to determine the hydrogen ion concentration of coffee, which has a pH of 5.0.

Hou Yu/Color China Photo/AP Images

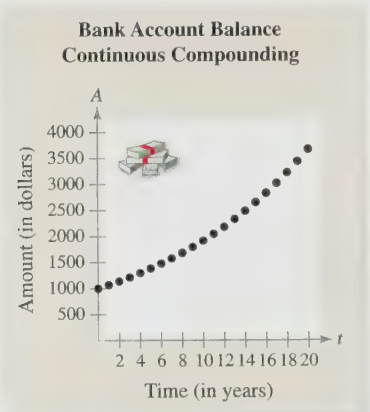
Comparing Models

So far, you have been given the type of model to use for a data set. Now, you will use the general trends of the graphs of the five models presented in this section to choose appropriate models for real-life situations.

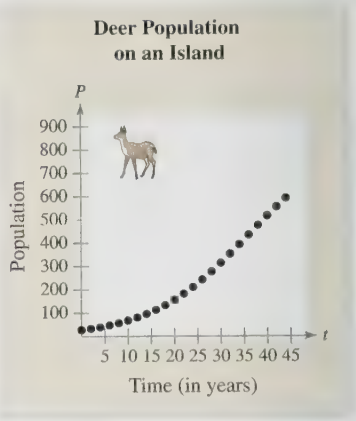
Example 8 Choosing an Appropriate Model

Decide whether to use an exponential growth model or a logistic growth model to represent each data set.

a.



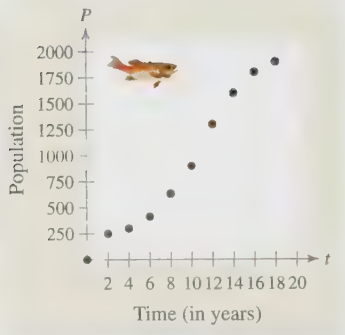
b.



SOLUTION

a. As long as withdrawals and deposits are not made and the interest rate remains constant, the bank account balance will grow exponentially. So, an exponential growth model is an appropriate model.

b. The growth of the deer population will slow as the population approaches the carrying capacity of the island. So, a logistic growth model is an appropriate model.



Checkpoint 8

Decide whether to use an exponential growth model or a logistic growth model to represent the data for the fish population of a lake, shown in the figure at the left.



SUMMARIZE (Section 4.6)

1. State the functions for an exponential growth model and an exponential decay model *(page 387)*. For examples of finding and using exponential growth and decay models, see Examples 1, 2, and 3.
2. State the function for a Gaussian model and describe its shape *(page 391)*. For an example of using a Gaussian model, see Example 4.
3. State the function for a logistic growth model *(page 392)*. For an example of using a logistic growth model, see Example 5.
4. State the functions for logarithmic models *(page 387)*. For examples of using logarithmic models, see Examples 6 and 7.

Mrs.Blondy/Shutterstock.com

SKILLS WARM UP 4.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.2, 4.3, and 4.5.

In Exercises 1–6, sketch the graph of the equation.

1. $y = e^{0.1x}$

2. $y = e^{-0.25x}$

3. $y = e^{-x^2/5}$

4. $y = \dfrac{2}{1 + e^{-x}}$

5. $y = \log_{10} 2x$

6. $y = \ln 4x$

In Exercises 7 and 8, solve the equation algebraically.

7. $3e^{2x} = 7$

8. $4 \ln 5x = 14$

In Exercises 9 and 10, solve the equation graphically.

9. $2e^{-0.2x} = 0.002$

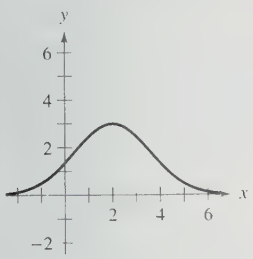
10. $6 \ln 2x = 12$

Exercises 4.6

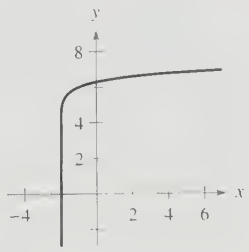
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–6, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

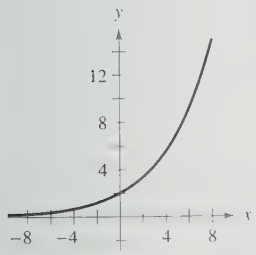
(a)



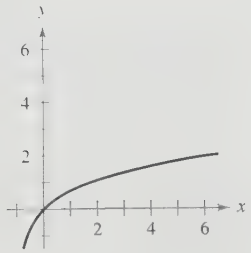
(b)



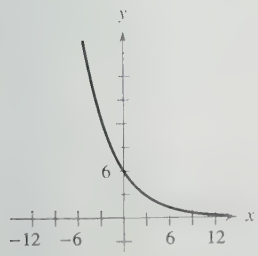
(c)



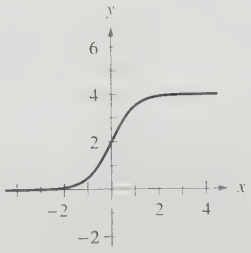
(d)



(e)



(f)



1. $y = 2e^{x/4}$

2. $y = 6e^{-x/4}$

3. $y = 6 + \log_{10}(x + 2)$

4. $y = 3e^{-(x-2)^2/5}$

5. $y = \ln(x + 1)$

6. $y = \dfrac{4}{1 + e^{-2x}}$

Classifying an Exponential Model In Exercises 7–10, classify the model as an exponential growth model or an exponential decay model.

7. $y = 3e^{0.5t}$

8. $y = 2e^{-0.6t}$

9. $y = 20e^{-1.5t}$

10. $y = 4e^{0.07t}$

11. Population The population P of a city is given by

$$P = 120{,}000e^{0.016t}$$

where t represents the year, with $t = 0$ corresponding to 2000. Sketch the graph of this equation. Use the model to predict the year in which the population of the city will reach 180,000.

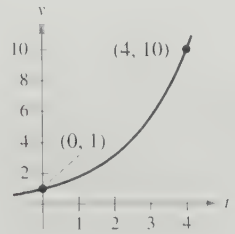
12. Population The population P of a city is given by

$$P = 240{,}360e^{0.012t}$$

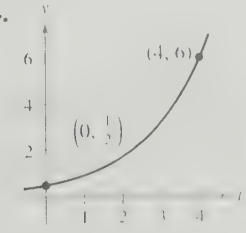
where t represents the year, with $t = 0$ corresponding to 2000. Sketch the graph of this equation. Use the model to predict the year in which the population of the city will reach 300,000.

Finding an Exponential Growth or Decay Model In Exercises 13–16, find the constants C and k such that the exponential function $y = Ce^{kt}$ passes through the points on the graph. *See Example 2.*

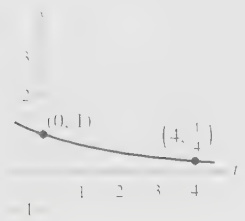
13.



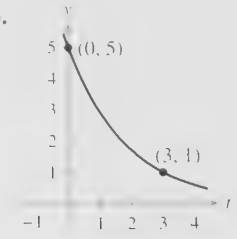
14.



15.



16.



Compound Interest In Exercises 17–26, use the account balance formula $A = Pe^{rt}$ to complete the table for a savings account in which interest is compounded continuously.

	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 Years*
17.	$5000	7%		
18.	$1000	$9\frac{1}{4}$%		
19.	$500		10 yr	
20.	$10,000		5 yr	
21.	$1000			$2281.88
22.	$2000			$3000
23.		11%		$19,205
24.		8%		$20,000
25.	$5000			$11,127.70
26.	$250			$600

Radioactive Decay In Exercises 27–32, use the exponential decay model $y = ae^{-bt}$ to complete the table for the radioactive isotope.

	Isotope	*Half-Life (Years)*	*Initial Quantity*	*Amount After 1000 Years*
27.	^{226}Ra	1599	4 g	
28.	^{226}Ra	1599		0.15 g
29.	^{14}C	5715		3.5 g
30.	^{14}C	5715	8 g	
31.	^{239}Pu	24,100		1.6 g
32.	^{239}Pu	24,100		0.38 g

33. Population The population P of a city is given by

$$P = 25{,}000e^{kt}$$

where t represents the year, with $t = 0$ corresponding to 2000. In 1980, the population was 15,000. Find the value of k and use this result to predict the population in 2015.

34. Population The population P of a city is given by

$$P = 52{,}500e^{kt}$$

where t represents the year, with $t = 0$ corresponding to 2000. In 2002, the population was 54,000. Find the value of k and use this result to predict the population in 2015.

35. Bacteria Growth The number N of bacteria in a culture is given by the model $N = 100e^{kt}$, where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 100$. When $t = 6$, there are 140 bacteria. How long does it take the bacteria population to double in size? To triple in size?

36. Bacteria Growth The number N of bacteria in a culture is given by the model $N = 250e^{kt}$, where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 250$. When $t = 10$, there are 320 bacteria. How long does it take the bacteria population to double in size? To triple in size?

37. Carbon Dating The ratio of carbon 14 to carbon 12 in a piece of wood discovered in a cave is $R = 1/8^{14}$. Estimate the age of the piece of wood.

38. Carbon Dating The ratio of carbon 14 to carbon 12 in a piece of paper buried in a tomb is $R = 1/13^{11}$. Estimate the age of the piece of paper.

39. Radioactive Decay What percent of a present amount of radioactive cesium (^{137}Cs) will remain after 100 years? Use the fact that radioactive cesium has a half-life of 30 years.

40. Radioactive Decay Find the half-life of radioactive iodine (^{131}I) if, after 20 days, 0.53 kilogram of an initial 3 kilograms remains.

41. Learning Curve The management at a factory has found that the maximum number of units a worker can produce in a day is 40. The learning curve for the number of units N produced per day after a new employee has worked t days is given by

$$N = 40(1 - e^{kt}).$$

After 20 days on the job, a particular worker produced 25 units in 1 day.

(a) Find the learning curve for this worker (first find the value of k).

(b) How many days should pass before this worker is producing 35 units per day?

42. Learning Curve The management at a customer service center has found that the maximum number of customer calls an employee can process effectively in a day is 90. The learning curve for the number N of calls processed per day after a new employee has worked t days is given by

$$N = 90(1 - e^{kt}).$$

After 15 days on the job, a particular employee processed 60 calls in 1 day.

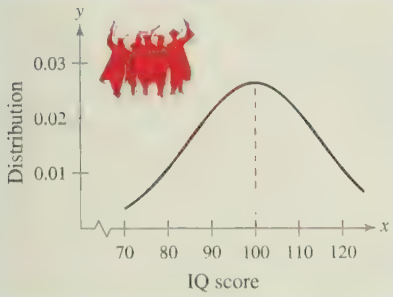
(a) Find the learning curve for this worker (first find the value of k).

(b) How many days should pass before this employee will process 80 calls per day?

43. IQ Scores The IQ scores for the students at a college roughly follow the normal distribution

$$y = 0.0266e^{-(x-100)^2/450}, \quad 70 \le x \le 125$$

where x is the IQ score (see figure). Use the graph to estimate the average IQ score for the students.



44. Education The amount of time (in hours per week) a student spends at a math-tutoring center roughly follows the normal distribution

$$y = 0.7979e^{-(x-5.4)^2/0.5}, \quad 4 \le x \le 7$$

where x is the number of hours.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average number of hours per week a student spends at the tutoring center.

45. Stocking a Lake with Fish A lake is stocked with 500 fish, and the fish population P increases according to the logistic curve

$$P = \frac{10{,}000}{1 + 19e^{-t/5}}, \quad t \ge 0$$

where t is the time (in months).

(a) Use a graphing utility to graph the logistic curve.

(b) Find the fish population after 5 months.

(c) After how many months will the fish population reach 2000?

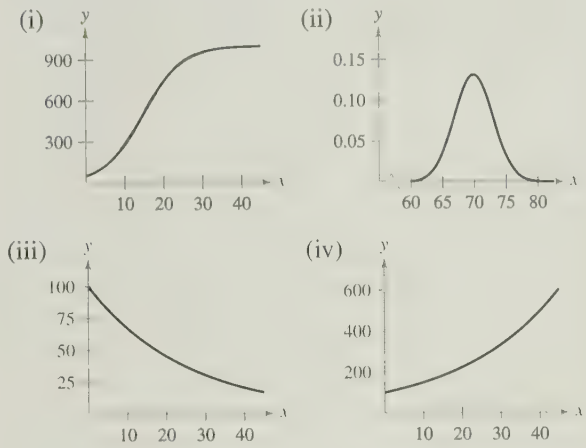
46. Endangered Species A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the herd will be modeled by the logistic curve

$$p = \frac{1000}{1 + 9e^{-kt}}, \quad t \ge 0$$

where p is the number of animals and t is the time (in years). The herd size is 134 after 2 years. Find k. Then find the population after 5 years.

47. Think About It Does the graph of a logistic growth model have a relative maximum value?

48. HOW DO YOU SEE IT? Match the description of each situation with its graph. Explain your reasoning. [The graphs are labeled (i), (ii), (iii), and (iv).]



(a) The number of trout in a lake over time

(b) The amount of a radioactive substance over time

(c) The distribution of heights of men

(d) The balance of an interest-bearing account over time

49. Aged Population The table shows the projected U.S. populations P (in thousands) of people who are 85 years old or older for several years from 2020 through 2050. *(Source: U.S. Census Bureau)*

Year	Population, P
2020	6597
2025	7239
2030	8745
2035	11,450
2040	14,198
2045	16,985
2050	19,041

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 20$ corresponding to 2020.

(b) Use the *regression* feature of the graphing utility to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Use the graphing utility to graph the exponential model in base e.

(d) Use the exponential model in base e to estimate the populations of people who are 85 years old or older in 2022 and 2042.

50. Number of Stores The table shows the number N of stores operated by Wal-Mart each year from 2000 through 2009. *(Source: Wal-Mart Stores, Inc.)*

Year	Stores, N	Year	Stores, N
2000	4189	2005	6141
2001	4414	2006	6779
2002	4688	2007	7262
2003	4906	2008	7720
2004	5289	2009	8416

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Use the graphing utility to graph the exponential model in base e.

(d) Use the exponential model in base e to predict the numbers of stores Wal-Mart will operate in 2010 and 2012.

51. Island Population The table shows the population P of an island town every five years for forty years.

Year	Population, P	Year	Population, P
0	280	25	2340
5	390	30	2830
10	650	35	3100
15	1160	40	3240
20	1730		

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year.

(b) Based on the scatter plot, do you think the data can be better represented by an exponential growth model or a logistic growth model? Explain.

(c) Use the *regression* feature of the graphing utility to find a model of the type you chose in part (b) for the data.

(d) Use the graphing utility to graph the model with the scatter plot. Does the model fit the data well?

(e) Use the model to predict the population of the island town in year 50.

52. Population The populations P of Utah (in thousands) from 1997 through 2009 are shown in the table. *(Source: U.S. Census Bureau)*

Year	Population, P
1997	2120
1998	2166
1999	2203
2000	2244
2001	2291
2002	2334
2003	2380
2004	2439
2005	2500
2006	2584
2007	2664
2008	2727
2009	2785

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use the *regression* feature of the graphing utility to find an exponential model for the data. Use the Inverse Property

$$b = e^{\ln b}$$

to rewrite the model as an exponential model in base e.

(c) Use the *regression* feature of the graphing utility to find a linear model and a quadratic model for the data.

(d) Use the graphing utility to graph the exponential model in base e and the models in part (c) with the scatter plot.

(e) Use each model to predict the populations in 2010, 2012, and 2014. Do all of the models give reasonable predictions? Explain.

Earthquake Magnitudes In Exercises 53 and 54, use the Richter scale for measuring the magnitudes of earthquakes. *See Example 6.*

53. Find the magnitude R (on the Richter scale) of an earthquake of intensity I. (Let $I_0 = 1$.)

(a) $I = 80{,}500{,}000$

(b) $I = 48{,}275{,}000$

54. Find the intensity I of an earthquake measuring R on the Richter scale. (Let $I_0 = 1$.)

(a) In 2011, an earthquake in the Vanuatu Islands had a magnitude of $R = 6.3$.

(b) In 2011, an earthquake near the coast of Japan had a magnitude of $R = 9.0$.

Intensity of Sound In Exercises 55 and 56, find the level of sound using the following information for determining sound intensity. The level of sound L (in decibels) of a sound with an intensity of I is given by

$$L = 10 \log_{10} \frac{I}{I_0}$$

where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear.

55. (a) $I = 10^{-10}$ watt per square meter (quiet room)

(b) $I = 10^{-5}$ watt per square meter (busy street corner)

56. (a) $I = 10^{-3}$ watt per square meter (loud car horn)

(b) $I \approx 10^{0}$ watt per square meter (threshold of pain)

pH Levels In Exercises 57–60, use the model $\text{pH} = -\log_{10}[\text{H}^+]$. *See Example 7.*

57. Compute the hydrogen ion concentration $[\text{H}^+]$ of a solution for which pH = 5.8.

58. Compute the hydrogen ion concentration $[\text{H}^+]$ of a solution for which pH = 7.3.

59. A grape has a pH of 3.5, and baking soda has a pH of 8.0. The hydrogen ion concentration of the grape is how many times that of the baking soda?

60. The pH of a solution is decreased by one unit. The hydrogen ion concentration is increased by what factor?

61. Estimating the Time of Death At 8:30 A.M., a coroner was called to the home of a person who had died during the night. The coroner assumed that the person had a normal body temperature of 98.6°F at death, and that the room temperature was a constant 70°F. From these two temperatures, the coroner was able to determine that the time elapsed since death and the body temperature are related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time (in hours) elapsed since the person died, and T is the temperature (in degrees Fahrenheit) of the person's body. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. Use this formula to estimate the time of death of the person.

62. Thawing a Package of Steaks You take a three-pound package of steaks out of the freezer at 11 A.M. and place it in the refrigerator. Will the steaks be thawed in time to be grilled at 6 P.M.? Assume that the refrigerator temperature is 40°F and that the freezer temperature is 0°F. Use the formula for Newton's Law of Cooling

$$t = -5.05 \ln \frac{T - 40}{0 - 40}$$

where t is the time in hours (with $t = 0$ corresponding to 11 A.M.) and T is the temperature of the package of steaks (in degrees Fahrenheit).

63. Worker's Productivity The numbers n of units per day that a new worker can produce after t days on the job are listed in the table.

Days, t	5	10	15	20	25
Units, n	6	13	22	34	56

(a) Use a graphing utility to create a scatter plot of the data. Do the data fit an exponential model or a logarithmic model?

(b) Use the *regression* feature of the graphing utility to find the model of the type you chose in part (a).

(c) Use the graphing utility to graph the model with the data. Does the model fit the data well? Can you think of a better model to use for these data? Explain.

64. Chemical Reaction The table shows the yield y (in milligrams) of a chemical reaction after x minutes.

Minutes, x	Yield, y
1	1.5
2	7.4
3	10.2
4	13.4
5	15.8
6	16.3
7	18.2
8	18.3

(a) Use a graphing utility to create a scatter plot of the data. Do the data fit an exponential model or a logarithmic model?

(b) Use the *regression* feature of the graphing utility to find the model of the type you chose in part (a).

(c) Use the graphing utility to graph the model with the data. Does the model fit the data well?

ALGEBRA TUTOR

Using Properties of Exponents

Much of the algebra in Chapter 4 involves evaluating expressions by using the properties of exponents. On this page, you can review some of these properties.

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (Assume all denominators and bases are nonzero.)

1. $a^m a^n = a^{m+n}$ Product of Powers

2. $\dfrac{a^m}{a^n} = a^{m-n}$ Quotient of Powers

3. $(ab)^m = a^m b^m$ Power of a Product

4. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ Power of a Quotient

5. $(a^m)^n = a^{mn}$ Power of a Power

6. $a^{-n} = \dfrac{1}{a^n}$ Definition of negative exponent

7. $a^0 = 1, \quad a \neq 0$ Definition of zero exponent

8. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n, \quad a \neq 0, b \neq 0$

9. $|a^2| = |a|^2 = a^2$

Example 1 Using Properties of Exponents

a. $(2^4)^3 = 2^{4 \cdot 3}$ Power of a Power

$= 2^{12}$ Multiply exponents.

$= 4096$ Simplify.

b. $10^{12} \cdot \dfrac{1}{10^{13}} = \dfrac{10^{12}}{10^{13}}$ Multiply.

$= 10^{12-13}$ Quotient of Powers

$= 10^{-1}$ Subtract exponents.

$= \dfrac{1}{10}$ Definition of negative exponent

c. $8^{-2} \cdot 4^0 = \dfrac{1}{8^2} \cdot 4^0$ Definition of negative exponent

$= \dfrac{1}{8^2} \cdot 1$ Definition of zero exponent

$= \dfrac{1}{64}$ Simplify.

d. $\left(\dfrac{2}{3}\right)^{4(2)} = \left(\dfrac{2}{3}\right)^8$ Multiply exponents.

$= \dfrac{2^8}{3^8}$ Power of a Quotient

$= \dfrac{256}{6561}$ Simplify.

e. $\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$ Property 8

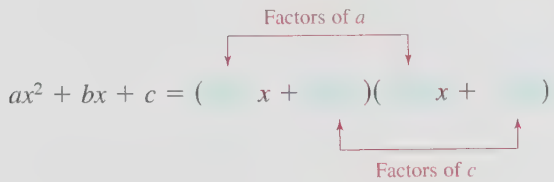
$= \dfrac{16}{9}$ Power of a Quotient

STUDY TIP

Be sure to remember that these properties apply for all integers m and n, not just positive integers.

Factoring Trinomials

To solve equations that are of quadratic type, you must be good at factoring trinomials. Some examples are presented on this page. Remember that to factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.



The goal is to find a combination of factors of a and c such that the outer and inner products add up to the middle term bx.

Example 2 Factoring Trinomials

Factor each trinomial.

a. $x^2 + 7x + 12$ **b.** $x^2 - 6x + 5$ **c.** $x^2 - 14x + 45$ **d.** $2x^2 + 11x - 21$

SOLUTION

a. For $x^2 + 7x + 12$, you have $a = 1$, $b = 7$, and $c = 12$. Because b and c are positive, both factors of 12 must be positive. That is, $12 = (1)(12)$, $12 = (2)(6)$, or $12 = (3)(4)$. So, the possible factorizations of $x^2 + 7x + 12$ are

$$(x + 1)(x + 12), \quad (x + 2)(x + 6), \quad \text{and} \quad (x + 3)(x + 4).$$

Testing the middle term, you can find the correct factorization to be

$$x^2 + 7x + 12 = (x + 3)(x + 4).$$

b. For $x^2 - 6x + 5$, you have $a = 1$, $b = -6$, and $c = 5$. Because b is negative and c is positive, both factors of 5 must be negative. That is, $5 = (-1)(-5)$. So, the factorization is

$$x^2 - 6x + 5 = (x - 1)(x - 5).$$

c. For $x^2 - 14x + 45$, you have $a = 1$, $b = -14$, and $c = 45$. Because b is negative and c is positive, both factors of 45 must be negative. That is, $45 = (-1)(-45)$, $45 = (-3)(-15)$, or $45 = (-5)(-9)$. So, the possible factorizations of $x^2 - 14x + 45$ are

$$(x - 1)(x - 45), \quad (x - 3)(x - 15), \quad \text{and} \quad (x - 5)(x - 9).$$

Testing the middle term, you can find the correct factorization to be

$$x^2 - 14x + 45 = (x - 5)(x - 9).$$

d. For $2x^2 + 11x - 21$, you have $a = 2$ and $c = -21$, which means that the factors of -21 must have unlike signs. The eight possible factorizations are shown below.

$$\begin{array}{ll} (2x - 1)(x + 21) & (2x + 1)(x - 21) \\ (2x - 3)(x + 7) & (2x + 3)(x - 7) \\ (2x - 7)(x + 3) & (2x + 7)(x - 3) \\ (2x - 21)(x + 1) & (2x + 21)(x - 1) \end{array}$$

Testing the middle term, you can find the correct factorization to be

$$2x^2 + 11x - 21 = (2x - 3)(x + 7).$$

SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 404. Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 4.1

	Review Exercises
■ Determine whether a function has an inverse function.	*1–6*
■ Find the inverse function of a function.	*3–6*
■ Graph a function and its inverse function.	*3–6*

Section 4.2

■ Evaluate an exponential expression. *7–12*

■ Sketch the graph of an exponential function. *13–24*

Graph of $y = a^x$, $a > 1$

- Domain: $(-\infty, \infty)$, Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Increasing
- x-axis is a horizontal asymptote: $(a^x \to 0 \text{ as } x \to -\infty)$
- Continuous

Graph of $y = a^{-x}$, $a > 1$

- Domain: $(-\infty, \infty)$, Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Decreasing
- x-axis is a horizontal asymptote: $(a^{-x} \to 0 \text{ as } x \to \infty)$
- Continuous
- Reflection of graph of $y = a^x$ in y-axis

■ Use the compound interest formulas. *25–29*

For n compoundings per year: $A = P(1 + r/n)^{nt}$

For continuous compounding: $A = Pe^{rt}$

■ Use an exponential model to solve an application problem. *29, 30*

Section 4.3

■ Recognize and evaluate a logarithmic function. *31–44*

$y = \log_a x$ if and only if $x = a^y$ $\qquad$ $y = \log_e x = \ln x$

■ Sketch the graph of a logarithmic function. *45–54*

Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Intercept: $(1, 0)$
- Increasing
- One-to-one; therefore has an inverse function
- y-axis is a vertical asymptote $(\log_a x \to -\infty \text{ as } x \to 0^+)$
- Continuous
- Reflection of graph of $y = a^x$ about the line $y = x$

■ Use a logarithmic model to solve an application problem. *55–58*

* A wide range of valuable study aids are available to help you master the material in this chapter. The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *www.cengagebrain.com* offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

Section 4.4

- Evaluate a logarithm using the change-of-base formula. *59–62*

$$\log_a x = \frac{\log_b x}{\log_b a}, \quad \log_a x = \frac{\log_{10} x}{\log_{10} a}, \quad \log_a x = \frac{\ln x}{\ln a}$$

- Use properties of logarithms to evaluate or rewrite a logarithmic expression. *63–70*

$$\log_a(uv) = \log_a u + \log_a v \qquad \ln(uv) = \ln u + \ln v$$

$$\log_a \frac{u}{v} = \log_a u - \log_a v \qquad \ln \frac{u}{v} = \ln u - \ln v$$

$$\log_a u^n = n \log_a u \qquad \ln u^n = n \ln u$$

- Use properties of logarithms to expand or condense a logarithmic expression. *71–82*
- Use logarithmic functions to model and solve real-life applications. *83, 84*

Section 4.5

- Solve an exponential equation. *85–90*
- Solve a logarithmic equation. *91–98*
- Use an exponential or a logarithmic model to solve an application problem. *99, 100*

Section 4.6

- Construct and use a model for exponential growth or exponential decay. *101–107*

$y = ae^{bx}, \quad b > 0 \qquad y = ae^{-bx}, \quad b > 0$

- Use a Gaussian model to solve an application problem. *108*

$y = ae^{-(x-b)^2/c}$

- Use a logistic growth model to solve an application problem. *109*

$$y = \frac{a}{1 + be^{-rx}}$$

- Use a logarithmic model to solve an application problem. *110, 111*

$y = a + b \ln x, \quad y = a + b \log_{10} x$

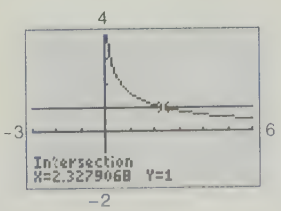
- Choose an appropriate model involving exponential or logarithmic functions for a real-life situation. *112*

Study Strategies

- **Solve Problems Algebraically or Graphically** When solving an exponential or logarithmic equation, you can use a variety of problem-solving strategies. For instance, you can solve the logarithmic equation $\ln(x + 4) - \ln x = 1$ *algebraically* or *graphically*.

To solve the equation *algebraically*, you can use the properties of logarithms to rewrite the equation, exponentiate each side, use the Inverse Property, and solve the resulting equation to determine that $x \approx 2.328$.

To solve the equation *graphically*, you can use a graphing utility to graph $y_1 = \ln(x + 4) - \ln x$ and $y_2 = 1$ in the same viewing window. Then use the *intersect* feature or the *zoom* and *trace* features to determine that the solution of the original equation is $x \approx 2.328$. (See figure.)



Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Verifying Inverse Functions In Exercises 1 and 2, show that f and g are inverse functions of each other.

1. $f(x) = 3x + 5, \quad g(x) = \dfrac{x - 5}{3}$
2. $f(x) = \sqrt[3]{x - 3}, \quad g(x) = x^3 + 3$

Finding Inverse Functions In Exercises 3 and 4, determine whether the function has an inverse function. If it does, find its inverse function and graph f and f^{-1} in the same coordinate plane.

3. $f(x) = \dfrac{1}{x}$
4. $f(x) = \dfrac{x^2}{x^2 - 9}$

Finding and Sketching Inverse Functions In Exercises 5 and 6, (a) find f^{-1}, (b) sketch the graphs of f and f^{-1} in the same coordinate plane, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

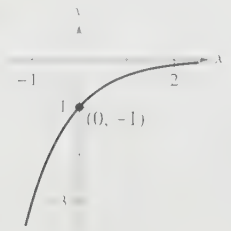
5. $f(x) = x^2, \quad x \geq 0$
6. $f(x) = \sqrt[3]{x - 1}$

Evaluating an Exponential Expression In Exercises 7–12, use a calculator to evaluate the expression. Round your result to three decimal places.

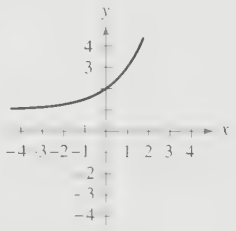
7. 4^{12}
8. $1.02^{-\pi}$
9. $2(100)^{-1/3}$
10. $7^{\sqrt{2}}$
11. $e^{3/4}$
12. $e^{-7.5}$

Matching In Exercises 13–16, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

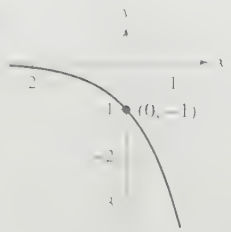
(a)



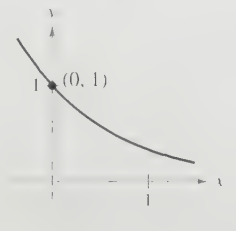
(b)



(c)



(d)



13. $f(x) = -3^x$
14. $f(x) = 3^{-x}$
15. $f(x) = -3^{-x}$
16. $f(x) = 2^x + 1$

Graphing Exponential Functions In Exercises 17–24, sketch the graph of the function.

17. $f(x) = 4^x$
18. $f(x) = 4^{x-1}$
19. $f(x) = \left(\frac{1}{2}\right)^x$
20. $f(x) = \left(\frac{1}{2}\right)^{x+1}$
21. $f(x) = 3e^{0.2x}$
22. $f(x) = 10e^{-0.1x}$
23. $f(x) = 3^{-x^2}$
24. $f(x) = 2^{1-x^2}$

Compound Interest In Exercises 25 and 26, complete the table to find the balance A for P dollars invested at rate r for t years, compounded n times per year.

n	1	2	4	12	365	Continuous
A						

25. $P = \$5000, \quad r = 8.5\%, \quad t = 12$ years
26. $P = \$8000, \quad r = 6.75\%, \quad t = 25$ years

Compound Interest In Exercises 27 and 28, complete the table to find the amount P that must be invested at rate r to obtain a balance of $A = \$200{,}000$ in t years.

t	1	10	20	30	40	50
P						

27. $r = 7.75\%$, compounded continuously
28. $r = 9.25\%$, compounded quarterly

29. **Investment Plan** You deposit \$6000 in a fund that yields 5.75% interest, compounded continuously. How much money will be in the fund after 6 years?
30. **Population** The population P of a town increases according to the model

$$P(t) = 15{,}000e^{0.025t}$$

where t is the time in years, with $t = 8$ corresponding to 2008. Use the model to approximate the population in 2009 and 2011.

Writing Logarithmic Equations In Exercises 31–34, write the exponential equation in logarithmic form.

31. $4^3 = 64$
32. $25^{3/2} = 125$
33. $e^2 = 7.3890\ldots$
34. $e^x = 8$

Writing Exponential Equations In Exercises 35–38, write the logarithmic equation in exponential form.

35. $\log_3 81 = 4$

36. $\log_5 0.2 = -1$

37. $\ln 1 = 0$

38. $\ln 4 = 1.3862\ldots$

Evaluating Logarithmic Expressions In Exercises 39–44, evaluate the expression without using a calculator.

39. $\log_2 32$

40. $\log_9 3$

41. $\ln e^7$

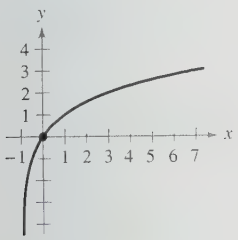
42. $\log_4 \frac{1}{4}$

43. $\ln e^{-1/2}$

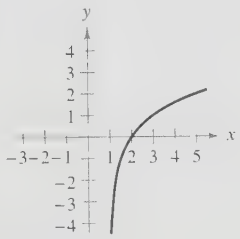
44. $\ln 1$

Matching In Exercises 45–48, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

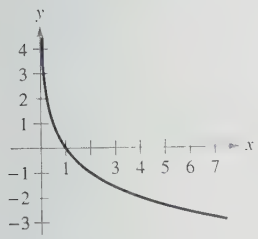
(a)



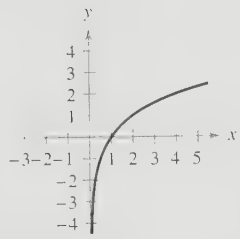
(b)



(c)



(d)



45. $f(x) = \log_2 x$

46. $f(x) = \log_2(x - 1)$

47. $f(x) = -\log_2 x$

48. $f(x) = \log_2(x + 1)$

Graphs of Exponential and Logarithmic Functions In Exercises 49 and 50, use the fact that f and g are inverse functions of each other to sketch their graphs in the same coordinate plane.

49. $f(x) = 10^x,\ g(x) = \log_{10} x$

50. $f(x) = e^x,\ g(x) = \ln x$

Sketching the Graphs of Logarithmic Functions In Exercises 51–54, find the domain, vertical asymptote, and x-intercept of the logarithmic function. Then sketch its graph.

51. $f(x) = \log_2(x - 3)$

52. $f(x) = 5 - 2\log_{10} x$

53. $g(x) = 2\ln x$

54. $g(x) = \ln(4 - x)$

55. Human Memory Model Students in a sociology class were given an exam and then retested monthly for 6 months with an equivalent exam. The average score for the class is given by the human memory model

$$f(t) = 82 - 16\log_{10}(t + 1), \quad 0 \le t \le 6$$

where t is the time (in months). How did the average score change over the six-month period?

56. Investment Time A principal P, invested at 5.85% interest compounded continuously, increases to an amount that is K times the principal after t years, where t is given by

$$t = \frac{\ln K}{0.0585}.$$

Complete the table and describe the result.

K	1	2	3	4	6	8	10
t							

57. Antler Spread The antler spread a (in inches) and shoulder height h (in inches) of an adult American elk are related by the model

$$h = 116\log_{10}(a + 40) - 176.$$

(a) Approximate the shoulder height of an elk with an antler spread of 55 inches.

(b) Use a graphing utility to graph the model.

58. Snow Removal The number of miles s of roads cleared of snow is approximated by the model

$$s = 25 - \frac{13\ln(h/12)}{\ln 3}, \quad 2 \le h \le 15$$

where h is the depth of the snow in inches.

(a) Use the model to find s when $h = 10$ inches.

(b) Use a graphing utility to graph the model.

Changing Bases In Exercises 59–62, evaluate the logarithm using the change-of-base formula. Evaluate each problem twice, once with common logarithms and once with natural logarithms. (Round your answer to three decimal places.)

59. $\log_3 10$

60. $\log_{1/4} 7$

61. $\log_{12} 200$

62. $\log_3 0.28$

Using Properties of Logarithms In Exercises 63–66, approximate the logarithm using the properties of logarithms, given $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.

63. $\log_b 6$

64. $\log_b \frac{4}{25}$

65. $\log_b \sqrt{3}$

66. $\log_b 30$

Using Inverse Properties In Exercises 67–70, find the *exact* value of the logarithm.

67. $\log_7 49$

68. $\log_6 \frac{1}{36}$

69. $\ln e^{3.2}$

70. $\ln \sqrt[5]{e^3}$

Expanding Logarithmic Expressions In Exercises 71–76, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms. (Assume that all variables are positive.)

71. $\log_{10} \frac{x}{y}$

72. $\log_{10} \frac{xy^3}{z^2}$

73. $\ln\left(x\sqrt{x-3}\right)$

74. $\ln \sqrt[3]{\frac{x^3}{y^2}}$

75. $\log_5 (y-3)^4$

76. $\log_2 2xy^2z$

Condensing Logarithmic Expressions In Exercises 77–82, condense the expression to the logarithm of a single quantity.

77. $\log_4 2 + \log_4 3$

78. $\ln y + 2 \ln z$

79. $\frac{1}{2} \ln x$

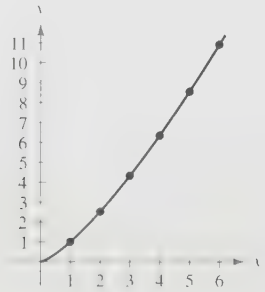
80. $4 \log_3 x + \log_3 y - 2 \log_3 z$

81. $\ln x - \ln(x-3) - \ln(x+1)$

82. $\log_{10}(x+2) + 2 \log_{10} x - 3 \log_{10}(x+4)$

83. Finding a Mathematical Model Find a logarithmic equation that relates y and x (see figure). Explain the steps used to find the equation.

x	1	2	3	4	5	6
y	1	2.520	4.327	6.350	8.550	10.903



84. Human Memory Model Students in a learning theory study were given an exam and then retested monthly for 6 months with an equivalent exam. The average scores for the class are shown in the table, with $t = 1$ representing 1 month after the initial exam. Use the table to find a logarithmic equation that relates s and t.

Month, t	1	2	3	4	5	6
Score, s	87.9	79.7	74.8	71.3	68.6	66.5

Solving Exponential and Logarithmic Equations In Exercises 85–98, solve the equation. Approximate the result to three decimal places.

85. $e^x = 8$

86. $2e^{x+1} = 7$

87. $3^{4x-1} - 4 = 23$

88. $2^{3x+1} + 5 = 133$

89. $e^{2x} - 3e^x - 4 = 0$

90. $e^{2x} - 8e^x + 12 = 0$

91. $\ln 3x = 8.2$

92. $2 \log_3 4x = 15$

93. $-2 + \ln 5x = 0$

94. $\ln 4x^2 = 21$

95. $\ln x - \ln 3 = 2$

96. $\log_3 x - \log_3 4 = 5$

97. $\log_2 \sqrt[3]{x+1} = 1$

98. $\ln \sqrt{x+1} = 2$

99. Demand Function The demand x (in units) for a desk is given by

$$p = 6000\left(1 - \frac{5}{5 + e^{-0.004x}}\right)$$

where p represents the price (in dollars) of the desk.

(a) Find the demand x for a price of $p = \$500$.

(b) Find the demand x for a price of $p = \$400$.

100. Demand Function The demand x (in units) for a bicycle is given by

$$p = 4000\left(1 - \frac{3}{3 + e^{-0.004x}}\right)$$

where p represents the price (in dollars) of the bicycle.

(a) Find the demand x for a price of $p = \$700$.

(b) Find the demand x for a price of $p = \$400$.

Radioactive Decay In Exercises 101 and 102, complete the table for the radioactive isotope.

	Isotope	*Half-Life (Years)*	*Initial Quantity*	*Amount After 1000 Years*
101.	^{14}C	5715	12 g	
102.	^{239}Pu	24,100		3.1 g

103. Population The population P of a city is given by



$$P = 185{,}000e^{0.018t}$$

where t represents the year, with $t = 10$ corresponding to 2010.

(a) Use a graphing utility to graph this equation.

(b) Use the model to predict the year in which the population of the city will reach 270,000.

104. Population The population P of a city is given by

$$P = 50{,}000e^{kt}$$

where t represents the year, with $t = 0$ corresponding to 2000. In 1990, the population was 34,500.

(a) Find the value of k and use this result to predict the population in the year 2025.

(b) Use a graphing utility to confirm the result of part (a).

105. Bacteria Growth The number of bacteria N in a culture is given by the model $N = 250e^{kt}$, where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 250$. When $t = 6$, there are 380 bacteria. How long does it take the bacteria population to double in size? To triple in size?

106. Bacteria Growth The number of bacteria N in a culture is given by the model

$$N = 200e^{kt}$$

where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 200$. When $t = 5$, there are 325 bacteria. How long does it take for the bacteria population to double in size? To triple in size?

107. Learning Curve The management at a factory has found that the maximum number of units a worker can produce in a day is 50. The learning curve for the number of units N produced per day after a new employee has worked t days is given by

$$N = 50(1 - e^{kt}).$$

After 20 days on the job, a particular worker produced 31 units in 1 day.

(a) Find the learning curve for this worker.

(b) How many days should pass before this worker is producing 45 units per day?

108. Test Scores The scores on a general aptitude test roughly follow a normal distribution given by

$$y = 0.0040e^{-[(x-300)^2]/20{,}000}, \quad 100 \le x \le 500.$$

Sketch the graph of this function. Estimate the average score on this test.

109. Wildlife Management A state parks and wildlife department releases 100 deer into a wilderness area. The department believes that the carrying capacity of the area is 400 deer and that the growth of the herd will be modeled by the logistic curve

$$P = \frac{400}{1 + 3e^{-kt}}, \quad t \ge 0$$

where P is the number of deer and t is the time (in years).

(a) The herd size is 135 after 2 years. Find k.

(b) Find the populations after 5 years, after 10 years, and after 20 years.

110. Perrigo Sales The annual sales S (in millions of dollars) for the Perrigo Company from 2004 through 2010 are shown in the table. *(Source: Perrigo Company)*

Year	Sales, S
2004	898.2
2005	1024.1
2006	1366.8
2007	1447.4
2008	1822.1
2009	2006.9
2010	2268.9

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 4$ corresponding to 2004.

(b) Use the *regression* feature of the graphing utility to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Use the *regression* feature of the graphing utility to find a logarithmic model for the data.

(d) Use the exponential model in base e and the logarithmic model to predict sales in 2011. It is projected that sales in 2011 will be \$2740 million. Do the predictions from the two models agree with this projection? Explain.

111. Thawing a Package of Steaks You take a package of steaks out of a freezer at 10 A.M. and place it in the refrigerator. Will the steaks be thawed in time to be grilled at 6 P.M.? Assume that the refrigerator temperature is 40°F and the freezer temperature is 0°F. Use the formula

$$t = -3.95 \ln \frac{T - 40}{0 - 40}$$

where t is the time in hours (with $t = 0$ corresponding to 10 A.M.) and T is the temperature of the package of steaks (in degrees Fahrenheit).

112. Earthquake Magnitudes On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity per unit of area for each value of R.

(a) $R = 8.4$ (b) $R = 6.85$ (c) $R = 9.1$

TEST YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, determine whether the function has an inverse function. If it does, find the inverse function and graph f and f^{-1} in the same coordinate plane.

1. $f(x) = 5x - 7$

2. $f(x) = x^2 + 1$

In Exercises 3–6, sketch the graph of the function.

3. $y = 2^x$

4. $y = e^{-2x}$

5. $y = \ln x$

6. $y = \log_3(x - 1)$

In Exercises 7 and 8, students in a psychology class were given an exam and then retested monthly with an equivalent exam. The average score $f(t)$ for the class is given by the human memory model

$f(t) = 87 - 15 \log_{10}(t + 1), \quad 0 \le t \le 18$

where t is the time (in months).

7. What was the average score on the original exam? After 2 months? After 4 months?

8. Find the average scores after 6 months, 12 months, and 18 months. What might these scores indicate about human memory?

In Exercises 9–12, expand the logarithmic expression. (Assume all variables are positive).

9. $\ln \dfrac{x^2y^3}{z}$

10. $\log_{10} 3xyz^2$

11. $\log_2\left(x\sqrt[3]{x-2}\right)$

12. $\log_8 \sqrt[5]{x^2+1}$

In Exercises 13 and 14, condense the logarithmic expression.

13. $2 \ln x + 3 \ln y - \ln z$

14. $\frac{2}{3}(\log_{10} x + \log_{10} y)$

In Exercises 15–18, solve the equation. Approximate the result to three decimal places.

15. $2^{4x} = 21$

16. $e^{2x} - 8e^x + 12 = 0$

17. $\log_2(x + 1) - 7 = 0$

18. $\ln \sqrt{x + 2} = 3$

19. You deposit \$40,000 in a fund that pays 6.75% interest, compounded continuously. When will the balance be greater than \$120,000?

20. The population P of a city is given by

$P = 85{,}000e^{0.025t}$

where t represents the year, with $t = 8$ corresponding to 2008. When will the city have a population of 125,000? Explain.

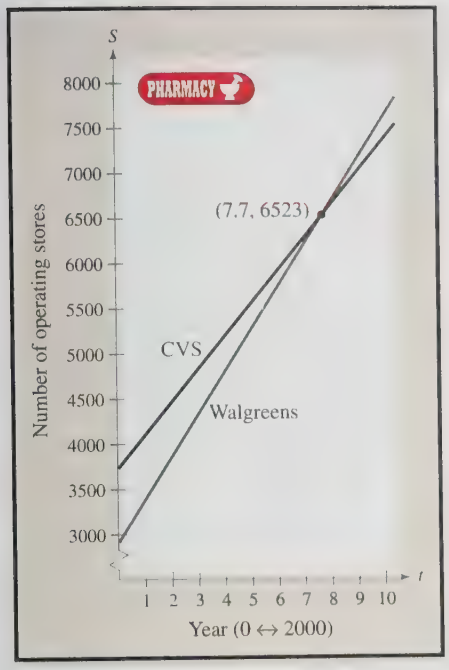
21. The number of bacteria N in a culture is given by

$N = 100e^{kt}$

where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 100$. When $t = 8$, $N = 175$. How long does it take the bacteria population to double?

22. Carbon 14 has a half-life of 5715 years. You have an initial quantity of 10 grams. How many grams will remain after 10,000 years? After 20,000 years?

23. Scientists release several bears on a small Alaskan island to start a bear population. Is the population likely to grow exponentially or logistically? Explain your reasoning.



Example 7 on page 416 shows how a system of linear equations can be used to estimate the year when the number of operating stores of one pharmacy chain exceeded the number of stores of another chain.

5 Systems of Equations and Inequalities

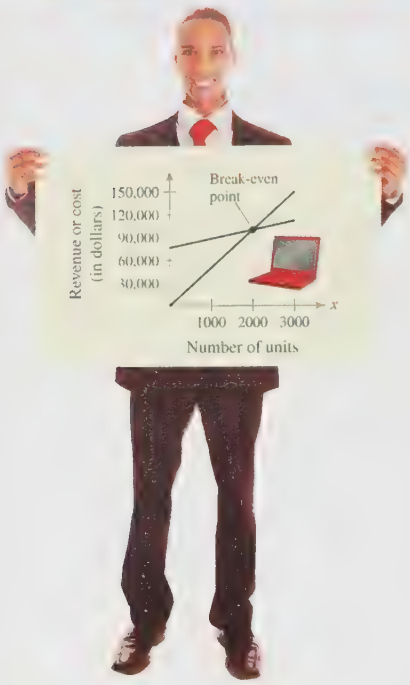
5.1 Solving Systems Using Substitution
5.2 Solving Systems Using Elimination
5.3 Linear Systems in Three or More Variables
5.4 Systems of Inequalities
5.5 Linear Programming





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5.1 Solving Systems Using Substitution



In Exercise 73 on page 419, you will use a system of equations to find the amount of sales a security software company needs to make in order to break even.

- Solve a system of equations by the method of substitution.
- Solve a system of equations graphically.
- Construct and use a system of equations to solve an application problem.

The Method of Substitution

So far, most problems in this text have involved either a function of one variable or a single equation in two variables. However, many problems in science, business, and engineering involve two or more equations in two or more variables. To solve such a problem, you need to find the solutions of a **system of equations.** Here is an example of a system of two equations in x and y.

$$\begin{cases} 2x + y = 5 & \text{Equation 1} \\ 3x - 2y = 4 & \text{Equation 2} \end{cases}$$

A **solution** of this system is an ordered pair that satisfies each equation in the system. The ordered pair $(2, 1)$ is a solution of this system. To check this, you can substitute 2 for x and 1 for y in *each* equation.

$$\begin{aligned} 2x + y &= 5 && \text{Write Equation 1.} \\ 2(2) + 1 &\stackrel{?}{=} 5 && \text{Substitute 2 for } x \text{ and 1 for } y. \\ 4 + 1 &\stackrel{?}{=} 5 && \text{Multiply.} \\ 5 &= 5 && \text{Solution checks in Equation 1.} \checkmark \end{aligned}$$

$$\begin{aligned} 3x - 2y &= 4 && \text{Write Equation 2.} \\ 3(2) - 2(1) &\stackrel{?}{=} 4 && \text{Substitute 2 for } x \text{ and 1 for } y. \\ 6 - 2 &\stackrel{?}{=} 4 && \text{Multiply.} \\ 4 &= 4 && \text{Solution checks in Equation 2.} \checkmark \end{aligned}$$

Finding the set of all solutions is called **solving the system of equations.** There are several different ways to solve systems of equations. In this chapter, you will study four of the most common techniques:

- *method of substitution,*
- *graphical approach,*
- *method of elimination,* and
- *Gaussian elimination.*

This section begins with the **method of substitution.**

Method of Substitution

1. *Solve* one of the equations for one variable in terms of the other.
2. *Substitute* the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. *Solve* the equation obtained in Step 2.
4. *Back-substitute* the value found in Step 3 into the expression obtained in Step 1 to find the value of the other variable.
5. *Check* that the solution satisfies *each* of the original equations.

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When using the method of substitution to solve a system of equations, it does not matter which variable you solve for first. You will obtain the same solution regardless. When making your choice, you should choose the variable that is easier to work with. For instance, solve for a variable that has a coefficient of 1 or -1 to avoid working with fractions.

Example 1 Solving a System of Two Equations by Substitution

Solve the system of equations.

$$\begin{cases} x + y = 4 & \text{Equation 1} \\ x - y = 2 & \text{Equation 2} \end{cases}$$

SOLUTION Begin by solving for y in Equation 1.

$$y = 4 - x \qquad \text{Revised Equation 1}$$

Next, substitute the expression $4 - x$ for y into Equation 2 and solve the resulting single-variable equation for x.

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ x - (4 - x) &= 2 && \text{Substitute } 4 - x \text{ for } y. \\ x - 4 + x &= 2 && \text{Distributive Property} \\ 2x &= 6 && \text{Combine like terms.} \\ x &= 3 && \text{Divide each side by 2.} \end{aligned}$$

Finally, you can solve for y by *back-substituting* $x = 3$ into the equation $y = 4 - x$.

$$\begin{aligned} y &= 4 - x && \text{Write revised Equation 1.} \\ y &= 4 - 3 && \text{Substitute 3 for } x. \\ y &= 1 && \text{Solve for } y. \end{aligned}$$

The solution is the ordered pair $(3, 1)$.

CHECK

Substitute $(3, 1)$ into Equation 1:

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\stackrel{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1.} \checkmark \end{aligned}$$

Substitute $(3, 1)$ into Equation 2:

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\stackrel{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2.} \checkmark \end{aligned}$$

TECH TUTOR

To check the solution in Example 1 with your graphing utility, graph $y_1 = -x + 4$ and $y_2 = x - 2$ in the same viewing window. Then use the *trace* feature to find the coordinates of the point of intersection. For instructions on how to use the *trace* feature, see Appendix A. Consult the user's guide for your graphing utility for more information.

STUDY TIP

Because many steps are required to solve a system of equations, it is easy to make errors in arithmetic. So, you should always check your solution by substituting it into *each* equation in the original system.

✓ Checkpoint 1

Solve the system of equations.

$$\begin{cases} x + y = 6 \\ x - y = 4 \end{cases}$$

The term *back-substitution* implies that you work *backwards*. First you solve for one of the variables, and then you substitute that value *back* into one of the equations in the system to find the value of the other variable.

Example 2 Solving a System by Substitution

A total of \$10,500 is invested in two funds paying 3% and 5% simple interest. (Recall that the formula for simple interest is $I = Prt$, where P is the principal, r is the annual interest rate, and t is the time.) The total annual interest is \$465. How much is invested at each rate?

SOLUTION

Verbal Model:

Amount in 3% fund + Amount in 5% fund = Total investment

Interest for 3% fund + Interest for 5% fund = Total interest

Labels:

Amount in 3% fund = x (dollars)
Interest for 3% fund = $0.03x$ (dollars)
Amount in 5% fund = y (dollars)
Interest for 5% fund = $0.05y$ (dollars)
Total investment = 10,500 (dollars)
Total interest = 465 (dollars)

System:

$$\begin{cases} x + y = 10{,}500 & \text{Equation 1} \\ 0.03x + 0.05y = 465 & \text{Equation 2} \end{cases}$$

To begin, it is convenient to multiply each side of Equation 2 by 100. This eliminates the need to work with decimals.

$100(0.03x + 0.05y) = 100(465)$ Multiply each side by 100.

$3x + 5y = 46{,}500$ Revised Equation 2

To solve this system, you can begin by solving for x in Equation 1.

$x = 10{,}500 - y$ Revised Equation 1

Then, substitute the expression $10{,}500 - y$ for x into revised Equation 2 and solve the resulting equation for y.

$3x + 5y = 46{,}500$ Write revised Equation 2.

$3(10{,}500 - y) + 5y = 46{,}500$ Substitute $10{,}500 - y$ for x.

$31{,}500 - 3y + 5y = 46{,}500$ Distributive Property

$2y = 15{,}000$ Combine like terms.

$y = 7500$ Divide each side by 2.

Next, back-substitute $y = 7500$ into the equation $x = 10{,}500 - y$ to solve for x.

$x = 10{,}500 - y$ Write revised Equation 1.

$x = 10{,}500 - 7500$ Substitute 7500 for y.

$x = 3000$ Solve for x.

The solution is (3000, 7500). So, \$3000 is invested at 3% and \$7500 is invested at 5%. Check this in the original system.

✓ Checkpoint 2

In Example 2, a total of \$15,000 is invested in the same two funds. The total annual interest is \$670. How much is invested at each rate? ■

The equations in Examples 1 and 2 are linear. The method of substitution can also be used to solve systems in which one or both of the equations are nonlinear.

Example 3 **Substitution: Two-Solution Case**

Solve the system of equations.

$$\begin{cases} x^2 - x - y = 1 & \text{Equation 1} \\ -x + y = -1 & \text{Equation 2} \end{cases}$$

SOLUTION Begin by solving for y in Equation 2 to obtain $y = x - 1$. Next, substitute the expression $x - 1$ for y into Equation 1 and solve for x.

$$\begin{aligned} x^2 - x - y &= 1 && \text{Write Equation 1} \\ x^2 - x - (x - 1) &= 1 && \text{Substitute for } y \\ x^2 - 2x + 1 &= 1 && \text{Simplify.} \\ x^2 - 2x &= 0 && \text{General form} \\ x(x - 2) &= 0 && \text{Factor.} \\ x &= 0 && \text{Set 1st factor equal to 0} \\ x - 2 = 0 \quad \Rightarrow \quad x &= 2 && \text{Set 2nd factor equal to 0} \end{aligned}$$

Back-substituting these values of x to solve for the corresponding values of y produces the two solutions $(0, -1)$ and $(2, 1)$. Check these solutions in the original system.

STUDY TIP

Note that in Example 3, solving Equation 2 for y instead of x makes substitution into Equation 1 less tedious.

When using the method of substitution, you may encounter an equation that has no solution, as shown in Example 4.

Example 4 **Substitution: No-Real-Solution Case**

Solve the system of equations.

$$\begin{cases} -x + y = 4 & \text{Equation 1} \\ x^2 + y = 3 & \text{Equation 2} \end{cases}$$

SOLUTION Begin by solving for y in Equation 1 to obtain $y = x + 4$. Next, substitute the expression $x + 4$ for y into Equation 2 and solve for x.

$$\begin{aligned} x^2 + y &= 3 && \text{Write Equation 2.} \\ x^2 + (x + 4) &= 3 && \text{Substitute } x + 4 \text{ for } y \\ x^2 + x + 1 &= 0 && \text{Simplify.} \\ x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} && \text{Use the Quadratic Formula} \\ x &= \frac{-1 \pm \sqrt{-3}}{2} && \text{Simplify.} \end{aligned}$$

Because the discriminant is negative, the equation $x^2 + x + 1 = 0$ has no (real) solution. So, this system has no (real) solution.

✓ Checkpoints 3 and 4

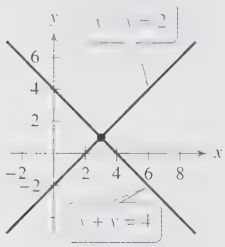
Solve each system of equations.

a. $\begin{cases} x^2 + 4x - y = 7 \\ 2x - y = -1 \end{cases}$

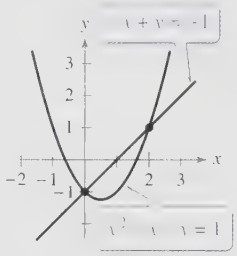
b. $\begin{cases} 2x^2 - y = 1 \\ x + y = -2 \end{cases}$

Graphical Approach to Finding Solutions

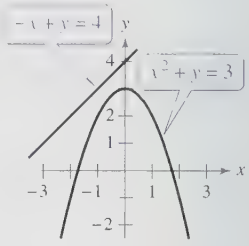
From Examples 1, 3, and 4, you can see that a system of two equations in two unknowns can have exactly one solution, more than one solution, or no solution. In practice, you can gain insight about the location and number of solutions of a system of equations by graphing each of the equations in the same coordinate plane. The solution(s) of the system correspond to the **point(s) of intersection** of the graphs. For instance, the graph of the system in Example 1 is two lines with a *single point* of intersection, as shown in Figure 5.1(a). The graph of the system in Example 3 is a parabola and a line with *two points* of intersection, as shown in Figure 5.1(b). The graph of the system in Example 4 is a line and a parabola that have *no points* of intersection, as shown in Figure 5.1(c).


One point of intersection

(a) One solution


Two points of intersection

(b) Two solutions


No points of intersection

(c) No solution

FIGURE 5.1

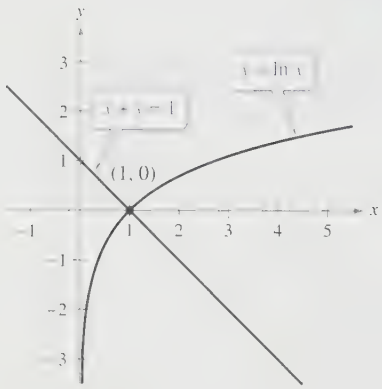


FIGURE 5.2

Example 5 **Solving a System of Equations Graphically**

Solve the system of equations graphically.

$$\begin{cases} y = \ln x & \text{Equation 1} \\ x + y = 1 & \text{Equation 2} \end{cases}$$

SOLUTION The graph of each equation is shown in Figure 5.2. From the graph, you can see that there is only one point of intersection. So, it appears that $(1, 0)$ is the solution point. You can confirm this by substituting 1 for x and 0 for y in *both* equations.

CHECK

Equation 1: $0 = \ln 1$ ✓

Equation 2: $1 + 0 = 1$ ✓

✓ Checkpoint 5

Solve the system of equations graphically.

$$\begin{cases} 2x + 2y = 8 \\ y = \ln \frac{1}{4}x \end{cases}$$

TECH TUTOR

Your graphing utility may have an *intersect* feature that approximates the point(s) of intersection of two graphs. Use the *intersect* feature to verify the solution of Example 5. For instructions on how to use the *intersect* feature, see Appendix A. Consult the user's guide for your graphing utility for more information.

Applications

The total cost C of producing x units of a product typically has two components—the initial cost and the cost per unit. When enough units have been sold so that the total revenue R equals the total cost C, the sales are said to have reached the **break-even point.** You will find that the break-even point corresponds to the point of intersection of the cost and revenue graphs.



In 2009, the average price for cross-training shoes in the United States was $50.23. *(Source: National Sporting Goods Association)*

Example 6 Break-Even Analysis

A shoe company invests $300,000 in equipment to produce cross-training shoes. Each pair of shoes costs $10 to produce and is sold for $75. How many pairs of shoes must be sold before the company breaks even?

SOLUTION The total cost of producing x units is

$$\begin{matrix}\text{Total}\\ \text{cost}\end{matrix} = \begin{matrix}\text{Cost per}\\ \text{unit}\end{matrix} \cdot \begin{matrix}\text{Number}\\ \text{of units}\end{matrix} + \begin{matrix}\text{Initial}\\ \text{cost}\end{matrix}$$

$$C = 10x + 300{,}000.$$

The total revenue obtained by selling x units is

$$\begin{matrix}\text{Total}\\ \text{revenue}\end{matrix} = \begin{matrix}\text{Price per}\\ \text{unit}\end{matrix} \cdot \begin{matrix}\text{Number}\\ \text{of units}\end{matrix}$$

$$R = 75x.$$

Because the break-even point occurs when $R = C$, you have $C = 75x$, and the system of equations to solve is

$$\begin{cases} C = 10x + 300{,}000 & \text{Equation 1} \\ C = 75x & \text{Equation 2} \end{cases}$$

Now, you can solve by substitution.

$$C = 10x + 300{,}000 \qquad \text{Write Equation 1.}$$

$$75x = 10x + 300{,}000 \qquad \text{Substitute } 75x \text{ for } C.$$

$$65x = 300{,}000 \qquad \text{Subtract } 10x \text{ from each side.}$$

$$x = \frac{300{,}000}{65} \qquad \text{Divide each side by 65.}$$

$$x \approx 4615 \qquad \text{Use a calculator.}$$

The company must sell about 4615 pairs of shoes to break even. Note in Figure 5.3 that sales less than the break-even point correspond to an overall loss, whereas sales greater than the break-even point correspond to a profit.

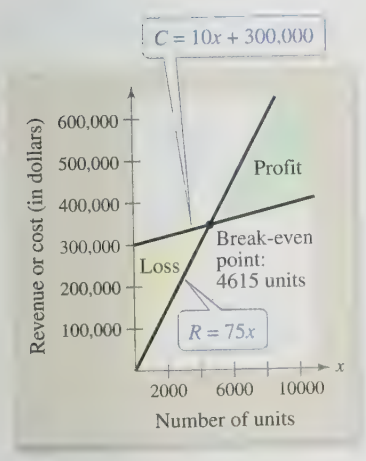


FIGURE 5.3

✓ Checkpoint 6

In Example 6, each pair of shoes costs $12 to produce. How many pairs of shoes must be sold before the business breaks even?

Another way to view the solution in Example 6 is to consider the profit function

$$P = R - C. \qquad \text{Profit function}$$

The break-even point occurs when the profit is 0, which is the same as saying that

$$R = C. \qquad \text{Profit is 0.}$$

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Example 7 Operating Stores

From 2000 through 2010, the number of Walgreens stores in operation increased more quickly than the number of CVS stores in operation. Models that represent the numbers of operating stores of the two pharmacy chains are given by

$S = 362.24t + 3728.0$ CVS Caremark Corporation

$S = 467.94t + 2912.5$ Walgreen Company

where S is the number of operating stores and t represents the year, with $t = 0$ corresponding to 2000. Use the models to estimate when the number of Walgreens stores in operation first exceeded the number of CVS stores in operation. *(Source: CVS Caremark Corporation and Walgreen Company)*

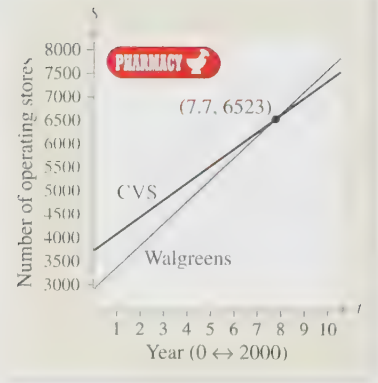


FIGURE 5.4

SOLUTION Both equations have already been solved for S in terms of t. So, substitute the value of S from the first equation into the second equation and solve for t.

$S = 467.94t + 2912.5$ Write second equation.

$362.24t + 3728.0 = 467.94t + 2912.5$ Substitute $362.24t + 3728.0$ for S.

$-105.7t + 3728.0 = 2912.5$ Subtract $467.94t$ from each side.

$-105.7t = -815.5$ Subtract 3728.0 from each side.

$t = \frac{-815.5}{-105.7}$ Divide each side by -105.7.

$t \approx 7.7$ Use a calculator.

Because the equations are equal when $t \approx 7.7$, you can conclude that the number of Walgreens stores first exceeded the number of CVS stores in 2007, as shown in Figure 5.4.

✓Checkpoint 7

From 2002 through 2009, the number of Abercrombie & Fitch stores in operation increased more quickly than the number of American Eagle stores in operation. Models that represent the numbers of operating stores of the two clothing companies are given by

$S = 76.81t + 469.5$ Abercrombie & Fitch Company

$S = 52.14t + 634.7$ American Eagle Outfitters, Inc.

where S is the number of operating stores and t represents the year, with $t = 2$ corresponding to 2002. Use the models to estimate when the number of Abercrombie & Fitch stores in operation first exceeded the number of American Eagle stores in operation. *(Source: Abercrombie & Fitch Company and American Eagle Outfitters, Inc.)*



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SUMMARIZE (Section 5.1)

1. Describe how to solve a system of equations using the method of substitution *(page 410)*. For examples of solving systems of equations using the method of substitution, see Examples 1, 2, 3, and 4.
2. Describe how to solve a system of equations graphically *(page 414)*. For an example of solving a system of equations graphically, see Example 5.
3. State the definition of a break-even point *(page 415)*. For an example of using a system of equations to find a break-even point, see Example 6.

SKILLS WARM UP 5.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.5, 1.1, 1.3, and 2.1.

In Exercises 1–4, sketch the graph of the equation.

1. $y = -\frac{1}{3}x + 6$
2. $y = 2(x - 3)$
3. $x^2 + y^2 = 4$
4. $y = 5 - (x - 3)^2$

In Exercises 5–8, perform the indicated operations and simplify.

5. $(3x + 2y) - 2(x + y)$
6. $(-10u + 3v) + 5(2u - 8v)$
7. $x^2 + (x - 3)^2 + 6x$
8. $y^2 - (y + 1)^2 + 2y$

In Exercises 9 and 10, solve the equation.

9. $3x + (x - 5) = 15 + 4$
10. $y^2 + (y - 2)^2 = 2$

Exercises 5.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Checking Solutions In Exercises 1–6, determine whether each ordered pair is a solution of the system of equations.

1. $\begin{cases} x + 4y = -3 \\ 5x - y = 6 \end{cases}$
   (a) $(-1, -1)$
   (b) $(1, -1)$
2. $\begin{cases} 2x - y = 2 \\ x + 3y = 8 \end{cases}$
   (a) $(2, 1)$
   (b) $(2, 2)$
3. $\begin{cases} 2x + 5y = -5 \\ 2x - y^2 = 1 \end{cases}$
   (a) $(5, -3)$
   (b) $(0, -1)$
4. $\begin{cases} 4x^2 + y = 3 \\ -x - y = 11 \end{cases}$
   (a) $(-2, -9)$
   (b) $(2, -13)$
5. $\begin{cases} y = -2e^x \\ 3x - y = 2 \end{cases}$
   (a) $(-2, 0)$
   (b) $(-1, 2)$
6. $\begin{cases} -\log_{10} x + 3 = y \\ \frac{1}{9}x + y = \frac{28}{9} \end{cases}$
   (a) $(1, 3)$
   (b) $\left(9, \frac{37}{9}\right)$

Solving a System by Substitution In Exercises 7–18, solve the system by the method of substitution. *See Example 1.*

7. $\begin{cases} 2x - y = -3 \\ -3x - 4y = -1 \end{cases}$
8. $\begin{cases} x + 2y = 1 \\ 5x - 4y = -23 \end{cases}$
9. $\begin{cases} 2x - y + 2 = 0 \\ 4x + y - 5 = 0 \end{cases}$
10. $\begin{cases} 6x - 3y - 4 = 0 \\ x + 2y - 4 = 0 \end{cases}$
11. $\begin{cases} x - y = 7 \\ 2x + y = 23 \end{cases}$
12. $\begin{cases} x - 2y = -2 \\ 3x - y = 6 \end{cases}$
13. $\begin{cases} 0.3x - 0.4y - 0.33 = 0 \\ 0.1x + 0.2y - 0.21 = 0 \end{cases}$
14. $\begin{cases} 1.5x + 0.8y = 2.3 \\ 0.3x - 0.2y = 0.1 \end{cases}$
15. $\begin{cases} \frac{1}{5}x + \frac{1}{2}y = 8 \\ x + y = 20 \end{cases}$
16. $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 10 \\ \frac{3}{2}x - y = 4 \end{cases}$
17. $\begin{cases} 6x + 5y = -3 \\ -x - \frac{5}{6}y = -7 \end{cases}$
18. $\begin{cases} -\frac{2}{3}x + y = 2 \\ 2x - 3y = 6 \end{cases}$

Solving a System by Substitution In Exercises 19–22, you are given the total annual interest earned from a total of $12,000 invested in two funds paying the given rates of simple interest. Write and solve a system of equations to find the amount invested at each rate. *See Example 2.*

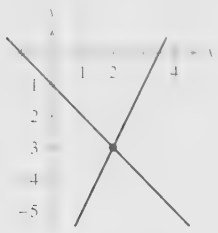
	Annual Interest	*Rate 1*	*Rate 2*
19.	$500	3%	5%
20.	$630	4%	6%
21.	$396	2.8%	3.8%
22.	$254	1.75%	2.25%

Solving a System with a Nonlinear Equation In Exercises 23–32, solve the system by the method of substitution. *See Examples 3 and 4.*

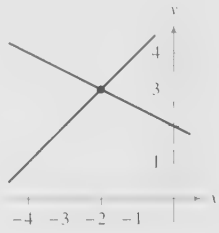
23. $\begin{cases} y = 2x \\ y = x^2 - 1 \end{cases}$
24. $\begin{cases} x + y = 4 \\ x^2 - y = 2 \end{cases}$
25. $\begin{cases} 3x - 7y + 6 = 0 \\ x^2 - y^2 = 4 \end{cases}$
26. $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$
27. $\begin{cases} x - 2y = 4 \\ x^2 - y = 0 \end{cases}$
28. $\begin{cases} x^2 + y^2 = 9 \\ x - y = -5 \end{cases}$
29. $\begin{cases} y = x^4 - 2x^2 + 1 \\ y = 1 - x^2 \end{cases}$
30. $\begin{cases} y = x^3 - 2x^2 + x - 1 \\ y = -x^2 + 3x - 1 \end{cases}$
31. $\begin{cases} xy - 2 = 0 \\ y = \sqrt{x - 1} \end{cases}$
32. $\begin{cases} xy = 3 \\ y = \sqrt{x - 2} \end{cases}$

Solving a System by Substitution In Exercises 33–38, solve the system by the method of substitution. Then use the graph to confirm your solution.

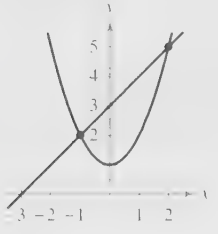
33. $\begin{cases} x + y = -1 \\ -2x + y = -7 \end{cases}$



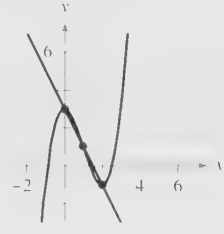
34. $\begin{cases} x - y = -5 \\ x + 2y = 4 \end{cases}$



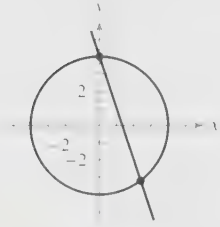
35. $\begin{cases} x - y = -3 \\ x^2 - y = -1 \end{cases}$



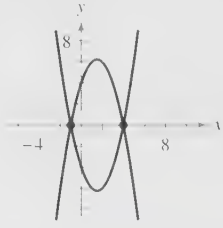
36. $\begin{cases} y = x^3 - 3x^2 + 3 \\ 2x + y = 3 \end{cases}$



37. $\begin{cases} 3x + y = 4 \\ x^2 + y^2 = 16 \end{cases}$



38. $\begin{cases} y = x^2 - 3x - 4 \\ y = -x^2 + 3x + 4 \end{cases}$



Solving a System of Equations Graphically In Exercises 39–50, solve the system graphically. *See Example 5.*

39. $\begin{cases} -x + 2y = 2 \\ 3x + y = 15 \end{cases}$

40. $\begin{cases} x + y = 0 \\ 3x - 2y = 10 \end{cases}$

41. $\begin{cases} x - 3y = -2 \\ 5x + 3y = 17 \end{cases}$

42. $\begin{cases} -x + 2y = 1 \\ x - y = 2 \end{cases}$

43. $\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$

44. $\begin{cases} -x + y = 3 \\ x^2 - 6x - 27 + y^2 = 0 \end{cases}$

45. $\begin{cases} x - y + 3 = 0 \\ x^2 - 4x + 7 = y \end{cases}$

46. $\begin{cases} y^2 - 4x + 11 = 0 \\ -\frac{1}{2}x + y = -\frac{1}{2} \end{cases}$

47. $\begin{cases} 7x + 8y = 24 \\ x - 8y = 8 \end{cases}$

48. $\begin{cases} x - y = 0 \\ 5x - 2y = 6 \end{cases}$

49. $\begin{cases} 3x - 2y = 0 \\ x^2 - y^2 = 4 \end{cases}$

50. $\begin{cases} 2x - y + 3 = 0 \\ x^2 + y^2 - 4x = 0 \end{cases}$

Finding the Number of Solutions In Exercises 51–56, use a graphing utility to determine whether the system of equations has *one solution, two solutions,* or *no solution.*

51. $\begin{cases} y = -5x + 1 \\ y = x + 3 \end{cases}$

52. $\begin{cases} -\frac{1}{2}x + y = -1 \\ 7x + y = 2 \end{cases}$

53. $\begin{cases} y = x^2 + 2x - 1 \\ y = 2x + 5 \end{cases}$

54. $\begin{cases} x^2 + 3x + y = 4 \\ 3x + y = -5 \end{cases}$

55. $\begin{cases} y = x^2 + 3x + 7 \\ y = -x^2 - 3x + 1 \end{cases}$

56. $\begin{cases} -10x + y = 2 \\ -10x + y = -3 \end{cases}$

Solving a System of Equations Graphically In Exercises 57–62, use a graphing utility to find the point(s) of intersection of the graphs. Then confirm your solution algebraically.

57. $\begin{cases} y = x^2 + 3x - 1 \\ y = -x^2 - 2x + 2 \end{cases}$

58. $\begin{cases} y = -2x^2 + x - 1 \\ y = x^2 - 2x - 1 \end{cases}$

59. $\begin{cases} x - y + 3 = 0 \\ x^2 - 4x + 7 = y \end{cases}$

60. $\begin{cases} x - y = 3 \\ x - y^2 = 1 \end{cases}$

61. $\begin{cases} y = e^x \\ x - y + 1 = 0 \end{cases}$

62. $\begin{cases} x^2 + y^2 = 8 \\ y = x^2 + 4 \end{cases}$

Choosing a Solution Method In Exercises 63–66, solve the system graphically or algebraically. Explain your choice of method.

63. $\begin{cases} 4x - y = 0 \\ x^2 - y = -3 \end{cases}$

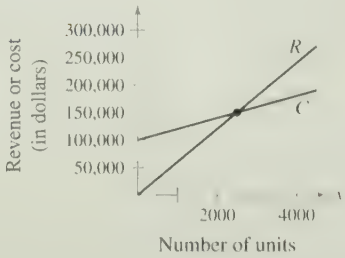
64. $\begin{cases} x^2 + y^2 = 4 \\ x + y = 6 \end{cases}$

65. $\begin{cases} x^2 + y^2 = 25 \\ \frac{1}{2}x + y = \frac{5}{2} \end{cases}$

66. $\begin{cases} y = \sqrt{x + 7} \\ y = x + 1 \end{cases}$

67. Think About It Create systems of two equations in two variables that have (a) one solution, (b) two solutions, and (c) no solution. (There are many correct answers.)

68. HOW DO YOU SEE IT? The cost C of producing x units and the revenue R obtained by selling x units are shown in the figure.



(a) Estimate the point of intersection. What does this point represent?

(b) Identify the area in the figure that corresponds to an overall loss. Identify the area that corresponds to a profit. Explain your reasoning.

Break-Even Analysis In Exercises 69–72, find the sales necessary to break even ($R = C$) for the cost C of producing x units and the revenue R obtained by selling x units. (Round your answer to the nearest whole unit.) *See Example 6.*

69. $C = 8650x + 250{,}000$; $R = 9950x$

70. $C = 5.5\sqrt{x} + 10{,}000$; $R = 3.29x$

71. $C = 2.65x + 350{,}000$; $R = 4.15x$

72. $C = 0.08x + 50{,}000$; $R = 0.25x$

73. Break-Even Analysis A security software company invests \$76,500 to produce software packages that will sell for \$49.95 each. Each software package can be produced for \$11.70. How many software packages must the company sell to break even?

74. Break-Even Analysis You invest \$3000 in a fishing lure business. A lure costs \$1.06 to produce and will be sold for \$5.86. How many lures must you sell to break even?

75. Comparing Populations From 2000 through 2009, the population of Alabama grew more slowly than that of Colorado. Models that represent the populations of the two states are given by

$$\begin{cases} P = 30.2t + 4420 & \text{Alabama} \\ P = 73.6t + 4332 & \text{Colorado} \end{cases}$$

where P is the population (in thousands) and t represents the year, with $t = 0$ corresponding to 2000. Use the models to estimate when the population of Colorado first exceeded the population of Alabama. *(Source: U.S. Census Bureau)*

76. Comparing Populations From 2000 through 2009, the population of Indiana grew more slowly than that of Arizona. Models that represent the populations of the two states are given by

$$\begin{cases} P = 37.6t + 6078 & \text{Indiana} \\ P = 168.5t + 5132 & \text{Arizona} \end{cases}$$

where P is the population (in thousands) and t represents the year, with $t = 0$ corresponding to 2000. Use the models to estimate when the population of Arizona first exceeded the population of Indiana. *(Source: U.S. Census Bureau)*

77. Body Mass Index Body mass index (BMI) is a measure of body fat based on height and weight. Models that represent the 85th percentile BMI for males and females, ages 9 to 20, are given by

$$\begin{cases} B = 0.77a + 11.7 & \text{Males} \\ B = 0.68a + 13.5 & \text{Females} \end{cases}$$

where B is the BMI (kg/m^2) and a represents the age, with $a = 9$ corresponding to 9 years old. Use a graphing utility to determine whether the BMI for males exceeds the BMI for females. *(Source: Centers for Disease Control and Prevention)*

78. Sales Per Share From 2000 through 2009, the sales per share of Ross Stores, Inc. grew faster than those of TJX Companies, Inc. Models that represent the sales per share of the two companies are given by

$$\begin{cases} S = 4.576t + 13.56 & \text{Ross Stores, Inc} \\ S = 3.745t + 16.14 & \text{TJX Companies, Inc} \end{cases}$$

where S is the sales per share (in dollars) and t represents the year, with $t = 0$ corresponding to 2000. Use a graphing utility to determine whether the sales per share of Ross Stores, Inc. will exceed the sales per share of TJX Companies, Inc. *(Source: Ross Stores, Inc. and TJX Companies, Inc.)*

79. Job Choices You are offered two jobs. Company A offers an annual salary of \$48,000 plus a year-end bonus of 2.5% of your total sales. Company B offers a salary of \$44,000 plus a year-end bonus of 6.5% of your total sales. What is the amount you must sell in one year to earn the same salary working for either company?

80. Camping You are choosing between camping outfitters. Outfitter A charges a reservation fee of \$150 plus a daily guide fee of \$70. Outfitter B charges a reservation fee of \$75 plus a daily guide fee of \$90. Estimate when the cost of Outfitter A equals the cost of Outfitter B.

81. Annual Sales The annual sales of food and beverage stores and general merchandise stores in the United States from 1992 through 2008 can be approximated by

$$\begin{cases} S = 0.026t^3 - 0.36t^2 + 10.3t + 348 & \text{Food \& beverage} \\ S = -0.024t^3 + 1.12t^2 + 8.2t + 231 & \text{General merchandise} \end{cases}$$

where S is the annual sales (in billions of dollars) and t represents the year, with $t = 2$ corresponding to 1992. Use a graphing utility to determine when the sales of general merchandise stores first exceeded the sales of food and beverage stores. Do you think these models will continue to be accurate? Explain your reasoning. *(Source: U.S. Census Bureau)*

82. Consumer Spending The amount spent per person per year on video games and newspapers in the United States from 1996 through 2008 can be approximated by

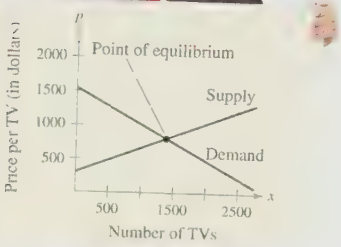
$$\begin{cases} A = 0.077t^3 - 2.63t^2 + 31.1t - 99 & \text{Video games} \\ A = -0.10t^2 + 2.0t + 44 & \text{Newspapers} \end{cases}$$

where A is the amount spent (in dollars) and t represents the year, with $t = 6$ corresponding to 1996. Use a graphing utility to determine when the amount spent per person per year on video games first exceeded the amount spent per person per year on newspapers. Do you think these models will continue to be accurate? Explain your reasoning. *(Source: Veronis Suhler Stevenson)*

5.2 Solving Systems Using Elimination

- Solve a linear system by the method of elimination.
- Interpret the solution of a linear system graphically.
- Construct and use a linear system to solve an application problem.







In Exercise 59 on page 430, you will use the method of elimination to solve a system of linear equations to find the equilibrium point that satisfies supply and demand equations.

The Method of Elimination

In Section 5.1, you studied two methods for solving a system of equations: substitution and graphing. In this section, you will study a third method called the **method of elimination.** The key step in the method of elimination is to obtain, for one of the variables, coefficients that differ only in sign, so that *adding* the two equations eliminates this variable. Here is an example.

$$\begin{aligned} 3x + 5y &= 7 && \text{Equation 1} \\ -3x - 2y &= -1 && \text{Equation 2} \\ \hline 3y &= 6 && \text{Add equations.} \end{aligned}$$

Note that by adding the two equations, you eliminate the variable x and obtain a single equation in y. Solving this equation for y produces $y = 2$, which you can then back-substitute into one of the original equations to solve for x.

Example 1 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 3x + 2y = 4 & \text{Equation 1} \\ 5x - 2y = 8 & \text{Equation 2} \end{cases}$$

SOLUTION Because the coefficients of the y-terms differ only in sign, you can eliminate the y-terms by adding the two equations. This leaves you with a single equation in x.

$$\begin{aligned} 3x + 2y &= 4 && \text{Write Equation 1.} \\ 5x - 2y &= 8 && \text{Write Equation 2.} \\ \hline 8x &= 12 && \text{Add equations.} \end{aligned}$$

Solving the equation $8x = 12$ produces $x = \frac{3}{2}$. By back-substituting $x = \frac{3}{2}$ into Equation 1, you can solve for y.

$$\begin{aligned} 3x + 2y &= 4 && \text{Write Equation 1.} \\ 3\left(\tfrac{3}{2}\right) + 2y &= 4 && \text{Substitute } \tfrac{3}{2} \text{ for } x. \\ \tfrac{9}{2} + 2y &= 4 && \text{Multiply.} \\ 2y &= -\tfrac{1}{2} && \text{Subtract } \tfrac{9}{2} \text{ from each side.} \\ y &= -\tfrac{1}{4} && \text{Divide each side by 2.} \end{aligned}$$

The solution is $\left(\frac{3}{2}, -\frac{1}{4}\right)$. Check this in the original system.

Checkpoint 1

Solve the system of linear equations.

$$\begin{cases} 2x - 3y = 5 \\ 5x + 3y = 9 \end{cases}$$

STUDY TIP

It is important that you understand both the method of substitution and the method of elimination. In some cases, it is more efficient to use one method instead of the other. For instance, when two equations are written such that one variable is represented in terms of the other variable, substitution is a more efficient solution method.

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To obtain coefficients (for one of the variables) that differ only in sign, you may need to multiply one or both of the equations by a suitable constant or constants, as demonstrated in Example 2.

Example 2 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 2x - 3y = -7 & \text{Equation 1} \\ 3x + y = -5 & \text{Equation 2} \end{cases}$$

SOLUTION For this system, you can obtain coefficients that differ only in sign by multiplying Equation 2 by 3. Then, by adding the two equations, you can eliminate the y-terms. This leaves you with a single equation in x.

$$\begin{aligned} 2x - 3y &= -7 \quad \Rightarrow \quad 2x - 3y = -7 && \text{Write Equation 1.} \\ 3x + y &= -5 \quad \Rightarrow \quad 9x + 3y = -15 && \text{Multiply Equation 2 by 3.} \\ & \qquad\quad 11x \quad\;\; = -22 && \text{Add equations.} \end{aligned}$$

Solving the equation $11x = -22$ produces $x = -2$. By back-substituting $x = -2$ into Equation 1, you can solve for y.

$$\begin{aligned} 2x - 3y &= -7 && \text{Write Equation 1.} \\ 2(-2) - 3y &= -7 && \text{Substitute } -2 \text{ for } x. \\ -4 - 3y &= -7 && \text{Multiply.} \\ -3y &= -3 && \text{Add 4 to each side.} \\ y &= 1 && \text{Divide each side by } -3. \end{aligned}$$

The solution is $(-2, 1)$.

CHECK

$$2(-2) - 3(1) \stackrel{?}{=} -7 \qquad 3(-2) + 1 \stackrel{?}{=} -5$$
$$-4 - 3 \stackrel{?}{=} -7 \qquad -6 + 1 \stackrel{?}{=} -5$$
$$-7 = -7 \checkmark \qquad -5 = -5 \checkmark$$

Checkpoint 2

Solve the system of linear equations.

$$\begin{cases} 3x - 5y = -1 \\ x - 2y = -1 \end{cases}$$

In Example 2, the two systems of linear equations

$$\begin{cases} 2x - 3y = -7 \\ 3x + y = -5 \end{cases} \quad \text{and} \quad \begin{cases} 2x - 3y = -7 \\ 9x + 3y = -15 \end{cases}$$

are called **equivalent systems** because they have precisely the same solution set. The operations that can be performed on a system of linear equations to produce an equivalent system are:

(1) interchanging any two equations,

(2) multiplying an equation by a nonzero constant, and

(3) adding a multiple of one equation to any other equation in the system.

The Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in x and y, use the following steps.

1. Examine the system to determine which variable is easiest to eliminate.
2. Obtain coefficients of x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
3. Add the equations to eliminate one variable and solve the resulting equation.
4. Back-substitute the value obtained in Step 3 into either of the original equations and solve for the other variable.
5. Check your solution in both of the original equations.

Example 3 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 5x + 3y = 9 & \text{Equation 1} \\ 2x - 4y = 14 & \text{Equation 2} \end{cases}$$

SOLUTION You can obtain coefficients of y that differ only in sign by multiplying Equation 1 by 4 and multiplying Equation 2 by 3.

$$\begin{array}{rcll} 5x + 3y = 9 & \Rightarrow & 20x + 12y = 36 & \text{Multiply Equation 1 by 4.} \\ 2x - 4y = 14 & \Rightarrow & 6x - 12y = 42 & \text{Multiply Equation 2 by 3.} \\ & & 26x \quad\quad = 78 & \text{Add equations.} \end{array}$$

Solving the equation $26x = 78$ produces $x = 3$. By back-substituting $x = 3$ into Equation 2, you can solve for y.

$$\begin{array}{ll} 2x - 4y = 14 & \text{Write Equation 2.} \\ 2(3) - 4y = 14 & \text{Substitute 3 for } x. \\ y = -2 & \text{Solve for } y. \end{array}$$

The solution is $(3, -2)$.

CHECK

$$5(3) + 3(-2) \stackrel{?}{=} 9 \qquad 2(3) - 4(-2) \stackrel{?}{=} 14$$
$$15 + (-6) \stackrel{?}{=} 9 \qquad 6 - (-8) \stackrel{?}{=} 14$$
$$9 = 9 \checkmark \qquad 14 = 14 \checkmark$$

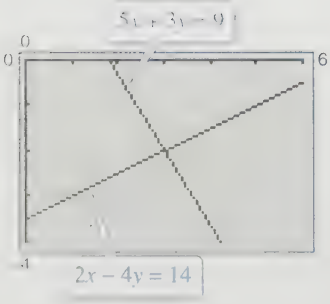


FIGURE 5.5

Checkpoint 3

Solve the system of linear equations.

$$\begin{cases} -2x + 7y = -15 \\ 3x - 5y = 6 \end{cases}$$

Remember that you can check the solution of a system of equations graphically. For instance, to check the solution found in Example 3, graph both equations in the same viewing window, as shown in Figure 5.5. Notice that the two lines intersect at $(3, -2)$.

Example 4 illustrates a strategy for solving a system of linear equations that has decimal coefficients.

Example 4 A Linear System Having Decimal Coefficients

Solve the system of linear equations.

$$\begin{cases} 0.02x - 0.05y = -0.38 & \text{Equation 1} \\ 0.03x + 0.04y = 1.04 & \text{Equation 2} \end{cases}$$

SOLUTION Because the coefficients in this system have two decimal places, you can begin by multiplying each equation by 100. This produces a system in which the coefficients are all integers.

$$\begin{cases} 2x - 5y = -38 & \text{Revised Equation 1} \\ 3x + 4y = 104 & \text{Revised Equation 2} \end{cases}$$

Now, to obtain coefficients of x that differ only in sign, multiply revised Equation 1 by 3 and revised Equation 2 by -2.

$$\begin{array}{rcrl} 2x - 5y = -38 & \Rightarrow & 6x - 15y = -114 & \text{Multiply by 3} \\ 3x + 4y = 104 & \Rightarrow & -6x - 8y = -208 & \text{Multiply by } -2 \\ & & -23y = -322 & \text{Add equations} \end{array}$$

Solving the equation $-23y = -322$ produces $y = 14$. Now, back-substitute $y = 14$ into any of the original or revised equations of the system that contain the variable y. By back-substituting $y = 14$ into revised Equation 2, you can solve for x.

$$\begin{array}{rl} 3x + 4y = 104 & \text{Write revised Equation 2} \\ 3x + 4(14) = 104 & \text{Substitute 14 for } y \\ 3x = 48 & \text{Subtract 56 from each side} \\ x = 16 & \text{Divide each side by 3} \end{array}$$

The solution is (16, 14). Check this in the original system.

✓ Checkpoint 4

Solve the system of linear equations.

$$\begin{cases} 0.03x + 0.04y = -0.13 \\ -0.04x + 0.05y = -0.24 \end{cases}$$



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TECH TUTOR

The general solution of the linear system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

is

$$x = (ce - bf)/(ae - bd) \quad \text{and} \quad y = (af - cd)/(ae - bd).$$

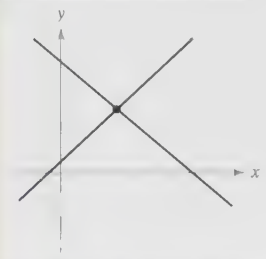
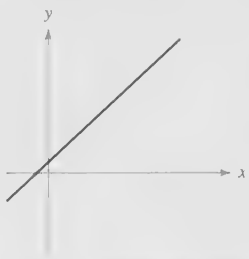
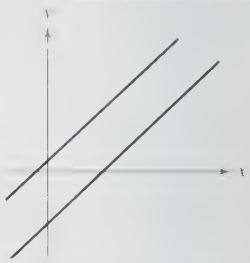
If $ae - bd = 0$, then the system does not have a unique solution. Several graphing utilities have programs for solving such a system. Try using one of these programs to check the solution of the system in Example 4.

Graphical Interpretation of Solutions

It is possible for a *general* system of equations to have

- exactly one solution,
- two or more solutions, or
- no solution.

If a system of *linear* equations has two different solutions, then it must have an *infinite* number of solutions. To see why this is true, consider the following graphical interpretations of systems of two linear equations in two variables. (Remember that the graph of a linear equation in two variables is a line.)

Graph			
Graphical interpretation	The two lines intersect.	The two lines coincide (are identical).	The two lines are parallel.
Intersection	Single point of intersection	Infinitely many points of intersection	No point of intersection
Slopes of lines	Slopes are not equal.	Slopes are equal.	Slopes are equal.
Number of solutions	Exactly one solution	Infinitely many solutions	No solution
Type of system	**Independent (consistent) system**	**Dependent (consistent) system**	**Inconsistent system**

STUDY TIP

Keep in mind that the terminology and methods discussed in this section and the following section apply only to systems of *linear* equations.

A system of linear equations is *consistent* if it has at least one solution. A consistent system with exactly one solution is *independent*, whereas a consistent system with infinitely many solutions is *dependent*. A system is *inconsistent* if it has no solution.

From the graphs above, you can see that a comparison of the slopes and y-intercepts of two lines is helpful in determining the number of solutions of the corresponding system of equations. For instance:

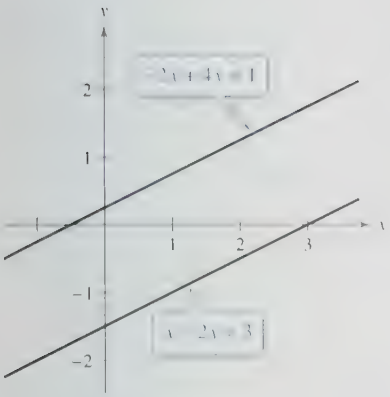
- *Independent (consistent) systems* have lines with slopes that are not equal.
- *Dependent (consistent) systems* have lines with equal slopes and the same y-intercept.
- *Inconsistent systems* have lines with equal slopes, but different y-intercepts.

So, when solving a system of linear equations graphically, it is helpful to know the slope of each line. Writing each linear equation in the slope-intercept form

$$y = mx + b \qquad \text{Slope-intercept form}$$

enables you to identify the slopes quickly.

In Examples 5 and 6, note how you can use the method of elimination to determine when a linear system has no solution or infinitely many solutions.



No Solution
FIGURE 5.6

Example 5 The Method of Elimination: No-Solution Case

Solve the system of linear equations.

$$\begin{cases} x - 2y = 3 & \text{Equation 1} \\ -2x + 4y = 1 & \text{Equation 2} \end{cases}$$

SOLUTION Obtain coefficients that differ only in sign.

$$\begin{array}{rcll} x - 2y = 3 & \Rightarrow & 2x - 4y = 6 & \text{Multiply Equation 1 by 2.} \\ -2x + 4y = 1 & \Rightarrow & -2x + 4y = 1 & \text{Write Equation 2.} \\ & & 0 = 7 & \text{False statement} \end{array}$$

Because there are no values of x and y for which $0 = 7$, you can conclude that the system is inconsistent and has no solution. The graphs of the equations are shown in Figure 5.6. Note that the two lines have equal slopes, but different y-intercepts. So, the lines are parallel and have no point of intersection.

Checkpoint 5

Solve the system of linear equations.

$$\begin{cases} -3x + 6y = 5 \\ x - 2y = 2 \end{cases}$$

In Example 5, note that the occurrence of a false statement, such as $0 = 7$, indicates that the system has no solution. In the next example, note that the occurrence of a statement that is true for all values of the variables, such as $0 = 0$, indicates that the system has infinitely many solutions.

Example 6 The Method of Elimination: Many-Solutions Case

Solve the system of linear equations.

$$\begin{cases} 2x - y = 1 & \text{Equation 1} \\ 4x - 2y = 2 & \text{Equation 2} \end{cases}$$

SOLUTION Obtain coefficients that differ only in sign.

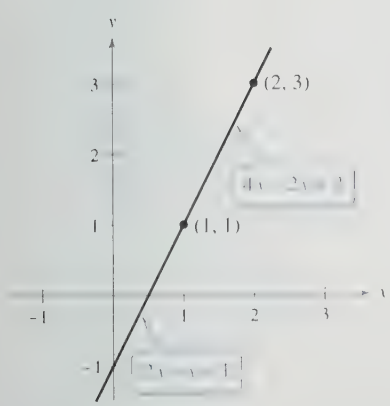
$$\begin{array}{rcll} 2x - y = 1 & \Rightarrow & 2x - y = 1 & \text{Write Equation 1.} \\ 4x - 2y = 2 & \Rightarrow & -2x + y = -1 & \text{Multiply Equation 2 by } -\frac{1}{2}. \\ & & 0 = 0 & \text{True statement} \end{array}$$

Because the two equations are equivalent (have the same solution set), you can conclude that the system is consistent and has infinitely many solutions. The solution set consists of all points (x, y) lying on the line

$$2x - y = 1$$

as shown in Figure 5.7. To represent the solution set as an ordered pair, let $x = a$, where a is any real number. Then $y = 2a - 1$, and the solution set can be written as

$$(a, 2a - 1).$$



Infinitely Many Solutions
FIGURE 5.7

Checkpoint 6

Solve the system of linear equations.

$$\begin{cases} -x - y = 5 \\ 4x + 4y = -20 \end{cases}$$

Applications

At this point, you may be asking the question, "How can I tell which application problems can be solved using a system of linear equations?" The answer comes from the following considerations.

1. Does the problem involve more than one unknown quantity?
2. Are there two (or more) equations or conditions to be satisfied?

If the answer to one or both of these questions is "yes," then the appropriate model for the problem may be a system of linear equations. Example 7 shows how to construct such a model.

Example 7 An Application of a Linear System

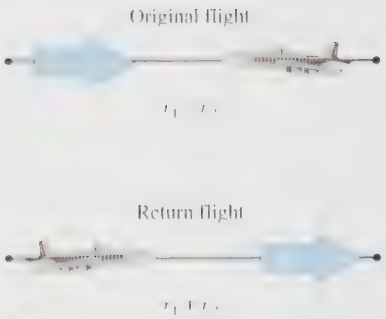


FIGURE 5.8

An airplane flying into a headwind travels the 2000-mile flying distance between Wilmington, Delaware and Tucson, Arizona in 4 hours and 24 minutes. On the return flight, the same distance is traveled in 4 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

SOLUTION The two unknown quantities are the speeds of the wind and the plane. If r_1 is the airspeed of the plane and r_2 is the speed of the wind, then

$r_1 - r_2$ = speed of the plane *against* the wind

$r_1 + r_2$ = speed of the plane *with* the wind

as shown in Figure 5.8. Using the formula distance = (rate)(time) for these two speeds, you obtain the following equations.

$$2000 = (r_1 - r_2)\left(4 + \frac{24}{60}\right)$$

$$2000 = (r_1 + r_2)(4)$$

These two equations simplify as shown.

$$\begin{cases} 5000 = 11r_1 - 11r_2 & \text{Equation 1} \\ 500 = r_1 + r_2 & \text{Equation 2} \end{cases}$$

To solve this system by elimination, multiply Equation 2 by 11.

$$\begin{array}{rcl} 5000 = 11r_1 - 11r_2 & \Rightarrow & 5000 = 11r_1 - 11r_2 \quad \text{Write Equation 1.} \\ \underline{500 = r_1 + r_2} & \Rightarrow & \underline{5500 = 11r_1 + 11r_2} \quad \text{Multiply Equation 2 by 11.} \\ & & 10{,}500 = 22r_1 \quad \text{Add equations.} \end{array}$$

Solving the equation $10{,}500 = 22r_1$ produces

$$r_1 = \frac{10{,}500}{22} = \frac{5250}{11} \approx 477.27.$$

By back-substituting $r_1 = 477.27$ into original Equation 2, you can solve for r_2.

$$500 = 477.27 + r_2 \quad \text{Substitute 477.27 for } r_1.$$

$$22.73 = r_2 \quad \text{Subtract 477.27 from each side.}$$

So, the airspeed of the plane is about 477.27 miles per hour and the speed of the wind is about 22.73 miles per hour. Check this solution in the original statement of the problem.

ALGEBRA TUTOR

For help with the algebra used to simplify the system of equations in Example 7, see the *Chapter 7 Algebra Tutor* on page 464.

Checkpoint 7

In Example 7, the return flight takes 4 hours and 6 minutes. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

In a free market, the demands for many products are related to the prices of the products. As the prices decrease, the demands by consumers increase and the amounts that producers are able or willing to supply decrease.

Example 8 Finding the Point of Equilibrium

The demand and supply equations for a video game console are given by

$$p = 350 - 0.00003x \quad \text{Demand equation}$$

$$p = 200 + 0.00003x \quad \text{Supply equation}$$

where p is the price (in dollars) and x is the number of video game consoles. For how many units will the quantity demanded equal the quantity supplied? What price corresponds to this value?

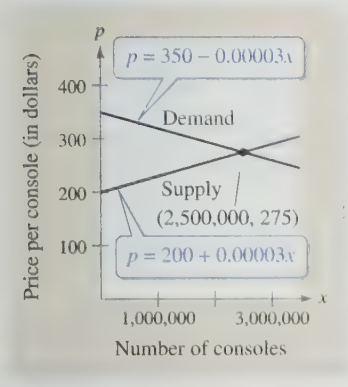


FIGURE 5.9

SOLUTION To obtain coefficients of p that differ only in sign, multiply the demand equation by -1.

$$p = 350 - 0.00003x \quad \Rightarrow \quad -p = -350 + 0.00003x \quad \text{Multiply demand equation by } -1.$$

$$p = 200 + 0.00003x \quad \Rightarrow \quad p = 200 + 0.00003x \quad \text{Write supply equation}$$

$$0 = -150 + 0.00006x \quad \text{Add equations}$$

By solving the equation $0 = -150 + 0.00006x$, you get $x = 2{,}500{,}000$. So, the quantity demanded equals the quantity supplied for 2,500,000 units (see Figure 5.9). The price that corresponds to this x-value is obtained by back-substituting $x = 2{,}500{,}000$ into either of the original equations. For instance, back-substituting into the demand equation produces

$$p = 350 - 0.00003x \quad \text{Demand equation}$$

$$= 350 - 0.00003(2{,}500{,}000) \quad \text{Substitute 2,500,000 for } x$$

$$= \$275. \quad \text{Simplify}$$

Back-substitute $x = 2{,}500{,}000$ into the supply equation to see that you obtain the same price. The solution (2,500,000, 275) is called the *point of equilibrium*. The **point of equilibrium** is the price p and the number of units x that satisfy both the demand and supply equations.



✓ Checkpoint 8

In Example 8, find the point of equilibrium when the supply equation is $p = 230 + 0.00003x$.

SUMMARIZE (Section 5.2)

1. Describe how to solve a system of linear equations using the method of elimination *(pages 420–422)*. For examples of solving systems of linear equations using the method of elimination, see Examples 1, 2, 3, and 4.
2. Explain what it means for a system of linear equations to be an independent system, a dependent system, or an inconsistent system *(page 424)*. For examples of solving these types of linear systems using the method of elimination, see Examples 5 and 6.
3. State the definition of the point of equilibrium *(page 427)*. For an example of solving a system of supply and demand equations to find the point of equilibrium, see Example 8.

SKILLS WARM UP 5.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 2.2.

In Exercises 1 and 2, sketch the graph of the equation.

1. $2x + y = 4$

2. $5x - 2y = 3$

In Exercises 3 and 4, find an equation of the line passing through the two points.

3. $(-1, 3), (4, 8)$

4. $(2, 6), (5, 1)$

In Exercises 5 and 6, determine the slope of the line.

5. $3x + 6y = 4$

6. $7x - 4y = 10$

In Exercises 7–10, determine whether the lines represented by the pair of equations are parallel, perpendicular, or neither.

7. $2x - 3y = -10$, $3x + 2y = 11$

8. $4x - 12y = 5$, $-2x + 6y = 3$

9. $5x + y = 2$, $3x + 2y = 1$

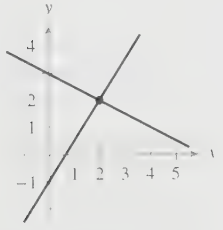
10. $x - 3y = 2$, $6x + 2y = 4$

Exercises 5.2

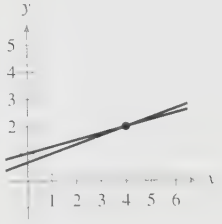
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Solving a System by Elimination In Exercises 1–8, solve the system by the method of elimination. Then use the graph to confirm your solution. Copy the graph and label each line with the appropriate equation. *See Examples 1, 2, 3, 5, and 6.*

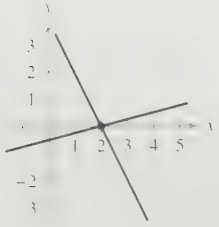
1. $\begin{cases} 3x - 2y = 2 \\ x + 2y = 6 \end{cases}$



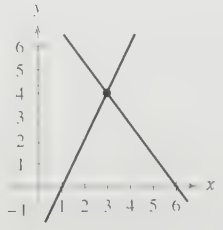
2. $\begin{cases} -x + 3y = 2 \\ x - 4y = -4 \end{cases}$



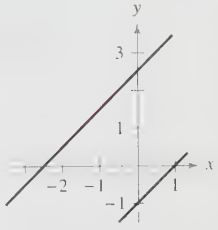
3. $\begin{cases} x - 4y = 2 \\ 2x + y = 4 \end{cases}$



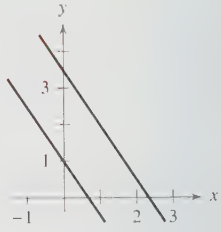
4. $\begin{cases} 2x - y = 2 \\ 4x + 3y = 24 \end{cases}$



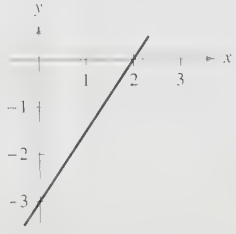
5. $\begin{cases} x - y = 1 \\ -2x + 2y = 5 \end{cases}$



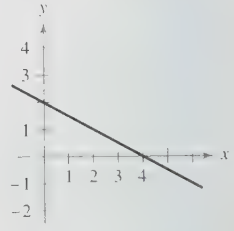
6. $\begin{cases} 3x + 2y = 2 \\ 6x + 4y = 14 \end{cases}$



7. $\begin{cases} 3x - 2y = 6 \\ -6x + 4y = -12 \end{cases}$



8. $\begin{cases} 2x + 4y = 8 \\ 6x + 12y = 24 \end{cases}$



Solving a System by Elimination In Exercises 9–24, solve the system by the method of elimination. Then state whether the system is consistent or inconsistent. *See Examples 1, 2, 3, 5, and 6.*

9. $\begin{cases} x + 2y = 3 \\ x - 2y = 1 \end{cases}$

10. $\begin{cases} 2x - 3y = 4 \\ -2x - y = 4 \end{cases}$

11. $\begin{cases} 4x - 3y = 11 \\ -6x + 3y = 3 \end{cases}$

12. $\begin{cases} 3x - 5y = 2 \\ 2x + 5y = 13 \end{cases}$

13. $\begin{cases} 3x - y = 17 \\ 5x + 5y = -5 \end{cases}$

14. $\begin{cases} x + 7y = 12 \\ 3x - 5y = 10 \end{cases}$

15. $\begin{cases} 3x + 2y = 10 \\ 2x + 5y = 3 \end{cases}$

16. $\begin{cases} 8r + 16s = 20 \\ 16r + 50s = 55 \end{cases}$

17. $\begin{cases} 2u + v = 120 \\ u + 2v = 120 \end{cases}$

18. $\begin{cases} 5u + 6v = 24 \\ 3u + 5v = 18 \end{cases}$

19. $\begin{cases} 4b + 3m = 3 \\ 3b + 11m = 13 \end{cases}$

20. $\begin{cases} 3b + 3m = 7 \\ 3b + 5m = 3 \end{cases}$

21. $\begin{cases} \frac{x}{4} + \frac{y}{6} = 1 \\ x - y = 3 \end{cases}$

22. $\begin{cases} \frac{1}{6}x - \frac{2}{3}y = 3 \\ 3x + y = 15 \end{cases}$

23. $\begin{cases} \frac{x+3}{4} + \frac{y-1}{3} = 1 \\ x - y = 3 \end{cases}$

24. $\begin{cases} \frac{x-1}{2} + \frac{y+2}{3} = 4 \\ x - 2y = 5 \end{cases}$

Writing and Solving a System In Exercises 25–28, use the given statements to write a system of equations. Solve the system by the method of elimination.

25. The sum of a number x and a number y is 13. The difference of x and y is 3.

26. The sum of a number a and a number b is 43. The difference of a and b is -27.

27. The sum of twice a number r and a number s is 8. The difference of r and s is 7.

28. The difference of a number m and twice a number n is 1. The sum of two times m and n is 22.

Solving a System Having Decimal Coefficients In Exercises 29–34, solve the system by the method of elimination. Then state whether the system is consistent or inconsistent. *See Examples 4, 5, and 6.*

29. $\begin{cases} 6r - 5s = 3 \\ -1.2r + s = 0.5 \end{cases}$

30. $\begin{cases} 1.8x + 1.2y = 4 \\ 9x + 6y = 3 \end{cases}$

31. $\begin{cases} 2.5x - 3y = 1.5 \\ 10x - 12y = 6 \end{cases}$

32. $\begin{cases} 1.5x + 2y = 3.75 \\ 7.5x + 10y = 18.75 \end{cases}$

33. $\begin{cases} 0.05x - 0.03y = 0.21 \\ 0.07x + 0.02y = 0.16 \end{cases}$

34. $\begin{cases} 0.02x - 0.05y = -0.19 \\ 0.03x + 0.04y = 0.52 \end{cases}$

Choosing a Solution Method In Exercises 35–42, use any method to solve the system. Explain your choice of method.

35. $\begin{cases} 8x + 12y = 8 \\ 16x + 30y = 22 \end{cases}$

36. $\begin{cases} x - 9y = -7 \\ 5x + 9y = -35 \end{cases}$

37. $\begin{cases} -3x + 10y = -84 \\ y = x - 7 \end{cases}$

38. $\begin{cases} \frac{1}{8}x - \frac{3}{4}y = -3 \\ 4x + y = 29 \end{cases}$

39. $\begin{cases} y = e^x \\ 2x - 3y + 3 = 0 \end{cases}$

40. $\begin{cases} x + 5y = -11 \\ 8x - 6y = 50 \end{cases}$

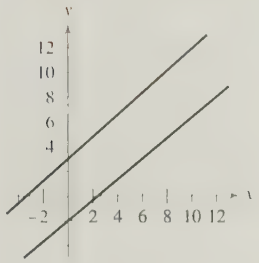
41. $\begin{cases} 4x + 3y = 140 \\ 3x + 4y = 140 \end{cases}$

42. $\begin{cases} y = x^3 - 2x^2 \\ 10x - y = -7 \end{cases}$

43. Think About It Find a system of linear equations that has the given solution. (There are many correct answers.)

(a) $(6, 0)$ (b) $(-3, 4)$ (c) $\left(-\frac{3}{4}, -7\right)$

44. HOW DO YOU SEE IT? Use the graphs of the two equations shown below.



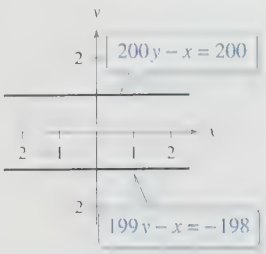
(a) Describe the graphs of the two equations.

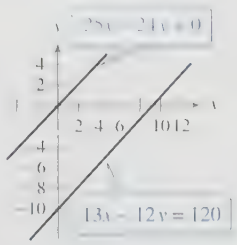
(b) Can you conclude that the system of equations shown in the graph is inconsistent? Explain.

Recognizing a System of Equations In Exercises 45 and 46, the two lines appear to be parallel. Are they? Justify your answer by using the method of elimination to solve the system.

45. $\begin{cases} 200y - x = 200 \\ 199y - x = -198 \end{cases}$

46. $\begin{cases} 25x - 24y = 0 \\ 13x - 12y = 120 \end{cases}$





47. Airplane Speed An airplane flying into a headwind travels the 1800-mile flying distance between Los Angeles, California and South Bend, Indiana in 3 hours and 36 minutes. On the return flight, the distance is traveled in 3 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

48. Airplane Speed Two planes start from the same airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 50 miles per hour faster. Find the airspeed of each plane if, 2 hours after the first plane departs, the planes are 2000 miles apart.

49. Acid Mixture Ten gallons of a 30% acid solution is obtained by mixing a 20% solution with a 50% solution. How much of each solution is required to obtain the specified concentration of the final mixture?

50. Fuel Mixture Five hundred gallons of 89-octane gasoline is obtained by mixing 87-octane gasoline with 92-octane gasoline. How much of each type of gasoline is required to obtain the specified mixture? (Octane ratings can be interpreted as percents.)

51. Investment Portfolio A total of $25,000 is invested in two corporate bonds that pay 4.75% and 5.5% simple interest. The total annual interest is $1315. How much is invested in each bond?

52. Shoe Sales You are the manager of a shoe store. On Sunday morning, you are going over the receipts for the previous week's sales. A total of 320 pairs of cross-training shoes were sold. One style sold for $56.95 and the other sold for $72.95. The total receipts were $21,024. The cash register that was supposed to keep track of the number of each type of shoe sold malfunctioned. Can you recover the information? If so, how many of each type were sold?

Finding the Point of Equilibrium In Exercises 53–56, find the point of equilibrium for the demand and supply equations. *See Example 8.*

	Demand	*Supply*
53.	$p = 56 - 0.0001x$	$p = 22 + 0.00001x$
54.	$p = 60 - 0.00001x$	$p = 15 + 0.00004x$
55.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
56.	$p = 400 - 0.0002x$	$p = 225 + 0.0005x$

57. Restaurants The sales y (in billions of dollars) of fast-food and full-service restaurants for the years 2002 through 2009 are shown in the table. *(Source: National Restaurant Association)*

Year	Fast-food	Full-service
2002	115.1	141.9
2003	120.5	148.3
2004	129.5	156.9
2005	136.9	165.2
2006	142.9	172.8
2007	150.8	179.1
2008	157.7	189.4
2009	160.0	182.0

(a) Use a spreadsheet software program to create a scatter plot of the data for fast-food sales and use the *regression* feature to find a linear model. Let x represent the year, with $x = 2$ corresponding to 2002. Repeat the procedure for the data for full-service sales.

(b) Assuming that the amounts for the given 8 years are representative of future years, will fast-food sales ever equal full-service sales? Discuss the appropriateness of using these models to determine whether fast-food sales will ever equal full-service sales.

58. Prescriptions The numbers of prescriptions y (in thousands) filled at two pharmacies in the years 2007 through 2012 are shown in the table.

Year	Pharmacy A	Pharmacy B
2007	18.6	19.9
2008	19.2	20.4
2009	19.6	20.8
2010	20.0	21.1
2011	20.4	21.4
2012	21.3	22.0

(a) Use a spreadsheet software program to create a scatter plot of the data for pharmacy A and use the *regression* feature to find a linear model. Let x represent the year, with $x = 7$ corresponding to 2007. Repeat the procedure for the data for pharmacy B.

(b) Assuming that the amounts for the given 6 years are representative of future years, will the number of prescriptions filled at pharmacy A ever exceed the number of prescriptions filled at pharmacy B?

59. Supply and Demand The supply and demand equations for a small LCD television are given by

$$\begin{cases} p + 0.53x = 1542 & \text{Demand} \\ p - 0.37x = 300 & \text{Supply} \end{cases}$$

where p is the price (in dollars) and x represents the number of televisions. For how many units will the quantity demanded equal the quantity supplied? What price corresponds to this value?

60. Supply and Demand The supply and demand equations for a microscope are given by

$$\begin{cases} p + 0.85x = 650 & \text{Demand} \\ p - 0.4x = 75 & \text{Supply} \end{cases}$$

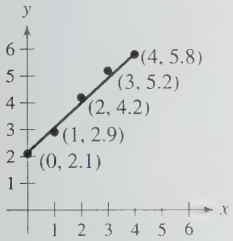
where p is the price (in dollars) and x represents the number of microscopes. For how many units will the quantity demanded equal the quantity supplied? What price corresponds to this value?

Fitting a Line to Data To find the least squares regression line $y = ax + b$ for a set of points (x_1, y_1), (x_2, y_2), . . ., (x_n, y_n), you can solve the following system for a and b. (If you are unfamiliar with summation notation, then look at the discussion in Section 7.1.)

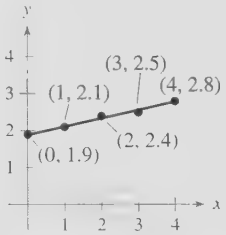
$$\begin{cases} nb + \left(\sum_{i=1}^{n} x_i\right)a = \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} x_i y_i \end{cases}$$

In Exercises 61–64, the sums have been evaluated. Solve the given system of linear equations for a and b to find the least squares regression line for the points. Use a graphing utility to confirm the result.

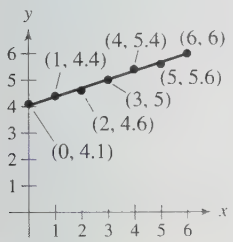
61. $\begin{cases} 5b + 10a = 20.2 \\ 10b + 30a = 50.1 \end{cases}$



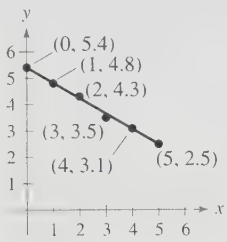
62. $\begin{cases} 5b + 10a = 11.7 \\ 10b + 30a = 25.6 \end{cases}$



63. $\begin{cases} 7b + 21a = 35.1 \\ 21b + 91a = 114.2 \end{cases}$



64. $\begin{cases} 6b + 15a = 23.6 \\ 15b + 55a = 48.8 \end{cases}$



65. Atmosphere The concentration y (in parts per million) of carbon dioxide in the atmosphere is measured at the Mauna Loa Observatory in Hawaii. The greatest monthly carbon dioxide concentrations for the years 2006 through 2010 are shown in the table. *(Source: Scripps CO2 Program)*

Year	t	Concentration, y
2006	0	384.94
2007	1	386.43
2008	2	388.49
2009	3	390.18
2010	4	393.22

(a) Solve the following system for a and b to find the least squares regression line $y = at + b$ for the data. Let t represent the year, with $t = 0$ corresponding to 2006.

$$\begin{cases} 5b + 10a = 1943.26 \\ 10b + 30a = 3906.83 \end{cases}$$

(b) Use a graphing utility to graph the regression line and predict the greatest monthly carbon dioxide concentration in 2016.

(c) Use the *regression* feature of the graphing utility to find a linear model for the data. Compare this model with the one you found in part (a).

66. Stores in Operation The numbers of stores y operated by BJ's Wholesale Club, Inc. for the years 2005 through 2009 are shown in the table. *(Source: BJ's Wholesale Club, Inc.)*

Year	2005	2006	2007	2008	2009
t	0	1	2	3	4
Stores, y	165	172	177	180	187

(a) Solve the following system for a and b to find the least squares regression line $y = at + b$ for the data. Let t represent the year, with $t = 0$ corresponding to 2005.

$$\begin{cases} 5b + 10a = 881 \\ 10b + 30a = 1814 \end{cases}$$

(b) Use a graphing utility to graph the regression line and predict the number of stores in operation in 2011.

(c) Use the *regression* feature of the graphing utility to find a linear model for the data. Compare this model with the one you found in part (a).



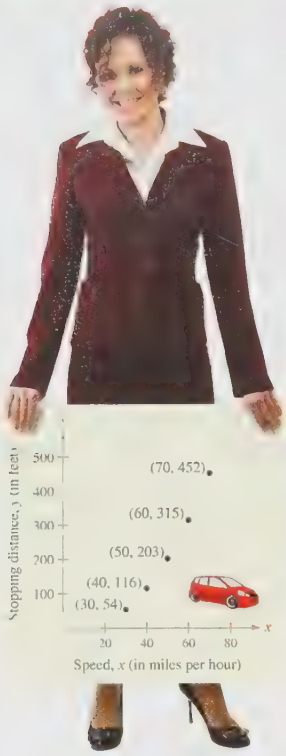
Business Capsule

SAS is a leader in business software and services. Using SAS forecasting technologies, customers can accurately analyze and forecast processes that take place over time. SAS/ETS software contains popular forecasting methods such as regression analysis and trend extrapolation.

67. Research Project Use your campus library, the Internet, or some other reference source to find information about a company or small business that generates software which uses regression analysis to predict trends. Write a brief paper about the company or small business.

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5.3 Linear Systems in Three or More Variables



In Exercise 69 on page 443, you will find the least squares regression parabola that models the stopping distance of an automobile based on its speed.

- Solve a linear system in row-echelon form using back-substitution.
- Use Gaussian elimination to solve a linear system.
- Solve a nonsquare linear system.
- Construct and use a linear system in three or more variables to solve an application problem.
- Find the equation of a parabola using a linear system in three or more variables.

Row-Echelon Form and Back-Substitution

The method of elimination can be applied to a system of linear equations in more than two variables. In fact, this method easily adapts to computer use for solving linear systems with dozens of variables.

When elimination is used to solve a system of linear equations, the goal is to rewrite the system in a form to which back-substitution can be applied. To see how this works, consider the following two systems of linear equations.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$ System of Three Linear Equations in Three Variables

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$ Equivalent System in Row-Echelon Form

The second system is said to be in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. After comparing the two systems, it should be clear that it is easier to solve the second system.

Example 1 Using Back-Substitution

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$ Equation 1, Equation 2, Equation 3

SOLUTION From Equation 3, you know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$y + 3(2) = 5 \quad \Rightarrow \quad y = -1.$$

Then substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$x - 2(-1) + 3(2) = 9 \quad \Rightarrow \quad x = 1.$$

The solution is $x = 1$, $y = -1$, and $z = 2$, which can be written as the **ordered triple** $(1, -1, 2)$. Check this in the original system of equations.

✓ Checkpoint 1

Solve the system of linear equations.

$$\begin{cases} x - 3y - 5z = -12 \\ y + z = 1 \\ z = 4 \end{cases}$$

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Gaussian Elimination

Two systems of equations are **equivalent** when they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that is in row-echelon form by using one or more of the **elementary row operations** shown below. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

Elementary Row Operations for Systems of Equations

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one equation to another equation.

Example 2 Using Elimination to Solve a System

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y = -4 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

SOLUTION Because the leading coefficient of Equation 1 is 1, you can begin by keeping the x in the upper left position and eliminating the other x-terms from the first column.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that all but the first x have been eliminated from the first column, work on the second column. (You need to eliminate y from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

Adding the second equation to the third equation produces a new third equation.

Finally, you need a coefficient of 1 for z in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system that was solved in Example 1, and, as in that example, you can conclude that the solution is $x = 1$, $y = -1$, and $z = 2$. Check this in the original system of equations.

STUDY TIP

Elementary row operations involve a lot of arithmetic. So that you can go back and check your work, you should note the elementary row operations performed in each step next to the row you are changing.

Checkpoint 2

Solve the system of linear equations.

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ 3x - z = 0 \end{cases}$$

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage in the elimination process, you obtain a false statement such as

$$0 = -2. \qquad \text{False statement}$$

Example 3 **An Inconsistent System**

Solve the system of linear equations.

$$\begin{cases} x - 3y + z = 1 & \text{Equation 1} \\ 2x - y - 2z = 2 & \text{Equation 2} \\ x + 2y - 3z = -1 & \text{Equation 3} \end{cases}$$

SOLUTION

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ x + 2y - 3z = -1 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 5y - 4z = -2 \end{cases}$$

Adding -1 times the first equation to the third equation produces a new third equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 0 = -2 \end{cases}$$

Adding -1 times the second equation to the third equation produces a new third equation.

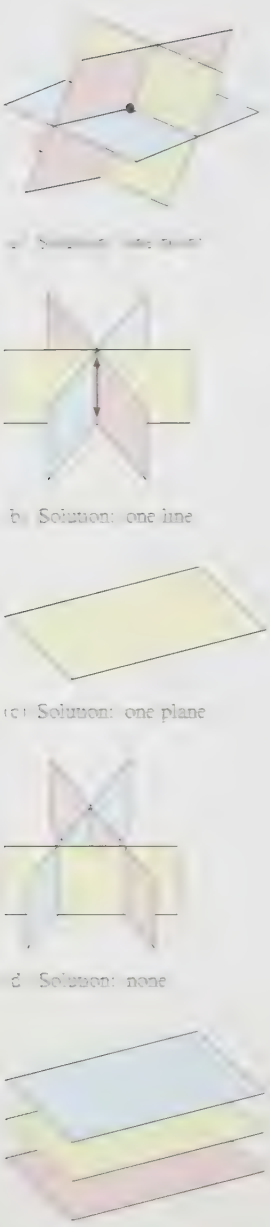
Because $0 = -2$ is a false statement, you can conclude that this system is inconsistent. So, it has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

Checkpoint 3

Solve the system of linear equations.

$$\begin{cases} 2x + y - z = 7 \\ x - 2y + 2z = -9 \\ 3x - y + z = 5 \end{cases}$$

As with a system of linear equations in two variables, the solution(s) of a system of linear equations in more than two variables must fall into one of three categories. Because an equation in three variables represents a plane in space, the possible solutions can be shown graphically. See Figure 5.10.



a. Solution: one point

b. Solution: one line

c. Solution: one plane

d. Solution: none

e. Solution: none

FIGURE 5.10

The Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

1. There is exactly one solution. [See Figure 5.10(a).]
2. There are infinitely many solutions. [See Figures 5.10(b) and (c).]
3. There is no solution. [See Figures 5.10(d) and (e).]

Recall from Section 5.2 that a system of linear equations is *consistent* if it has at least one solution. A consistent system with exactly one solution is *independent*, whereas a consistent system with infinitely many solutions is *dependent*. A system is *inconsistent* if it has no solution.

Example 4 **A System with Infinitely Many Solutions**

Solve the system of linear equations.

$$\begin{cases} x + y - 3z = -1 & \text{Equation 1} \\ y - z = 0 & \text{Equation 2} \\ -x + 2y = 1 & \text{Equation 3} \end{cases}$$

SOLUTION

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 3y - 3z = 0 \end{cases}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 0 = 0 \end{cases}$$

Adding -3 times the second equation to the third equation produces a new third equation.

This means that Equation 3 depends on Equations 1 and 2 in the sense that it gives us no additional information about the variables. Because $0 = 0$ is a true statement, you can conclude that this system has infinitely many solutions. So, the original system is equivalent to the system

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \end{cases}.$$

In the equation $y - z = 0$, solve for y in terms of z to obtain $y = z$. Back-substituting for y into the equation $x + y - 3z = -1$ produces $x = 2z - 1$. Finally, letting $z = a$, where a is a real number, the solutions of the original system are all of the form

$$x = 2a - 1, \quad y = a, \quad \text{and} \quad z = a.$$

So, every ordered triple of the form

$$(2a - 1, a, a) \qquad a \text{ is a real number.}$$

is a solution of the system.

STUDY TIP

There are an infinite number of solutions to Example 4, but they are all of a specific form. By selecting, for instance, a-values of 0, 1, and 3, you can verify that $(-1, 0, 0)$, $(1, 1, 1)$, and $(5, 3, 3)$ are specific solutions. It is incorrect to say simply that the solution to Example 4 is "infinite." You must also specify the form of the solutions.

Checkpoint 4

Solve the system of linear equations.

$$\begin{cases} 2x + y + 3z = 1 \\ 2x + 6y + 12z = 3 \\ 6x + 8y + 18z = 5 \end{cases}$$

In Example 4, there are other ways to write the same infinite set of solutions. For instance, the solutions could have been written as

$$\left(b, \tfrac{1}{2}(b + 1), \tfrac{1}{2}(b + 1)\right), \quad \text{where } b \text{ is a real number.}$$

This description produces the same set of solutions, as shown below.

Substitution	*Solution*
$a = 0$	$(2(0) - 1, 0, 0) = (-1, 0, 0)$
$b = -1$	$\left(-1, \tfrac{1}{2}(-1 + 1), \tfrac{1}{2}(-1 + 1)\right) = (-1, 0, 0)$
$a = 1$	$(2(1) - 1, 1, 1) = (1, 1, 1)$
$b = 1$	$\left(1, \tfrac{1}{2}(1 + 1), \tfrac{1}{2}(1 + 1)\right) = (1, 1, 1)$

In both cases, you obtain the same ordered triples. So, when comparing descriptions of an infinite solution set, keep in mind that there is more than one way to describe the set.

Nonsquare Systems

So far, each system of linear equations you have looked at has been **square,** which means that the number of equations is equal to the number of variables. In a **nonsquare** system, the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables in the system.

Example 5 A System with Fewer Equations than Variables

Solve the system of linear equations.

$$\begin{cases} x - 2y + z = 2 & \text{Equation 1} \\ 2x - y - z = 1 & \text{Equation 2} \end{cases}$$

SOLUTION Begin by rewriting the system in row-echelon form.

$$\begin{cases} x - 2y + z = 2 \\ 3y - 3z = -3 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + z = 2 \\ y - z = -1 \end{cases}$$

Multiplying the second equation by $\frac{1}{3}$ produces a new second equation.

Solving for y in terms of z, you obtain

$$y = z - 1.$$

Back-substitution into Equation 1 yields

$$x - 2(z - 1) + z = 2$$

$$x - 2z + 2 + z = 2$$

$$x - z = 0$$

$$x = z.$$

Finally, by letting $z = a$, where a is a real number, you have the solution

$$x = a, \quad y = a - 1, \quad \text{and} \quad z = a.$$

So, every ordered triple of the form

$$(a, a - 1, a) \qquad a \text{ is a real number.}$$

is a solution of the system. Because there were originally three variables and only two equations, the system cannot have a unique solution.

✓Checkpoint 5

Solve the system of linear equations.

$$\begin{cases} 2x - 2y + 5z = 2 \\ 4x \quad - \quad z = 0 \end{cases}$$

In Example 5, try choosing some values of a to obtain different solutions of the system, such as $(1, 0, 1)$, $(2, 1, 2)$, and $(3, 2, 3)$. Then check each of the solutions in the original system. For example, you can check the solution $(1, 0, 1)$ as shown.

Equation 1: $1 - 2(0) + 1 = 2$ ✓

Equation 2: $2(1) - 0 - 1 = 1$ ✓

Application

Example 6 Business Loans

A company borrows $1,000,000. Some of the money is borrowed at 5%, some at 7%, some at 8%, and some at 10% simple annual interest. The amount borrowed at 8% is the same as the amount borrowed at 10%. How much is borrowed at each rate when the total annual interest is $79,000?



SOLUTION Let x, y, z, and w represent the amounts borrowed at each rate. Because the total amount borrowed is $1,000,000, you can write the equation

$$x + y + z + w = 1{,}000{,}000.$$

A second equation can be derived from the fact that the total annual interest is $79,000.

$$0.05x + 0.07y + 0.08z + 0.10w = 79{,}000$$

Finally, because the amount borrowed at 8% is the same as the amount borrowed at 10%, you can write

$$z - w = 0.$$

These three equations make up the system shown.

$$\begin{cases} x + y + z + w = 1{,}000{,}000 & \text{Equation 1} \\ 0.05x + 0.07y + 0.08z + 0.10w = 79{,}000 & \text{Equation 2} \\ z - w = 0 & \text{Equation 3} \end{cases}$$

Using elimination, you find that the system has infinitely many solutions, which can be written as shown.

$$x = 2a - 450{,}000$$

$$y = -4a + 1{,}450{,}000$$

$$z = a$$

$$w = a$$

So, there are many different scenarios. One possible solution is to let $a = 250{,}000$. This yields the following scenario.

x = $50,000 at 5%

y = $450,000 at 7%

z = $250,000 at 8%

w = $250,000 at 10%

Checkpoint 6

In Example 6, the company borrows $1,200,000. How much is borrowed at each rate?

Remember that you can check solutions that are not unique by substituting several different values for a.

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Finding the Equation of a Parabola

Example 7 Data Analysis: Curve Fitting

Find a quadratic equation $y = ax^2 + bx + c$ whose graph passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$.

SOLUTION Because the graph of $y = ax^2 + bx + c$ passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$, you can substitute for x and y as shown below.

When $x = -1$ and $y = 3$: $a(-1)^2 + b(-1) + c = 3$

When $x = 1$ and $y = 1$: $a(1)^2 + b(1) + c = 1$

When $x = 2$ and $y = 6$: $a(2)^2 + b(2) + c = 6$

This produces the following system of linear equations.

$$\begin{cases} a - b + c = 3 & \text{Equation 1} \\ a + b + c = 1 & \text{Equation 2} \\ 4a + 2b + c = 6 & \text{Equation 3} \end{cases}$$

The solution of this system is

$$a = 2, \quad b = -1, \quad \text{and} \quad c = 0.$$

So, the equation of the parabola is

$$y = 2x^2 - x$$

as shown in Figure 5.11.

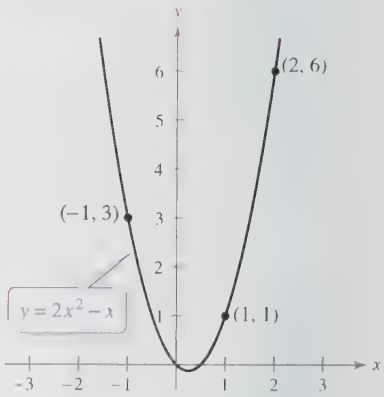


FIGURE 5.11

ALGEBRA TUTOR

For help with the algebra used to solve the system of equations in Example 7, see the *Chapter 5 Algebra Tutor* on page 464.

Checkpoint 7

Find a quadratic equation $y = ax^2 + bx + c$ whose graph passes through the points $(-1, 7)$, $(1, 3)$, and $(2, 7)$.



SUMMARIZE (Section 5.3)

1. State the definition of row-echelon form *(page 432)*. For an example of solving a linear system in row-echelon form, see Example 1.
2. Describe the process of Gaussian elimination *(page 433)*. For examples that use Gaussian elimination to solve a linear system, see Examples 2, 3, and 4.
3. Describe the difference between square and nonsquare systems of linear equations *(page 436)*. For an example of solving a nonsquare linear system, see Example 5.
4. Describe a real-life example of how a linear system in three or more variables can be used to analyze the amount of money borrowed by a company *(page 437, Example 6)*.
5. Describe how a linear system in three or more variables can be used to find the equation of a parabola given three points on the parabola *(page 438)*. For an example of finding the equation of a parabola given three points on the parabola, see Example 7.

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SKILLS WARM UP 5.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.2, 2.1, 5.1, and 5.2.

In Exercises 1–4, solve the system of linear equations.

1. $\begin{cases} x + y = 25 \\ y = 15 \end{cases}$

2. $\begin{cases} 2x - 3y = 4 \\ 6x = -18 \end{cases}$

3. $\begin{cases} x + y = 32 \\ x - y = 24 \end{cases}$

4. $\begin{cases} 2r - s = 5 \\ r + 2s = 10 \end{cases}$

In Exercises 5–8, determine whether the ordered triple is a solution of the equation.

5. $5x - 3y + 4z = 2$
$(-1, -2, 1)$

6. $x - 2y + 12z = 9$
$(6, 3, 2)$

7. $2x - 5y + 3z = -9$
$(a - 2, a + 1, a)$

8. $-5x + y + z = 21$
$(a - 4, 4a + 1, a)$

In Exercises 9 and 10, solve for x in terms of a.

9. $x + 2y - 3z = 4$
$y = 1 - a, z = a$

10. $x - 3y + 5z = 4$
$y = 2a + 3, z = a$

Exercises 5.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–4, match the system of equations with its solution. [The solutions are labeled (a), (b), (c), and (d).]

(a) $(-1, 0, 3)$ (b) $(6, 2, -2)$

(c) $(2, 1, -3)$ (d) $(4, -1, 5)$

1. $\begin{cases} -2x + 3y - 2z = 5 \\ 3x - 4y + z = -1 \\ x + 2y + 5z = -11 \end{cases}$

2. $\begin{cases} 5x + 2y - 4z = -17 \\ -8x + y - 5z = -7 \\ 4x + 3y - z = -7 \end{cases}$

3. $\begin{cases} 2x + 5y - 7z = 36 \\ x + 6y - 10z = 38 \\ x - 4y + 8z = -18 \end{cases}$

4. $\begin{cases} -x - 2y + 5z = 23 \\ -3x + y + 6z = 17 \\ 9x + 2y - 7z = -1 \end{cases}$

Identifying Row-Echelon Form In Exercises 5–8, determine whether the system of equations is in row-echelon form. Justify your answer.

5. $\begin{cases} x + 3y - 7z = -11 \\ y - 2z = -3 \\ z = 2 \end{cases}$

6. $\begin{cases} x - y + 3z = -11 \\ y + 8z = -12 \\ z = -2 \end{cases}$

7. $\begin{cases} x - 9y + z = 22 \\ 2y + z = -3 \\ z = 1 \end{cases}$

8. $\begin{cases} x - y - 8z = 12 \\ 2y - 2z = 2 \\ 7z = -7 \end{cases}$

Using Back-Substitution In Exercises 9 and 10, use back-substitution to solve the system of linear equations. *See Example 1.*

9. $\begin{cases} x - y + z = 4 \\ y + 2z = -6 \\ z = -2 \end{cases}$

10. $\begin{cases} x - 2y + z = 8 \\ y + z = 5 \\ z = 2 \end{cases}$

Solving a System of Linear Equations In Exercises 11–36, solve the system of linear equations. *See Examples 2, 3, 4, and 5.*

11. $\begin{cases} 4x + y - 3z = 11 \\ 2x - 3y + 2z = 9 \\ x + y + z = -3 \end{cases}$

12. $\begin{cases} 6y + 4z = -12 \\ 3x + 3y = 9 \\ 2x - 3z = 10 \end{cases}$

13. $\begin{cases} 3x + 2z = 13 \\ x + 2y + z = -5 \\ -3y - z = 10 \end{cases}$

14. $\begin{cases} 2x + 3y + z = -4 \\ 2x - 4y + 3z = 18 \\ 3x - 2y + 2z = 9 \end{cases}$

15. $\begin{cases} 3x - 2y + 4z = 1 \\ x + y - 2z = 3 \\ 2x - 3y + 6z = 8 \end{cases}$

16. $\begin{cases} 5x - 3y + 2z = 3 \\ 2x + 4y - z = 7 \\ x - 11y + 4z = 3 \end{cases}$

17. $\begin{cases} 3x + 3y + 5z = 1 \\ 3x + 5y + 9z = 0 \\ 5x + 9y + 17z = 0 \end{cases}$

18. $\begin{cases} 2x + y - z = 13 \\ x + 2y + z = 2 \\ 8x - 3y + 4z = -2 \end{cases}$

19. $\begin{cases} x + 2y - 7z = -4 \\ 2x + y + z = 13 \\ 3x + 9y - 36z = -33 \end{cases}$

20. $\begin{cases} 2x + y - 3z = 4 \\ 4x + 2z = 10 \\ -2x + 3y - 13z = -8 \end{cases}$

21. $\begin{cases} x + 4z = 13 \\ 4x - 2y + z = 7 \\ 2x - 2y - 7z = -19 \end{cases}$

22. $\begin{cases} 4x - y + 5z = 11 \\ x + 2y - z = 5 \\ 5x - 8y + 13z = 7 \end{cases}$

23. $\begin{cases} x + 4z = 1 \\ x + y + 10z = 10 \\ 2x - y + 2z = -5 \end{cases}$

24. $\begin{cases} 3x - 2y - 6z = 4 \\ -3x + 2y + 6z = 1 \\ x - y - 5z = 3 \end{cases}$

25. $\begin{cases} 4x + 3y + 5z = 10 \\ 5x + 2y + 10z = 13 \\ 3x + y - 2z = 9 \end{cases}$

26. $\begin{cases} 2x + 5y = 25 \\ 3x - 2y + 4z = 1 \\ 4x - 3y + z = 9 \end{cases}$

27. $\begin{cases} 2x + 3y - z = 1 \\ x - 2y + z = 7 \\ 3x + y + 2z = 12 \end{cases}$

28. $\begin{cases} 2x + 3y = 0 \\ 4x + 3y - z = 0 \\ 8x + 3y + 3z = 0 \end{cases}$

29. $\begin{cases} 12x + 5y + z = 0 \\ 12x + 4y - z = 0 \end{cases}$

30. $\begin{cases} x - 2y + 5z = 2 \\ 3x + 2y - z = -2 \end{cases}$

31. $\begin{cases} x - 3y + 2z = 18 \\ 5x - 13y + 12z = 80 \end{cases}$

32. $\begin{cases} 2x - 3y + z = -2 \\ -4x + 9y = 7 \end{cases}$

33. $\begin{cases} 3x - 3y + 6z = 7 \\ -x + y - 2z = 3 \\ 2x + 3y - 4z = 8 \end{cases}$

34. $\begin{cases} 4x + 3y = 7 \\ x - 2y + z = 0 \\ 2x + 4y - 2z = 13 \end{cases}$

35. $\begin{cases} x + 3w = -4 \\ 2y - z - w = 0 \\ 3y - 2w = 1 \\ 2x - y + 4z = 5 \end{cases}$

36. $\begin{cases} x + y + z + w = 6 \\ 2x + 3y - w = 0 \\ 3x + 4y + z + 2w = 4 \\ x + 2y - z + w = 0 \end{cases}$

Think About It In Exercises 37–40, find two systems of linear equations that have the ordered triple as a solution. (There are many correct answers.)

37. $(3, -1, 2)$

38. $\left(-\frac{1}{2}, 2, 4\right)$

39. $(1, -5, 3)$

40. $\left(0, 2, \frac{1}{2}\right)$

Writing Ordered Triples In Exercises 41–44, write three ordered triples of the given form.

41. $(a, a + 2, \frac{1}{3}a)$

42. $(3a, 5 - a, a)$

43. $\left(\frac{1}{2}a, 4a, 7\right)$

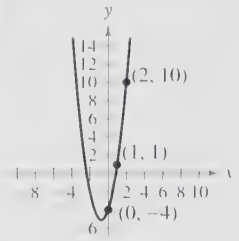
44. $\left(-\frac{1}{4}a + 3, a, 8\right)$

Data Analysis: Curve Fitting In Exercises 45–48, find the quadratic equation

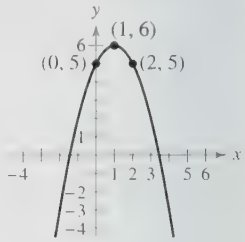
$y = ax^2 + bx + c$

whose graph passes through the points. *See Example 7.*

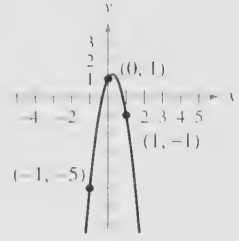
45.



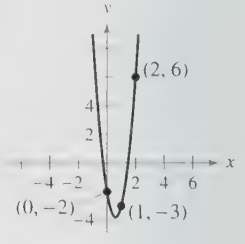
46.



47.



48.

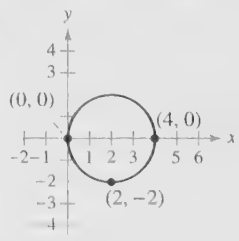


Data Analysis: Curve Fitting In Exercises 49–52, find the equation

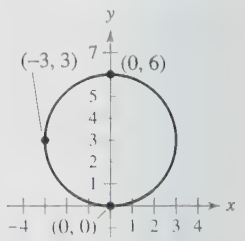
$x^2 + y^2 + Dx + Ey + F = 0$

of the circle that passes through the points.

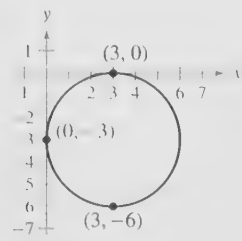
49.



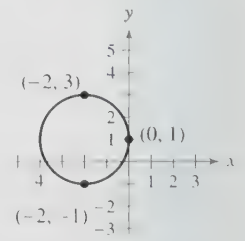
50.



51.



52.



53. Investment A real estate company borrows \$1,500,000. Some of the money is borrowed at 3%, some at 4%, and some at 6% simple annual interest. How much is borrowed at each rate when the total annual interest is \$53,000 and the amount borrowed at 4% is the same as the amount borrowed at 6%?

54. Investment A clothing company borrows \$700,000. Some of the money is borrowed at 8%, some at 9%, and some at 10% simple annual interest. How much is borrowed at each rate when the total annual interest is \$60,500 and the amount borrowed at 8% is three times the amount borrowed at 10%?

55. Candles A candle company sells three types of candles for \$15, \$10, and \$5 per unit. In one year, the total revenue for the three products was \$550,000, which corresponded to the sale of 50,000 units. The company sold half as many units of the \$15 candles as units of the \$10 candles. How many units of each type of candle were sold?

56. Hair Products A hair product company sells three types of hair products for \$30, \$20, and \$10 per unit. In one year, the total revenue for the three products was \$800,000, which corresponded to the sale of 40,000 units. The company sold half as many units of the \$30 product as units of the \$20 product. How many units of each product were sold?

57. Crop Spraying A mixture of 5 gallons of chemical A, 8 gallons of chemical B, and 12 gallons of chemical C is required to kill a crop-destroying insect. Commercial spray X contains 1, 2, and 3 parts of these chemicals, respectively. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray is needed to obtain the desired mixture?

58. Acid Mixture A chemist needs 10 liters of a 25% acid solution. The solution is to be mixed from three solutions whose acid concentrations are

10%, 20%, and 50%.

How many liters of each solution should the chemist use to satisfy each of the following requirements?

(a) Use as little as possible of the 50% solution.

(b) Use as much as possible of the 50% solution.

(c) Use 2 liters of the 50% solution.

Investment Portfolio In Exercises 59 and 60, you have a total of \$500,000 that is to be invested in certificates of deposit, municipal bonds, blue-chip stocks, and growth stocks. How much should be put in each type of investment? *See Example 6.*

59. The certificates of deposit pay 2.5% simple annual interest, and the municipal bonds pay 10% simple annual interest. Over a five-year period, you expect the blue-chip stocks to return 12% simple annual interest and the growth stocks to return 18% simple annual interest. You want a combined annual return of 10% and you also want to have only one-fourth of the portfolio invested in stocks.

60. The certificates of deposit pay 3% simple annual interest, and the municipal bonds pay 10% simple annual interest. Over a five-year period, you expect the blue-chip stocks to return 12% simple annual interest and the growth stocks to return 15% simple annual interest. You want a combined annual return of 10% and you also want to have only one-fourth of the portfolio invested in stocks.

Fitting a Parabola to Data To find the least squares regression parabola

$$y = ax^2 + bx + c$$

for a set of points

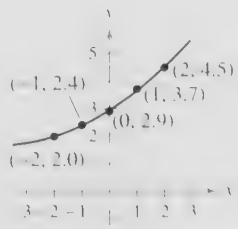
$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

you can solve the following system of linear equations for a, b, and c.

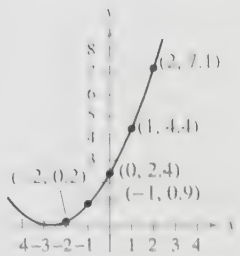
$$\begin{cases} nc + \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right)c + \left(\sum_{i=1}^{n} x_i^2\right)b + \left(\sum_{i=1}^{n} x_i^3\right)a = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i^2\right)c + \left(\sum_{i=1}^{n} x_i^3\right)b + \left(\sum_{i=1}^{n} x_i^4\right)a = \sum_{i=1}^{n} x_i^2 y_i \end{cases}$$

In Exercises 61–64, the sums have been evaluated. Solve the given system for a, b, and c to find the least squares regression parabola for the points. Use a graphing utility to confirm the result.

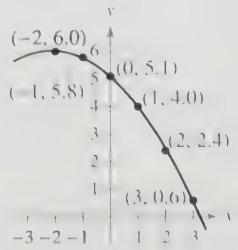
61.
$$\begin{cases} 5c + 10a = 15.5 \\ 10b = 6.3 \\ 10c + 34a = 32.1 \end{cases}$$



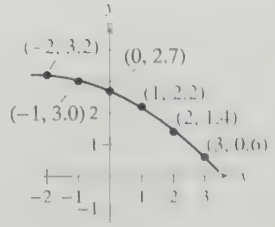
62.
$$\begin{cases} 5c + 10a = 15.0 \\ 10b = 17.3 \\ 10c + 34a = 34.5 \end{cases}$$



63.
$$\begin{cases} 6c + 3b + 19a = 23.9 \\ 3c + 19b + 27a = -7.2 \\ 19c + 27b + 115a = 48.8 \end{cases}$$



64.
$$\begin{cases} 6c + 3b + 19a = 13.1 \\ 3c + 19b + 27a = -2.6 \\ 19c + 27b + 115a = 29.0 \end{cases}$$



65. Consumer Spending The annual amounts y (in billions of dollars) spent on hunting equipment and firearms in the United States from 2005 through 2009 are shown in the table. In the table, x represents the year, with $x = -2$ corresponding to 2005. *(Source: National Sporting Goods Association)*

Year, x	Amount, y
-2	3.6
-1	3.7
0	3.9
1	4.5
2	5.2

(a) Find the least squares regression parabola

$$y = ax^2 + bx + c$$

for the data by solving the system below.

$$\begin{cases} 5c + 10a = 20.9 \\ 10b = 4.0 \\ 10c + 34a = 43.4 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Compare the model with the model found in part (a).

66. Trademark Registration The annual numbers y (in thousands) of trademarks issued in the United States from 2005 through 2009 are shown in the table. In the table, x represents the year, with $x = -2$ corresponding to 2005. *(Source: U.S. Patent and Trademark Office)*

Year, x	-2	-1	0	1	2
Number, y	155	194	219	234	222

(a) Find the least squares regression parabola

$$y = ax^2 + bx + c$$

for the data by solving the system below.

$$\begin{cases} 5c + 10a = 1024 \\ 10b = 174 \\ 10c + 34a = 1936 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Compare the model with the model found in part (a).

67. Gardening The percents y of households in the United States involved in vegetable gardening from 2005 through 2009 are shown in the table. In the table, x represents the year, with $x = -1$ corresponding to 2005. *(Source: The National Gardening Association)*

Year, x	Percent, y
-1	25
0	22
1	22
2	23
3	27

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the system below.

$$\begin{cases} 5c + 5b + 15a = 119 \\ 5c + 15b + 35a = 124 \\ 15c + 35b + 99a = 382 \end{cases}$$

(b) Use the model found in part (a) to predict the percent of households involved in vegetable gardening in 2010.

68. Sales Per Share The sales per share y (in dollars) of Hershey's from 2005 through 2009 are shown in the table. In the table, x represents the year, with $x = -1$ corresponding to 2005. *(Source: The Hershey Company)*

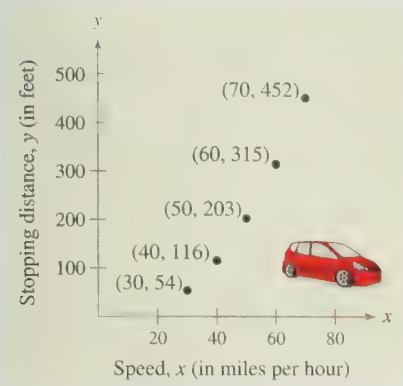
Year, x	Sales per share, y
-1	20.11
0	21.47
1	21.79
2	22.61
3	23.24

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the system below.

$$\begin{cases} 5c + 5b + 15a = 109.22 \\ 5c + 15b + 35a = 116.62 \\ 15c + 35b + 99a = 341.50 \end{cases}$$

(b) Use the model found in part (a) to predict Hershey's sales per share in 2010 and 2011.

69. Automobile Stopping Distance Engineers tested the braking system of a new automobile. The scatter plot shows the stopping distances y (in feet) of the automobile for several speeds x (in miles per hour).



(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the system below.

$$\begin{cases} 5c + 250b + 13{,}500a = 1140 \\ 250c + 13{,}500b + 775{,}000a = 66{,}950 \\ 13{,}500c + 775{,}000b + 46{,}590{,}000a = 4{,}090{,}500 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to check your answer to part (a).

(c) Use the model found in part (a) to predict the stopping distance of the automobile when traveling at a speed of 75 miles per hour.

70. HOW DO YOU SEE IT? The number of sides x and the combined number of sides and diagonals y for each of three regular polygons are shown below. How can you create a system of linear equations to find an equation of the form $y = ax^2 + bx + c$ that represents the relationship between x and y for the three polygons?


$x = 3, y = 3$


$x = 4, y = 6$


$x = 5, y = 10$

71. Geometry Find the equation of the form

$$y = ax^2 + bx + c$$

that represents the relationship between x and y for the three polygons shown in Exercise 70. Then sketch regular polygons with 6, 7, and 8 sides, and draw their diagonals. Does the equation represent the relationship between x and y for these polygons?

72. Think About It Is it possible for a square linear system to have no solution? Explain.

73. Think About It Is it possible for a square linear system to have infinitely many solutions? Explain.

74. Writing One solution for Exercise 30 is

$$(-a, 2a \quad 1, a).$$

A student gives

$$(b, -2b - 1, -b)$$

as a solution to the same exercise. Explain why both solutions are correct.

75. Think About It Find values of a, b, and c (if possible) such that the system of linear equations below has each number of solutions.

(a) A unique solution

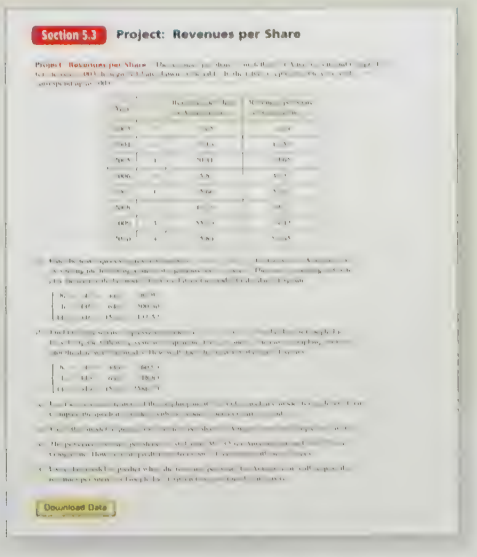
(b) No solution

(c) Infinitely many solutions

$$\begin{cases} x + y = 2 \\ y + z = 2 \\ x + z = 2 \\ ax + by + cz = 0 \end{cases}$$

76. Writing When using Gaussian elimination to solve a system of linear equations, explain how you can recognize that the system has no solution. Give an example that illustrates your answer.

77. Project: Revenues per Share For a project involving the revenues per share of Amazon.com and Google Inc., visit this text's website at *www.cengagebrain.com.* *(Source: Amazon.com and Google Inc.)*



QUIZ YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, solve the system by the method of substitution. Use a graphing utility to verify your solution.

1. $\begin{cases} 3x + y = 11 \\ x - 2y = -8 \end{cases}$

2. $\begin{cases} 4x + 8y = 8 \\ x + 2y = 6 \end{cases}$

3. $\begin{cases} x + y = 4 \\ y = 2\sqrt{x} + 1 \end{cases}$

4. $\begin{cases} x^2 + y^2 = 9 \\ y = 2x + 1 \end{cases}$

In Exercises 5 and 6, find the number of units x that need to be sold to break even.

5. $C = 10.50x + 9000, R = 16.50x$

6. $C = 3.79x + 400{,}000, R = 4.59x$

In Exercises 7 and 8, solve the system by the method of elimination.

7. $\begin{cases} 2.5x - y = 6 \\ 3x + 4y = 2 \end{cases}$

8. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 1 \\ x - 2y = -2 \end{cases}$

9. Find the point of equilibrium for the supply and demand equations. Verify your solution graphically.

Demand: $p = 50 - 0.002x$

Supply: $p = 20 + 0.004x$

10. The total numbers y (in millions) of Medicare enrollees in the years 2005 through 2009 are shown in the table at the left. Solve the following system for a and b to find the least squares regression line $y = at + b$ for the data. Let t represent the year, with $t = 0$ corresponding to 2005. *(Source: U.S. Centers for Medicare and Medicaid Services)*

$$\begin{cases} 5b + 10a = 221.4 \\ 10b + 30a = 451.2 \end{cases}$$

Year	t	Number, y
2005	0	42.6
2006	1	43.4
2007	2	44.3
2008	3	45.2
2009	4	45.9

Table for 10

In Exercises 11–13, solve the system of equations.

11. $\begin{cases} 2x + 3y - z = -7 \\ x + 3z = 10 \\ 2y + z = -1 \end{cases}$

12. $\begin{cases} x + y - 2z = 12 \\ 2x - y - z = 6 \\ y - z = 6 \end{cases}$

13. $\begin{cases} 3x + 2y + z = 17 \\ -x + y + z = 4 \\ x - y - z = 3 \end{cases}$

14. The average prices y (in dollars) of retail prescription drugs for the years 2005 through 2009 are shown in the table. In the table, x represents the year, with $x = -2$ corresponding to 2005. *(Source: National Association of Chain Drug Stores)*

Year, x	-2	-1	0	1	2
Average price, y	63.87	66.97	69.91	72.87	76.94

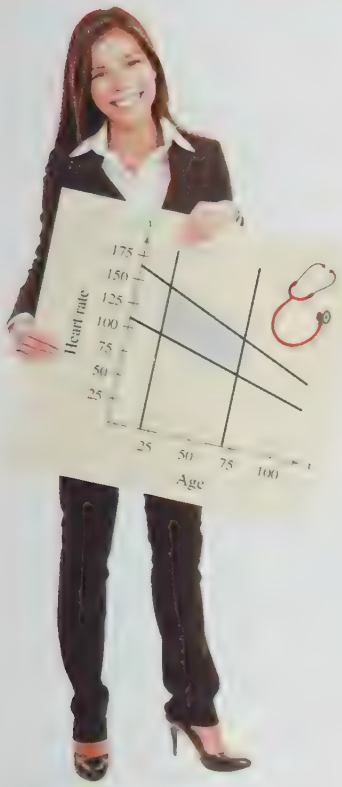
(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the system below.

$$\begin{cases} 5c + 10a = 350.56 \\ 10b = 32.04 \\ 10c + 34a = 703.08 \end{cases}$$

(b) Use the model found in part (a) to predict the average prices of retail prescription drugs in 2010 and 2011.

5.4 Systems of Inequalities

- Sketch the graph of an inequality in two variables.
- Solve a system of inequalities.
- Construct and use a system of inequalities to solve an application problem.



In Exercise 66 on page 453, you will use a graph to find the exercise target heart rate for individuals between the ages of 20 and 70 years old.

The Graph of an Inequality

The following statements are inequalities in two variables:

$$3x - 2y < 6 \quad \text{and} \quad 2x^2 + 3y^2 \geq 6.$$

An ordered pair (a, b) is a **solution of an inequality** in x and y if the inequality is true when a and b are substituted for x and y, respectively. The **graph of an inequality** is the collection of all solutions of the inequality. To sketch the graph of an inequality, begin by sketching the graph of the *corresponding equation*. The graph of the equation will normally separate the plane into two or more regions. In each such region, one of the following must be true.

1. *All* points in the region are solutions of the inequality.
2. *No* point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality simply by testing *one* point in the region. When possible, use test points that are convenient to substitute into the inequality, such as $(0, 0)$.

Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign with an equal sign, and sketch the graph of the resulting equation. (Use a dashed line for $<$ or $>$ and a solid line for $\leq$ or $\geq$.)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, then shade the entire region to denote that every point in the region satisfies the inequality.

Example 1 Sketching the Graph of an Inequality

Sketch the graph of the inequality $y \geq x^2 - 1$.

SOLUTION The graph of the corresponding *equation* $y = x^2 - 1$ is a parabola, as shown in Figure 5.12. Test a point *above* the parabola $(0, 0)$ and a point *below* the parabola $(0, -2)$.

The points that satisfy the inequality are those lying above (or on) the parabola.

$(0, 0)$: $0 \stackrel{?}{\geq} 0^2 - 1$

$0 \geq -1$ $(0, 0)$ is a solution.

$(0, -2)$: $-2 \stackrel{?}{\geq} 0^2 - 1$

$-2 \not\geq -1$ $(0, -2)$ is not a solution.

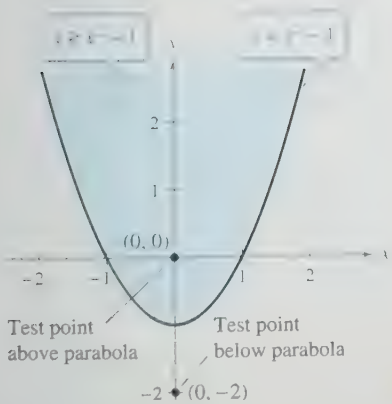


FIGURE 5.12

Checkpoint 1

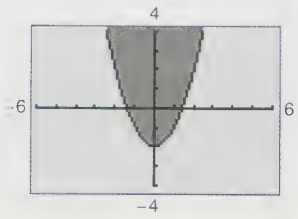
Sketch the graph of $y < x^2 + 2$.

The inequality in Example 1 is a nonlinear inequality in two variables. Most of the following examples involve **linear inequalities** such as $ax + by < c$ (a and b are not both zero). The graph of a linear inequality is a half-plane lying on one side of the line $ax + by = c$.

Mandav/Shutterstock.com

TECH TUTOR

A graphing utility can be used to graph an inequality or a system of inequalities. For instance, to graph $y \geq x^2 - 2$, enter $y = x^2 - 2$ and use the *shade* feature of the graphing utility to shade the part of the graph consisting of the points that satisfy the inequality. You should obtain the graph shown below. Consult the user's guide for your graphing utility for more information.



Example 2 Sketching the Graph of a Linear Inequality

Sketch the graph of each linear inequality.

a. $x > -2$ **b.** $y \leq 3$

SOLUTION

a. The graph of the corresponding equation $x = -2$ is a vertical line. The points that satisfy the inequality $x > -2$ are those lying to the right of this line, as shown in Figure 5.13.

b. The graph of the corresponding equation $y = 3$ is a horizontal line. The points that satisfy the inequality $y \leq 3$ are those lying below (or on) this line, as shown in Figure 5.14.

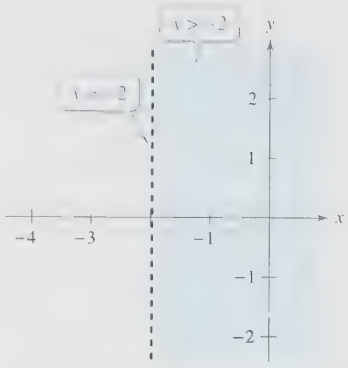


FIGURE 5.13

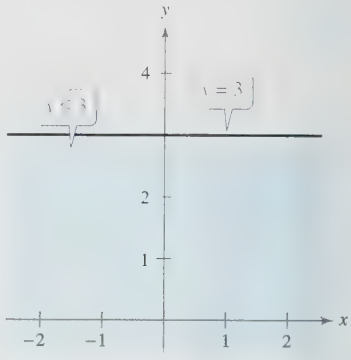


FIGURE 5.14

Checkpoint 2

Sketch the graph of $x \geq -1$.

Example 3 Sketching the Graph of a Linear Inequality

Sketch the graph of $x - y < 2$.

SOLUTION The graph of the corresponding equation $x - y = 2$ is a line, as shown in Figure 5.15. Because the origin (0, 0) satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point you choose, you will see that it does not satisfy the inequality.)

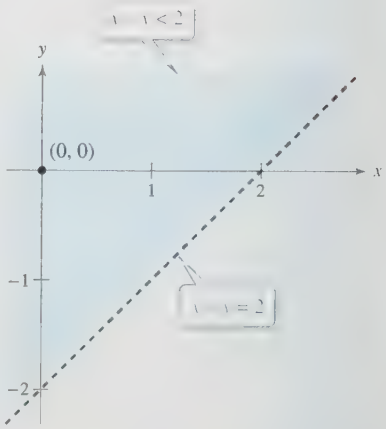


FIGURE 5.15

STUDY TIP

To graph a linear inequality, it can help to write the inequality in slope-intercept form. For instance, by writing $x - y < 2$ in the form

$$y > x - 2$$

you can see that the solution points lie above the line $x - y = 2$ (or $y = x - 2$), as shown in Figure 5.15.

Checkpoint 3

Sketch the graph of $x + y \leq 1$.

Systems of Inequalities

A **solution** of a system of inequalities in x and y is a point (x, y) that satisfies each inequality in the system. To solve a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. This region represents the **solution set** of the system. For systems of *linear* inequalities, it is helpful to find the vertices of the solution region.

Example 4 Solving a System of Inequalities

Solve the system of linear inequalities.

$$\begin{cases} x - y < 2 & \text{Inequality 1} \\ x > -2 & \text{Inequality 2} \\ y \le 3 & \text{Inequality 3} \end{cases}$$

SOLUTION To solve the system, sketch the graph of the solution set. The graphs of these inequalities are shown in Figures 5.15, 5.13, and 5.14 on page 446. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate plane, as shown in Figure 5.16. To find the vertices of the region, solve the three systems of equations obtained by taking the *pairs* of equations representing the boundaries of the individual regions.

Vertex A: $(-2, -4)$
Obtained by solving the system

$$\begin{cases} x - y = 2 \\ x = -2 \end{cases}$$

Vertex B: $(5, 3)$
Obtained by solving the system

$$\begin{cases} x - y = 2 \\ y = 3 \end{cases}$$

Vertex C: $(-2, 3)$
Obtained by solving the system

$$\begin{cases} x = -2 \\ y = 3 \end{cases}$$

STUDY TIP

Using pencils of different colors to shade the solutions of the inequalities in a system makes identifying the solution of the system of inequalities easier. The region common to every graph in the system is where all the shaded regions overlap.

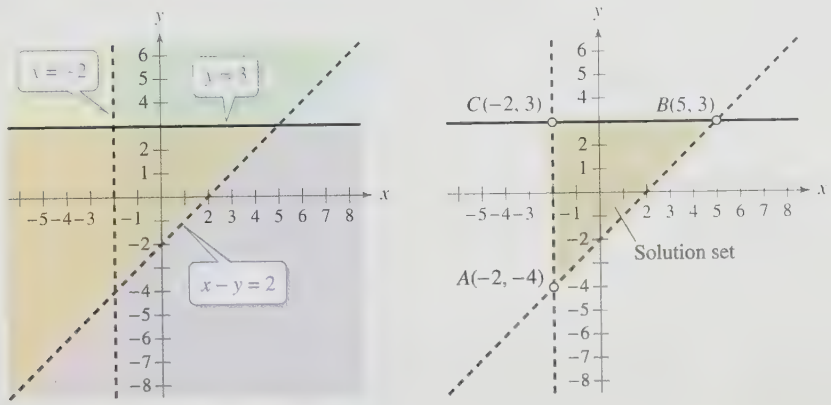


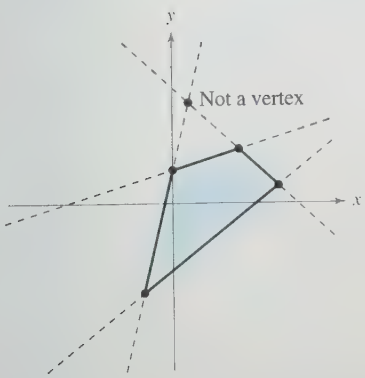
FIGURE 5.16

Checkpoint 4

Solve the system of linear inequalities.

$$\begin{cases} 2x - y > -3 \\ x \le 2 \\ y > -2 \end{cases}$$

For the triangular region shown in Figure 5.16, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is *not* a vertex of the region, as shown in Figure 5.17. To determine which points of intersection are actually vertices of the region, sketch the region and refer to your sketch as you find each point of intersection.



Boundary lines can intersect at a point that is not a vertex.

FIGURE 5.17

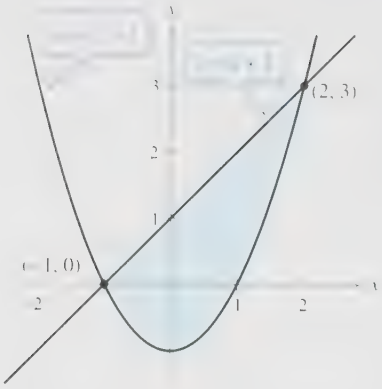


FIGURE 5.18

Example 5 **Solving a System of Inequalities**

Solve the system of inequalities.

$$\begin{cases} x^2 - y \le 1 & \text{Inequality 1} \\ -x + y \le 1 & \text{Inequality 2} \end{cases}$$

SOLUTION To solve the system, sketch the graph of the solution set. As shown in Figure 5.18, the points that satisfy the inequality $x^2 - y \le 1$ are the points lying above (or on) the parabola given by $y = x^2 - 1$. The points that satisfy the inequality $-x + y \le 1$ are the points lying below (or on) the line given by $y = x + 1$. To find the points of intersection of the parabola and the line, solve the system of corresponding equations.

$$\begin{cases} x^2 - y = 1 \\ -x + y = 1 \end{cases}$$

Using the method of substitution, you can find the points of intersection to be $(-1, 0)$ and $(2, 3)$. The graph of the solution set of the system is shown in Figure 5.18.

Checkpoint 5

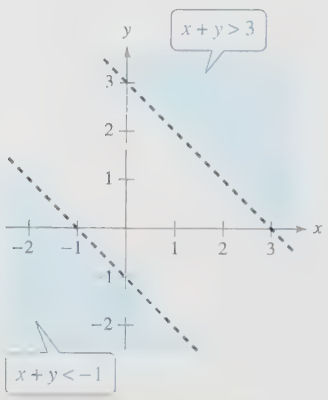
Solve the system of inequalities.

$$\begin{cases} x^2 + y < 3 \\ x + y > -3 \end{cases}$$

When solving a system of inequalities, you should be aware that the system might have no solution. For instance, the system

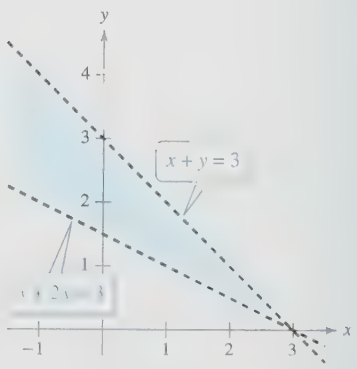
$$\begin{cases} x + y > 3 \\ x + y < -1 \end{cases}$$

has no solution points, because the quantity $(x + y)$ cannot be less than -1 *and* greater than 3, as shown in Figure 5.19.



No Solution

FIGURE 5.19



Unbounded Region

FIGURE 5.20

Another possibility is that the solution set of a system of inequalities can be unbounded. For instance, the solution set of

$$\begin{cases} x + y < 3 \\ x + 2y > 3 \end{cases}$$

forms an *infinite wedge*, as shown in Figure 5.20.

Applications

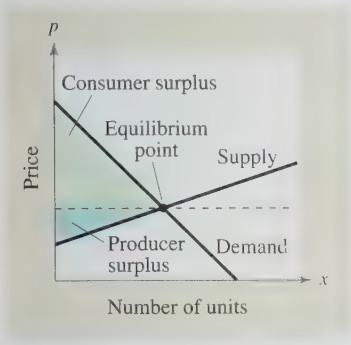


FIGURE 5.21

Example 8 in Section 5.2 discussed the *point of equilibrium* for a demand function and a supply function. The next example discusses two related concepts that economists call **consumer surplus** and **producer surplus.** As shown in Figure 5.21, the consumer surplus is defined as the area of the region that lies *below* the demand graph, *above* the horizontal line passing through the equilibrium point, and to the right of the p-axis. Similarly, the producer surplus is defined as the area of the region that lies *above* the supply graph, *below* the horizontal line passing through the equilibrium point, and to the right of the p-axis. The consumer surplus is a measure of the amount that consumers would have been willing to pay *above what they actually paid*, whereas the producer surplus is a measure of the amount that producers would have been willing to receive *below what they actually received.*

Example 6 Consumer and Producer Surpluses

The demand and supply equations for a video game console are given by

$$\begin{cases} p = 350 - 0.00003x & \text{Demand equation} \\ p = 200 + 0.00003x & \text{Supply equation} \end{cases}$$

where p is the price (in dollars) and x is the number of video game consoles. Find the consumer surplus and producer surplus for these two equations.

SOLUTION In Example 8 in Section 5.2, you saw that the point of equilibrium for these equations is

$$(2{,}500{,}000,\ 275).$$

So, the horizontal line passing through this point is $p = 275$. Now you can determine that the consumer surplus and producer surplus are the areas of the triangular regions given by the following systems of inequalities, respectively.

Consumer Surplus

$$\begin{cases} p \le 350 - 0.00003x \\ p \ge 275 \\ x \ge 0 \end{cases}$$

Producer Surplus

$$\begin{cases} p \ge 200 + 0.00003x \\ p \le 275 \\ x \ge 0 \end{cases}$$

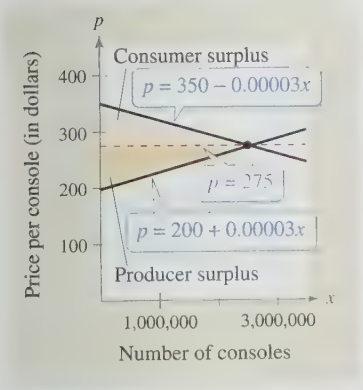


FIGURE 5.22

In Figure 5.22, you can see that the consumer and producer surpluses are defined as the areas of the shaded triangles. The base of the triangle representing the consumer surplus is 2,500,000 because the x-value of the point of equilibrium is 2,500,000. To find the height of this triangle, subtract the p-value of the point of equilibrium, 275, from the p-intercept of the demand equation, 350, to obtain 75. You can find the base and height of the triangle representing the producer surplus in a similar manner.

$$\begin{aligned} \text{Consumer surplus} &= \tfrac{1}{2}(\text{base})(\text{height}) \\ &= \tfrac{1}{2}(2{,}500{,}000)(75) \\ &= \$93{,}750{,}000 \\ \text{Producer surplus} &= \tfrac{1}{2}(\text{base})(\text{height}) \\ &= \tfrac{1}{2}(2{,}500{,}000)(75) \\ &= \$93{,}750{,}000 \end{aligned}$$

Checkpoint 6

In Example 6, the supply equation is given by $p = 230 + 0.00003x$ and the new point of equilibrium is (2,000,000, 290). Find the consumer surplus and producer surplus.

Example 7 Nutrition

The liquid portion of a diet requires at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C.

a. Set up a system of linear inequalities that describes how many cups of each drink should be consumed each day to meet or exceed the minimum daily requirements for calories and vitamins.

b. A nutritionist normally gives a patient 6 cups of dietary drink X and 1 cup of dietary drink Y per day. Supplies of dietary drink X are running low. Use the graph of the system of linear inequalities to determine other combinations of drinks X and Y that can be given that will meet the minimum daily requirements.

SOLUTION

a. Begin by letting x represent the number of cups of drink X and letting y represent the number of cups of drink Y. To meet or exceed the minimum daily requirements, the following inequalities must be satisfied.

$$\begin{cases} 60x + 60y \geq 300 & \text{Calories} \\ 12x + 6y \geq 36 & \text{Vitamin A} \\ 10x + 30y \geq 90 & \text{Vitamin C} \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The last two inequalities are included because x and y cannot be negative. The graph of this system of inequalities is shown in Figure 5.23. (This application is discussed again in Example 5 in Section 5.5.)

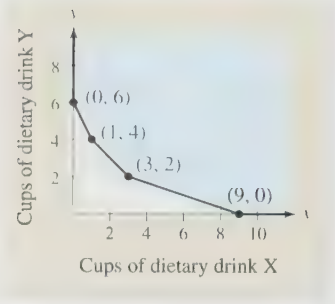
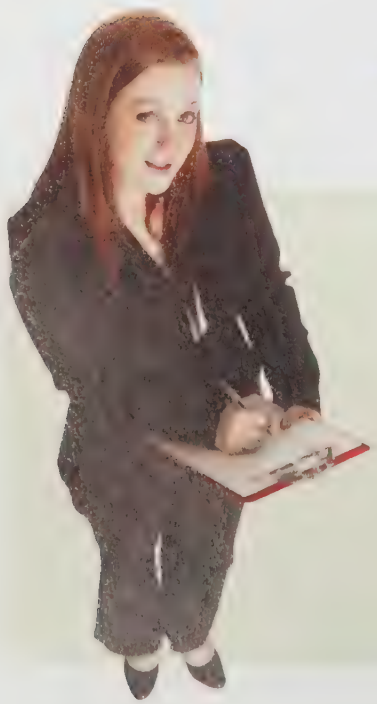


FIGURE 5.23

b. From Figure 5.23, there are many different possible substitutions that the nutritionist can make. Because supplies of dietary drink X are running low, the nutritionist should choose a combination that contains a small amount of drink X. For instance, 1 cup of dietary drink X and 4 cups of dietary drink Y will also meet the minimum daily requirements.

Checkpoint 7

In Example 7, should the nutritionist give a patient 4 cups of dietary drink X and 1 cup of dietary drink Y? Explain.



David Gilder/Shutterstock.com

SUMMARIZE (Section 5.4)

1. Describe how to sketch the graph of an inequality in two variables *(page 445)*. For examples of sketching the graph of an inequality, see Examples 1, 2, and 3.
2. Describe how to solve a system of inequalities *(page 447)*. For examples of solving a system of inequalities, see Examples 4 and 5.
3. State the definitions of consumer surplus and producer surplus *(page 449)*. For an example of finding consumer surplus and producer surplus, see Example 6.
4. Describe a real-life example of how a system of linear inequalities can be used to determine the amounts of dietary drinks that should be consumed each day to meet or exceed the minimum daily requirements *(page 450, Example 7)*.

SKILLS WARM UP 5.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.1, 2.2, 3.1, 5.1, and 5.2.

In Exercises 1–6, classify the graph of the equation as a line, a parabola, or a circle.

1. $x + y = 3$
2. $y = x^2 - 4$
3. $x^2 + y^2 = 9$
4. $y = -x^2 + 1$
5. $4x - y = 8$
6. $y^2 = 16 - x^2$

In Exercises 7–10, solve the system of equations.

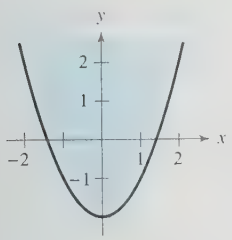
7. $\begin{cases} x + 2y = 3 \\ 4x - 7y = -3 \end{cases}$
8. $\begin{cases} 2x - 3y = 4 \\ x + 5y = 2 \end{cases}$
9. $\begin{cases} x^2 + y = 5 \\ 2x - 4y = 0 \end{cases}$
10. $\begin{cases} x^2 + y^2 = 13 \\ x + y = 5 \end{cases}$

Exercises 5.4

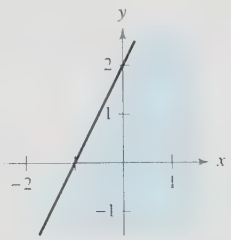
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–6, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

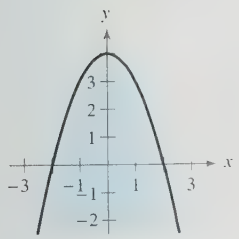
(a)



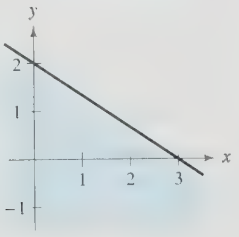
(b)



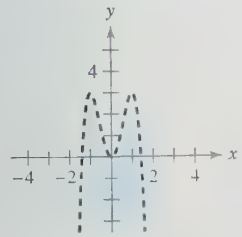
(c)



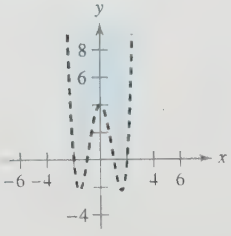
(d)



(e)



(f)



1. $2x + 3y \leq 6$
2. $2x - y \geq -2$
3. $x^2 - y \leq 2$
4. $y \leq 4 - x^2$
5. $y > x^4 - 5x^2 + 4$
6. $3x^4 + y < 6x^2$

Sketching the Graph of an Inequality In Exercises 7–22, sketch the graph of the inequality. *See Examples 1, 2, and 3.*

7. $x \geq 2$
8. $x < 4$
9. $y + 2x^2 > 0$
10. $y^2 - x < 0$
11. $y > -1$
12. $y \leq 3$
13. $y < 2 - x$
14. $y > 2x - 4$
15. $2y - x \geq 4$
16. $5x + 3y \geq -15$
17. $y < \ln x$
18. $y \geq -\ln x + 1$
19. $x^2 + y^2 \leq 4$
20. $x^2 + y^2 > 4$
21. $y - (x + 2)^3 \leq 0$
22. $y - (x - 3)^3 \geq 0$

Solving a System of Inequalities In Exercises 23–44, graph the solution set of the system of inequalities. *See Examples 4 and 5.*

23. $\begin{cases} x + y \leq 2 \\ -x + y \leq 2 \\ y \geq 0 \end{cases}$
24. $\begin{cases} 3x + 2y < 6 \\ x > 1 \\ y > 0 \end{cases}$
25. $\begin{cases} x + y \leq 5 \\ x \geq 2 \\ y \geq 0 \end{cases}$
26. $\begin{cases} 2x + y \geq 2 \\ x \leq 2 \\ y \leq 1 \end{cases}$
27. $\begin{cases} -3x + 2y < 6 \\ x + 4y < -2 \\ 2x + y < 3 \end{cases}$
28. $\begin{cases} x - 7y > -36 \\ 5x + 2y > 5 \\ 6x - 5y > 6 \end{cases}$
29. $\begin{cases} x^2 + y \leq 6 \\ x \geq -1 \\ y \geq 0 \end{cases}$
30. $\begin{cases} 2x^2 + y > 4 \\ x < -1 \\ y > -2 \end{cases}$
31. $\begin{cases} 2x + y > 2 \\ 6x + 3y < 2 \end{cases}$
32. $\begin{cases} 5x - 3y > -6 \\ 5x - 3y < -9 \end{cases}$
33. $\begin{cases} y \geq -3 \\ y \leq 1 - x^2 \end{cases}$
34. $\begin{cases} y < -(x - 1)^2 \\ y > (x - 3)^2 - 4 \end{cases}$
35. $\begin{cases} x^2 + y^2 \leq 16 \\ x^2 + y^2 < 1 \end{cases}$
36. $\begin{cases} x^2 + y^2 \leq 25 \\ x^2 + y^2 \geq 9 \end{cases}$
37. $\begin{cases} x > y^2 \\ x < y + 2 \end{cases}$
38. $\begin{cases} x < 2y - y^2 \\ 0 < x + y \end{cases}$
39. $\begin{cases} y \leq \sqrt{3x} + 1 \\ y \geq x + 1 \end{cases}$
40. $\begin{cases} y < \sqrt{2x} + 3 \\ y > x + 3 \end{cases}$

41. $\begin{cases} y < x^3 - 2x + 1 \\ y > -2x \\ x \le 1 \end{cases}$

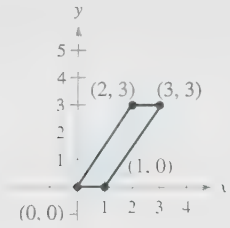
42. $\begin{cases} x^2 + y \le 4 \\ y \ge 2x \\ x \ge -1 \end{cases}$

43. $\begin{cases} y \le e^x \\ y \ge \ln x \\ x \ge \frac{1}{2} \\ x \le 2 \end{cases}$

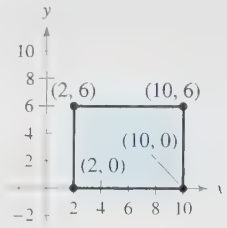
44. $\begin{cases} y \le e^{-x^2/2} \\ y \ge 0 \\ x \ge -1 \\ x \le 0 \end{cases}$

Geometry In Exercises 45–50, write a system of inequalities that corresponds to the solution set shown in the graph.

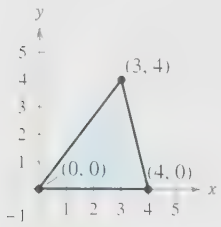
45. Parallelogram



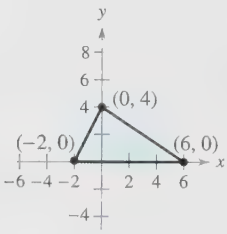
46. Rectangle



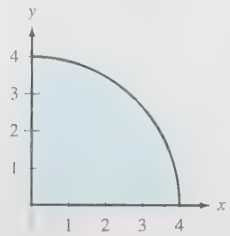
47. Triangle



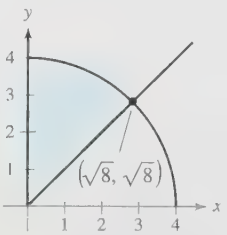
48. Triangle



49. Sector of a circle



50. Sector of a circle



51. Furniture Production A furniture company produces tables and chairs. Each table requires 2 hours in the assembly center and $1\frac{1}{2}$ hours in the finishing center. Each chair requires $1\frac{1}{2}$ hours in the assembly center and $\frac{3}{4}$ hour in the finishing center. The company's assembly center is available 18 hours per day, and its finishing center is available 12 hours per day. Let x and y be the numbers of tables and chairs produced per day, respectively. (a) Find a system of inequalities describing all possible production levels, and (b) sketch the graph of the system.

52. Kayak Inventory A store sells two models of kayaks. Because of the demand, it is necessary to stock at least twice as many units of model A as units of model B. The costs to the store for the two models are \$500 and \$700, respectively. The management does not want more than \$25,000 in kayak inventory at any one time, and it wants at least six model A kayaks and three model B kayaks in inventory at all times. (a) Find a system of inequalities describing all possible inventory levels, and (b) sketch the graph of the system.

Consumer and Producer Surpluses In Exercises 53–56, find the consumer surplus and producer surplus for the demand and supply equations. *See Example 6.*

	Demand	*Supply*
53.	$p = 56 - 0.0001x$	$p = 22 + 0.00001x$
54.	$p = 60 - 0.00001x$	$p = 15 + 0.00004x$
55.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
56.	$p = 600 - 0.0002x$	$p = 125 + 0.0006x$

57. Think About It Under what circumstances are the consumer surplus and producer surplus equal for a pair of linear supply and demand equations? Explain.

58. Think About It Under what circumstances is the consumer surplus greater than the producer surplus for a pair of linear supply and demand equations? Explain.

59. Investment You plan to invest up to \$30,000 in two different interest-bearing accounts. Each account is to contain at least \$7000. Moreover, one account should have at least twice the amount that is in the other account. (a) Find a system of inequalities that describes the amounts that you can invest in each account, and (b) sketch the graph of the system.

60. Concert Ticket Sales Two types of tickets are to be sold for a concert. One type costs \$30 per ticket and the other type costs \$40 per ticket. The promoter of the concert must sell at least 15,000 tickets, including at least 8000 of the \$30 tickets and at least 4000 of the \$40 tickets. Moreover, the gross receipts must total at least \$500,000 in order for the concert to be held. (a) Find a system of inequalities describing the different numbers of tickets that must be sold, and (b) sketch the graph of the system.

61. Diet Supplement A dietitian designs a special diet supplement using two different foods. Each ounce of food X contains 20 units of calcium, 10 units of iron, and 15 units of vitamin B. Each ounce of food Y contains 15 units of calcium, 20 units of iron, and 20 units of vitamin B. The minimum daily requirements for the diet are 400 units of calcium, 250 units of iron, and 220 units of vitamin B. (a) Find a system of inequalities describing the different amounts of food X and food Y that the dietitian can use in the diet, and (b) sketch the graph of the system.

62. Diet Supplement A dietitian designs a special diet supplement using two different foods. Each ounce of food X contains 12 units of calcium, 10 units of iron, and 20 units of vitamin B. Each ounce of food Y contains 15 units of calcium, 20 units of iron, and 12 units of vitamin B. The minimum daily requirements for the diet are 300 units of calcium, 280 units of iron, and 300 units of vitamin B.

(a) Find a system of inequalities describing the different amounts of food X and food Y that the dietitian can use in the diet.

(b) Sketch the graph of the system.

(c) A nutritionist normally gives a patient 10 ounces of food X and 12 ounces of food Y per day. Supplies of food Y are running low. What other combinations of foods X and Y can be given to the patient to meet the minimum daily requirements?

63. Nesting Falcons The numbers of nesting pairs y of falcons in a nature conservancy from 2006 through 2010 can be approximated by the linear model

$$y = 2.8t + 1.8, \quad 6 \le t \le 10$$

where t represents the year, with $t = 6$ corresponding to 2006.

(a) The *total* number of nesting pairs during this five-year period can be approximated by finding the area of the trapezoid represented by the system below.

$$\begin{cases} y \le 2.8t + 1.8 \\ y \ge 0 \\ t \ge 5.5 \\ t \le 10.5 \end{cases}$$

Graph this region using a graphing utility.

(b) Use the formula for the area of a trapezoid to approximate the total number of nesting pairs.

64. Sales The annual sales y (in millions of dollars) for WebMD Health from 2005 through 2010 can be approximated by the linear model

$$y = 69.50t - 169.5, \quad 5 \le t \le 10$$

where t represents the year, with $t = 5$ corresponding to 2005. *(Source: WebMD Health Corporation)*

(a) The *total* sales during this six-year period can be approximated by finding the area of the trapezoid represented by the system below.

$$\begin{cases} y \le 69.50t - 169.5 \\ y \ge 0 \\ t \ge 4.5 \\ t \le 10.5 \end{cases}$$

Graph this region using a graphing utility.

(b) Use the formula for the area of a trapezoid to approximate the total sales.

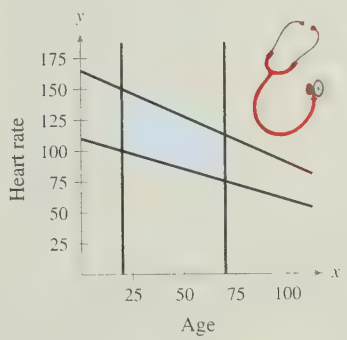
65. Writing Explain the difference between the graphs of the inequality $x \le 4$ on the real number line and on the rectangular coordinate system.

66. HOW DO YOU SEE IT? A person's maximum heart rate is

$$220 - x$$

where x is the person's age in years for $20 \le x \le 70$. When a person exercises, it is recommended that the person strive for a heart rate that is at least 50% of the maximum and at most 75% of the maximum. The exercise target heart rate region can be approximated by the system of inequalities below, where y represents a person's heart rate. *(Source: American Heart Association)*

$$\begin{cases} y \ge 0.5(220 - x) \\ y \le 0.75(220 - x) \\ x \ge 20 \\ x \le 70 \end{cases}$$



Using the graph of the system of inequalities above, find two solutions of the system and interpret the meanings of the solutions in the context of the problem.

67. Geometry Write a system of inequalities whose graphed solution set is a rectangle.

68. Geometry Write a system of inequalities whose graphed solution set is an isosceles triangle.

69. Graphical Reasoning Two concentric circles have radii x and y, where $y > x$. The area between the circles must be at least 10 square units.

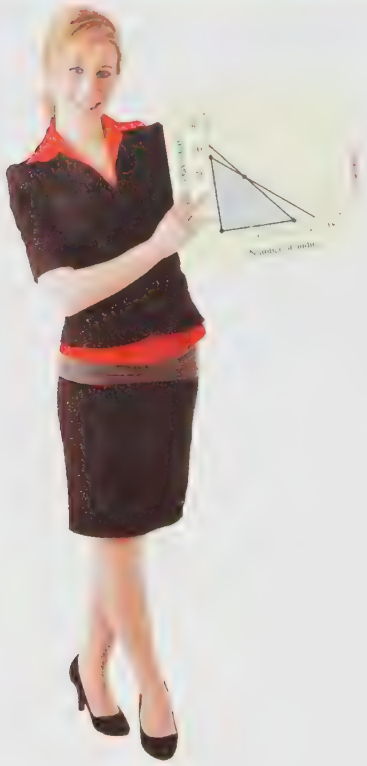
(a) Find a system of inequalities describing the constraints on the circles.

(b) Use a graphing utility to graph the system of inequalities in part (a). Graph the line $y = x$ in the same viewing window.

(c) Identify the graph of the line in relation to the boundary of the inequality. Explain its meaning in the context of the problem.

5.5 Linear Programming

- Use linear programming to minimize or maximize an objective function.
- Use linear programming to optimize an application.



In Exercise 43 on page 462, you will use linear programming to find the numbers of audits and tax returns that will bring in an optimal revenue for an accounting firm.

Linear Programming: A Graphical Approach

Many applications in business and economics involve a process called **optimization,** in which you are asked to find the minimum cost, the maximum profit, or the minimum use of resources. In this section, you will study an optimization strategy that applies systems of linear inequalities, called **linear programming.**

A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints.** The objective function gives the quantity that is to be maximized (or minimized), and the constraints determine the set of **feasible solutions.** For example, suppose you are asked to maximize the value of

$$z = ax + by \qquad \text{Objective function}$$

subject to a set of constraints that determines the region in Figure 5.24. Because every point in the region satisfies each constraint, it is not clear how you should go about finding the point that yields a maximum value of z. Fortunately, it can be shown that if there is an optimal solution, it must occur at one of the vertices of the region. This means that *you can find the maximum value by testing z at each of the vertices.*

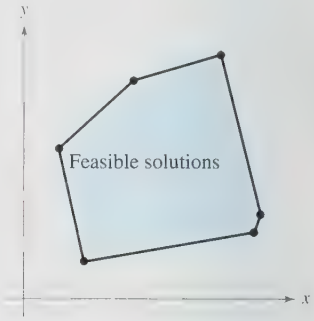


FIGURE 5.24

Optimal Solution of a Linear Programming Problem

If a linear programming problem has a solution, then it must occur at a vertex of the set of feasible solutions. If the problem has more than one solution, then at least one solution must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is unique.

A linear programming problem can include hundreds, and sometimes even thousands, of variables. However, in this section, you will solve linear programming problems that involve only two variables. The guidelines for solving a linear programming problem in two variables are listed below.

Guidelines for Solving a Linear Programming Problem

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are *feasible solutions.*)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and a maximum value will exist. (For an unbounded region, *if* an optimal solution exists, then it will occur at a vertex.)

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Example 1 **Solving a Linear Programming Problem**

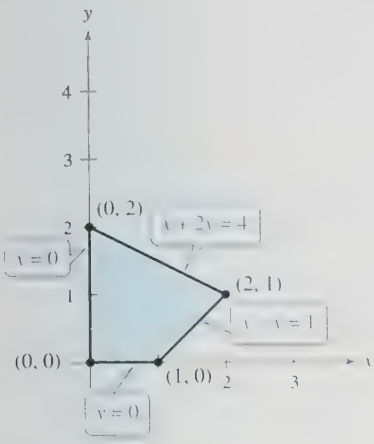


FIGURE 5.25

Find the maximum value of

$$z = 3x + 2y$$ Objective function

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\}$$ Constraints

SOLUTION The constraints form the region shown in Figure 5.25. At the four vertices of this region, the objective function has the values shown below.

At (0, 0): $z = 3(0) + 2(0) = 0$

At (1, 0): $z = 3(1) + 2(0) = 3$

At (2, 1): $z = 3(2) + 2(1) = 8$ Maximum value of z

At (0, 2): $z = 3(0) + 2(2) = 4$

So, the maximum value of z is 8, which occurs when $x = 2$ and $y = 1$.

Checkpoint 1

Find the maximum value of

$$z = 2x + 3y$$

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 3 \\ x - y &\leq 2 \end{aligned}\right\}$$

STUDY TIP

Remember that a vertex of a region can be found using a system of linear equations. The system will consist of the equations of the lines passing through the vertex.

In Example 1, try testing some of the *interior* points of the region. You will see that the corresponding values of z are less than 8. Here are some examples.

At (1, 1): $z = 3(1) + 2(1) = 5$

At $\left(1, \frac{1}{2}\right)$: $z = 3(1) + 2\left(\frac{1}{2}\right) = 4$

At $\left(\frac{1}{2}, \frac{3}{2}\right)$: $z = 3\left(\frac{1}{2}\right) + 2\left(\frac{3}{2}\right) = \frac{9}{2}$

At $\left(\frac{3}{2}, 1\right)$: $z = 3\left(\frac{3}{2}\right) + 2(1) = \frac{13}{2}$

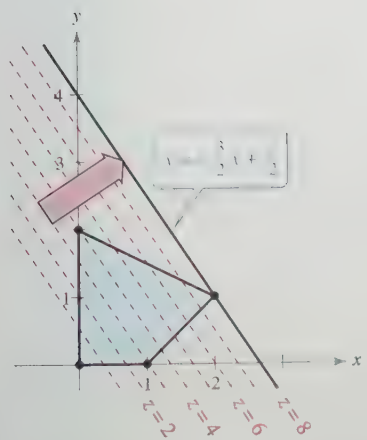


FIGURE 5.26

To see why the maximum value of the objective function in Example 1 must occur at a vertex, consider writing the objective function in slope-intercept form

$$y = -\frac{3}{2}x + \frac{z}{2}$$ Family of lines

where

$$\frac{z}{2}$$

is the y-intercept of the objective function. This equation represents a family of lines, each of slope $-\frac{3}{2}$. Of these infinitely many lines, you want the one that has the largest z-value while still intersecting the region determined by the constraints. In other words, of all the lines whose slope is $-\frac{3}{2}$, you want the one that has the largest y-intercept and intersects the given region, as shown in Figure 5.26. It should be clear that such a line will pass through one (or more) of the vertices of the region.

The guidelines for solving a linear programming problem will work whether the objective function is to be maximized or minimized. The steps used are precisely the same. In other words, once you have evaluated the objective function at the vertices of the set of feasible solutions, you simply choose the largest value as the maximum and the smallest value as the minimum. For instance, the same test used in Example 1 to find the maximum value of z can be used to conclude that the minimum value of z is 0, and that this value occurs at the vertex $(0, 0)$.

Example 2 Solving a Linear Programming Problem

Find (a) the maximum value and (b) the minimum value of

$$z = 4x + 6y$$ Objective function

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 11 \\ x + y &\leq 27 \\ 2x + 5y &\leq 90 \end{aligned}\right\}$$ Constraints

SOLUTION

a. The region bounded by the constraints is shown in Figure 5.27.

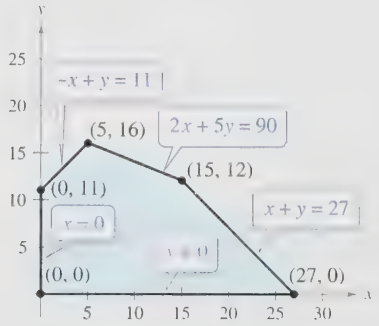


FIGURE 5.27

By testing the objective function at each vertex, you obtain the values shown below.

At $(0, 0)$: $z = 4(0) + 6(0) = 0$

At $(0, 11)$: $z = 4(0) + 6(11) = 66$

At $(5, 16)$: $z = 4(5) + 6(16) = 116$

At $(15, 12)$: $z = 4(15) + 6(12) = 132$ Maximum value of z

At $(27, 0)$: $z = 4(27) + 6(0) = 108$

So, the maximum value of z is 132, which occurs when $x = 15$ and $y = 12$.

b. Using the values of z at the vertices in part (a), you can conclude that the minimum value of z is 0, and that this value occurs when $x = 0$ and $y = 0$.

ALGEBRA TUTOR

For help with the algebra used to find the vertices of the region in Example 2, see the *Chapter 5 Algebra Tutor* on page 465.

Checkpoint 2

Find (a) the maximum value and (b) the minimum value of $z = 5x + 2y$ subject to the same constraints as in Example 2. ■

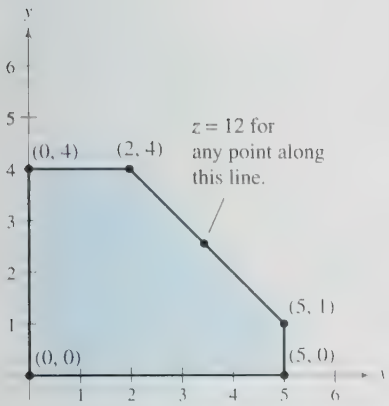


FIGURE 5.28

It is possible for the maximum (or minimum) value in a linear programming problem to occur at *two* different vertices. For instance, at the vertices of the region shown in Figure 5.28, the objective function

$$z = 2x + 2y \qquad \text{Objective function}$$

has the values shown below.

At (0, 0): $z = 2(0) + 2(0) = 0$

At (0, 4): $z = 2(0) + 2(4) = 8$

At (2, 4): $z = 2(2) + 2(4) = 12$ Maximum value of z

At (5, 1): $z = 2(5) + 2(1) = 12$ Maximum value of z

At (5, 0): $z = 2(5) + 2(0) = 10$

In this case, you can conclude that the objective function has a maximum value not only at the vertices (2, 4) and (5, 1); it also has a maximum value (of 12) at *any point on the line segment connecting these two vertices*. Note that the objective function

$$y = -x + \frac{1}{2}z$$

has the same slope as the line through the vertices (2, 4) and (5, 1).

Some linear programming problems have no optimal solution. This can occur when the region determined by the constraints is *unbounded*. This is shown in Example 3.

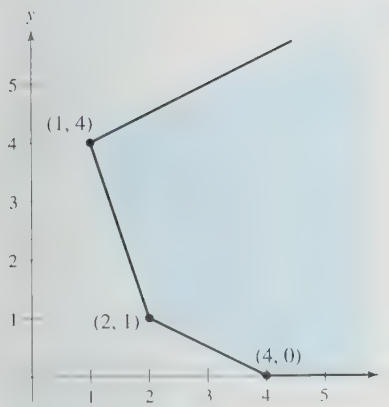


FIGURE 5.29

Example 3 **An Unbounded Region**

Find the maximum value of

$$z = 4x + 2y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} x + 2y &\geq 4 \\ 3x + y &\geq 7 \\ -x + 2y &\leq 7 \end{aligned}\right\} \qquad \text{Constraints}$$

SOLUTION The region determined by the constraints is shown in Figure 5.29. For this unbounded region, there is no maximum value of z. To see this, note that the point $(x, 0)$ lies in the region for all values of $x \geq 4$. By choosing x to be large, you can obtain values of

$$\begin{aligned} z &= 4(x) + 2(0) \\ &= 4x \end{aligned}$$

that are as large as you want. So, there is no maximum value of z. For this problem, there *is* a minimum value of z, $z = 10$, which occurs at the vertex (2, 1), as shown below.

At (1, 4): $z = 4(1) + 2(4) = 12$

At (2, 1): $z = 4(2) + 2(1) = 10$ Minimum value of z

At (4, 0): $z = 4(4) + 2(0) = 16$

✓Checkpoint 3

Find the maximum value of the objective function

$$z = x + 8y$$

where $x \geq 0$ and $y \geq 0$, subject to the same constraints as in Example 3.

Applications

Example 4 shows how linear programming can be used to find the maximum profit in a business application.

Example 4 Optimal Profit

A manufacturer wants to maximize the profit for two laboratory products. Product I yields a profit of \$1.50 per unit, and product II yields a profit of \$2.00 per unit. Market tests and available resources have indicated the following constraints.

1. The combined production level should not exceed 1200 units per month.
2. The demand for product II is no more than half the demand for product I.
3. The production level of product I is less than or equal to 600 units plus three times the production level of product II.

What is the optimal production level for each product?

SOLUTION Let x be the number of units of product I and let y be the number of units of product II. The objective function (for the combined profit) is given by

$$P = 1.5x + 2y.$$ Objective function

The three constraints translate into the following linear inequalities.

1. $x + y \leq 1200 \quad \Rightarrow \quad x + y \leq 1200$
2. $y \leq \frac{1}{2}x \quad \Rightarrow \quad -x + 2y \leq 0$
3. $x \leq 3y + 600 \quad \Rightarrow \quad x - 3y \leq 600$

Because neither x nor y can be negative, you also have the two additional constraints of $x \geq 0$ and $y \geq 0$.

So, the constraints are as shown.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 1200 \\ -x + 2y &\leq 0 \\ x - 3y &\leq 600 \end{aligned}\right\}$$ Constraints

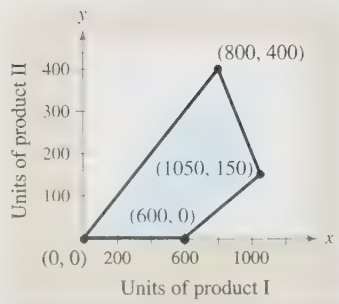


FIGURE 5.30

Figure 5.30 shows the region determined by the constraints. To find the maximum profit, test the value of P at each vertex of the region.

At (0, 0): $P = 1.5(0) + 2(0) = 0$

At (800, 400): $P = 1.5(800) + 2(400) = 2000$ Maximum profit

At (1050, 150): $P = 1.5(1050) + 2(150) = 1875$

At (600, 0): $P = 1.5(600) + 2(0) = 900$

So, the maximum profit is \$2000, and it occurs when the monthly production levels are 800 units of product I and 400 units of product II.

Checkpoint 4

In Example 4, the manufacturer improves the production of product I so that it yields a profit of \$2.50 per unit. How would this improvement affect the optimal number of units the manufacturer should sell in order to obtain a maximum profit? ■

Example 5 shows how linear programming can be used to find the optimal cost in a real-life application.

Example 5 Optimal Cost

The liquid portion of a diet requires at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X costs \$0.12 and provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y costs \$0.15 and provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. How many cups of each drink should be consumed each day to obtain an optimal cost and still meet the daily requirements?

SOLUTION As in Example 7 on page 450, let x be the number of cups of dietary drink X and let y be the number of cups of dietary drink Y.

$$\left.\begin{array}{lrcl} \text{For calories:} & 60x + 60y & \geq & 300 \\ \text{For vitamin A:} & 12x + 6y & \geq & 36 \\ \text{For vitamin C:} & 10x + 30y & \geq & 90 \\ & x & \geq & 0 \\ & y & \geq & 0 \end{array}\right\} \quad \text{Constraints}$$

The cost C is given by

$$C = 0.12x + 0.15y. \qquad \text{Objective function}$$

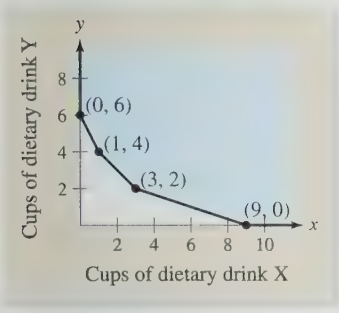


FIGURE 5.31

The graph of the region corresponding to the constraints is shown in Figure 5.31. Because you want to incur as little cost as possible, you want to determine the *minimum* cost. To determine the minimum cost, test C at each vertex of the region, as shown below.

At (0, 6): $C = 0.12(0) + 0.15(6) = 0.90$

At (1, 4): $C = 0.12(1) + 0.15(4) = 0.72$

At (3, 2): $C = 0.12(3) + 0.15(2) = 0.66$ Minimum value of C

At (9, 0): $C = 0.12(9) + 0.15(0) = 1.08$

So, the minimum cost is \$0.66 per day, and this cost occurs when 3 cups of dietary drink X and 2 cups of dietary drink Y are consumed each day.

Checkpoint 5

In Example 5, a cup of dietary drink Y costs \$0.11. How would this affect the number of cups of each drink that should be consumed each day to obtain an optimal cost and still meet the daily requirements?



SUMMARIZE (Section 5.5)

1. Describe the guidelines for solving a linear programming problem *(page 454)*. For an example of solving a linear programming problem, see Example 1.
2. Describe how to find the maximum and minimum values of an objective function of a linear programming problem *(page 456)*. For an example of maximizing and minimizing the objective function of a linear programming problem, see Example 2.
3. Describe a situation in which a linear programming problem has no optimal solution *(page 457)*. For an example of solving a linear programming problem that has no optimal solution, see Example 3.
4. Describe a real-life example of how linear programming can be used to find the maximum profit in a business application *(page 458, Example 4)*.

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SKILLS WARM UP 5.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.2, 5.1, and 5.4.

In Exercises 1–4, sketch the graph of the linear equation.

1. $y + x = 3$

2. $y - x = 12$

3. $x = 0$

4. $y = 4$

In Exercises 5–8, find the point of intersection of the two lines.

5. $x + y = 4$
$x = 0$

6. $x + 2y = 12$
$y = 0$

7. $x + y = 4$
$2x + 3y = 9$

8. $x + 2y = 12$
$2x + y = 9$

In Exercises 9 and 10, sketch the graph of the inequality.

9. $2x + 3y \geq 18$

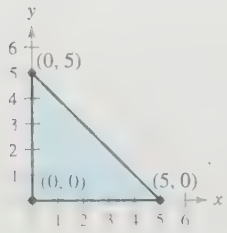
10. $4x + 3y \geq 12$

Exercises 5.5

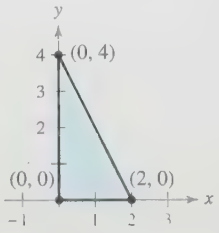
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Solving a Linear Programming Problem In Exercises 1–8, find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints. (For each exercise, the graph of the region determined by the constraints is provided.) *See Examples 1 and 2.*

1. *Objective function:*
$z = 3x + 4y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + y \leq 5$



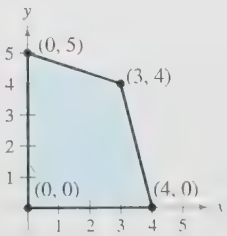
2. *Objective function:*
$z = 2x + 8y$
Constraints:
$x \geq 0$
$y \geq 0$
$2x + y \leq 4$



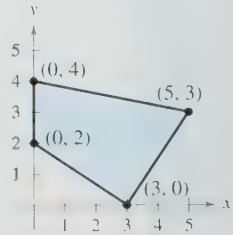
3. *Objective function:*
$z = 9x + 6y$
Constraints:
See Exercise 1.

4. *Objective function:*
$z = 7x + 3y$
Constraints:
See Exercise 2.

5. *Objective function:*
$z = 6x + 2y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + 3y \leq 15$
$4x + y \leq 16$



6. *Objective function:*
$z = 5x + 4y$
Constraints:
$x \geq 0$
$y \geq 0$
$2x + 3y \geq 6$
$3x - 2y \leq 9$
$x + 5y \leq 20$



7. *Objective function:*
$z = 5x + 0.5y$
Constraints:
See Exercise 5.

8. *Objective function:*
$z = x + 6y$
Constraints:
See Exercise 6.

Solving a Linear Programming Problem In Exercises 9–20, sketch the region determined by the indicated constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the constraints. *See Examples 1, 2, and 3.*

9. *Objective function:*
$z = 6x + 10y$
Constraints:
$x \geq 0$
$y \geq 0$
$3x + 5y \leq 15$

10. *Objective function:*
$z = 7x + 8y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + 2y \leq 8$

11. *Objective function:*
$z = 8x + 4y$
Constraints:
See Exercise 9.

12. *Objective function:*
$z = 10x + 3y$
Constraints:
See Exercise 10.

13. *Objective function:*
$z = 4x + 5y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + y \geq 8$
$3x + 5y \geq 30$

14. *Objective function:*
$z = 4x + 5y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + y \leq 5$
$x + 2y \leq 6$

15. *Objective function:*
$z = 2x + 7y$
Constraints:
See Exercise 13.

16. *Objective function:*
$z = 2x - y$
Constraints:
See Exercise 14.

17. *Objective function:*
$z = 3x + 6y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + 2y \leq 40$
$x + y \leq 30$
$2x + 3y \leq 65$

18. *Objective function:*
$z = x$
Constraints:
$x \geq 0$
$y \geq 0$
$2x + 3y \leq 60$
$2x + y \leq 28$
$4x + y \leq 48$

19. *Objective function:*
$z = x + y$
Constraints:
See Exercise 17.

20. *Objective function:*
$z = y$
Constraints:
See Exercise 18.

Maximizing the Objective Function In Exercises 21–24, maximize the objective function subject to the constraints $3x + y \leq 15$, $4x + 3y \leq 30$, $x \geq 0$, and $y \geq 0$.

21. $z = 2x + y$

22. $z = 5x + y$

23. $z = 4x + 3y$

24. $z = 3x + y$

Maximizing the Objective Function In Exercises 25–28, maximize the objective function subject to the constraints $x + 5y \leq 25$, $x + y \leq 9$, $4x + 2y \leq 32$, $x \geq 0$, and $y \geq 0$.

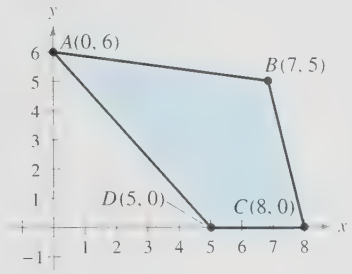
25. $z = 6x + 7y$

26. $z = 10x + 8y$

27. $z = 15x + y$

28. $z = 5x + 11y$

Think About It In Exercises 29–36, find an objective function that has a maximum or minimum value at the indicated vertex of the constraint region shown. (There are many correct answers.)



29. The maximum occurs at vertex A.

30. The maximum occurs at vertex B.

31. The minimum occurs at vertex C.

32. The minimum occurs at vertex D.

33. The maximum occurs at vertices A and B.

34. The maximum occurs at vertices B and C.

35. The minimum occurs at vertices A and D.

36. The minimum occurs at vertices C and D.

37. Optimal Profit A fruit grower raises crops A and B. The profit is \$185 per acre for crop A and \$245 per acre for crop B. Research and available resources indicate the following constraints.

- The fruit grower has 150 acres of land for raising the crops.
- It takes 1 day to trim an acre of crop A and 2 days to trim an acre of crop B, and there are 240 days per year available for trimming.
- It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking.

What is the optimal acreage for each fruit? What is the optimal profit?

38. Optimal Profit The costs to a store for two models of Global Positioning System (GPS) receivers are \$80 and \$100. The \$80 model yields a profit of \$25 and the \$100 model yields a profit of \$30. Market tests and available resources indicate the following constraints.

- The merchant estimates that the total monthly demand will not exceed 200 units.
- The merchant does not want to invest more than \$18,000 in GPS receiver inventory.

What is the optimal inventory level for each model? What is the optimal profit?

39. Optimal Cost A farming cooperative mixes two brands of cattle feed. Brand X costs \$30 per bag, and brand Y costs \$25 per bag. Research and available resources have indicated the following constraints.

- Brand X contains two units of nutritional element A, two units of element B, and two units of element C.
- Brand Y contains one unit of nutritional element A, nine units of element B, and three units of element C.
- The minimum requirements for nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively.

What is the optimal number of bags of each brand that should be mixed? What is the optimal cost?

40. Optimal Cost A humanitarian agency can use two models of vehicles for a refugee rescue mission. Each model A vehicle costs \$1000 and each model B vehicle costs \$1500. Mission strategies and objectives indicate the following constraints.

- A total of at least 20 vehicles must be used.
- A model A vehicle can hold 45 boxes of supplies. A model B vehicle can hold 30 boxes of supplies. The agency must deliver at least 690 boxes of supplies to the refugee camp.
- A model A vehicle can hold 20 refugees. A model B vehicle can hold 32 refugees. The agency must rescue at least 520 refugees.

What is the optimal number of vehicles of each model that should be used? What is the optimal cost?

41. Optimal Profit A manufacturer produces two models of bicycles. The times (in hours) required for assembling, painting, and packaging each model are shown in the table.

Process	Model A	Model B
Assembling	2	2.5
Painting	4	1
Packaging	1	0.75

The total times available for assembling, painting, and packaging are 4000 hours, 4800 hours, and 1500 hours, respectively. The profits per unit are \$50 for model A and \$75 for model B. What is the optimal production level for each model? What is the optimal profit?

42. Optimal Profit A company makes two models of doghouses. The times (in hours) required for assembling, painting, and packaging are shown in the table.

Process	Model A	Model B
Assembling	2.5	3
Painting	2	1
Packaging	0.75	1.25

The total times available for assembling, painting, and packaging are 4000 hours, 2500 hours, and 1500 hours, respectively. The profits per unit are \$60 for model A and \$75 for model B. What is the optimal production level for each model? What is the optimal profit?

43. Optimal Revenue An accounting firm charges \$2500 for an audit and \$350 for a tax return. Research and available resources have indicated the following constraints.

- The firm has 900 hours of staff time available each week.
- The firm has 155 hours of review time available each week.
- Each audit requires 75 hours of staff time and 10 hours of review time.
- Each tax return requires 12.5 hours of staff time and 2.5 hours of review time.

What numbers of audits and tax returns will bring in an optimal revenue?

44. Optimal Revenue The accounting firm in Exercise 43 lowers its charge for an audit to \$2000. What numbers of audits and tax returns will bring in an optimal revenue?

45. Media Selection A company has budgeted a maximum of \$4,800,000 for national advertising of an allergy medication. Each minute of television time costs \$400,000 and each one-page newspaper ad costs \$80,000. Each television ad is expected to be viewed by 20 million viewers, and each newspaper ad is expected to be seen by 5 million readers. The company's market research department recommends that at most 80% of the advertising budget be spent on television ads. What is the optimal amount that should be spent on each type of ad? What is the optimal total audience?

46. Optimal Profit A fruit juice company makes two drinks by blending apple and pineapple juices. The percents of apple juice and pineapple juice in each drink are shown in the table.

Mixture	Drink A	Drink B
Apple juice	30%	60%
Pineapple juice	70%	40%

There are 1000 liters of apple juice and 1500 liters of pineapple juice available. The profit for drink A is \$0.70 per liter and the profit for drink B is \$0.60 per liter. What is the optimal production level for each type of drink? What is the optimal profit?

47. Investments An investor has up to $250,000 to invest in two types of investments. Type A investments pay 7% annually and type B pay 12% annually. To have a well-balanced portfolio, the investor imposes the following conditions. At least one-fourth of the total portfolio is to be allocated to type A investments and at least one-fourth is to be allocated to type B investments. What is the optimal amount that should be invested in each type of investment? What is the optimal return?

48. Investments An investor has up to $450,000 to invest in two types of investments. Type A investments pay 8% annually and type B pay 14% annually. To have a well-balanced portfolio, the investor imposes the following conditions. At least one-half of the total portfolio is to be allocated to type A investments and at least one-fourth is to be allocated to type B investments. What is the optimal amount that should be invested in each type of investment? What is the optimal return?

49. Optimal Profit A company makes two models of a patio furniture set. The times for assembling, finishing, and packaging model A are 3 hours, 2.5 hours, and 0.6 hour, respectively. The times for model B are 2.75 hours, 1 hour, and 1.25 hours. The total times available for assembling, finishing, and packaging are 3000 hours, 2400 hours, and 1200 hours, respectively. The profit per unit for model A is $100 and the profit per unit for model B is $85. What is the optimal production level for each model? What is the optimal profit?

50. Optimal Profit A manufacturer produces two models of elliptical cross-training exercise machines. The times for assembling, finishing, and packaging model A are 3 hours, 3 hours, and 0.8 hour, respectively. The times for model B are 4 hours, 2.5 hours, and 0.4 hour. The total times available for assembling, finishing, and packaging are 6000 hours, 4200 hours, and 950 hours, respectively. The profits per unit are $300 for model A and $375 for model B. What is the optimal production level for each model? What is the optimal profit?

Recognizing Unusual Characteristics In Exercises 51–56, the given linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. Find the maximum value of the objective function and where it occurs.

51. *Objective function:*

$z = 2.5x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 3x + 5y &\leq 15 \\ 5x + 2y &\leq 10 \end{aligned}$$

52. *Objective function:*

$z = x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 1 \\ -x + 2y &\leq 4 \end{aligned}$$

53. *Objective function:*

$z = -x + 2y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 10 \\ x + y &\leq 7 \end{aligned}$$

54. *Objective function:*

$z = x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 1 \\ -3x + y &\geq 3 \end{aligned}$$

55. *Objective function:* $z = 3x + 4y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 1 \\ 2x + y &\leq 4 \end{aligned}$$

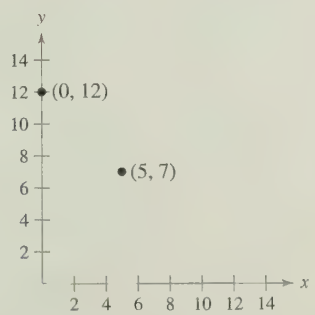
56. *Objective function:* $z = x + 2y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ 2x + y &\leq 4 \end{aligned}$$

57. Reasoning An objective function has a minimum value at the vertex (20, 0). Can you conclude that it also has a minimum value at the point (0, 0)?

58. HOW DO YOU SEE IT? A company trying to determine an optimal profit finds that the objective function has a maximum value at the vertices shown in the graph.



(a) Can you conclude that it also has a maximum value at the point (3, 9)? Explain.

(b) Can you conclude that it also has a maximum value at the point (6, 6)? Explain.

(c) Find two additional points that maximize the objective function.

59. Reasoning When solving a linear programming problem, you find that the objective function has a maximum value at more than one vertex. Can you assume that there are an infinite number of points that will produce the maximum value? Explain your reasoning.

ALGEBRA TUTOR

Simplifying and Solving Systems of Equations

On this page, you can review the algebra used in solving some of the examples in Chapter 5.

Example 1 Simplifying a System of Equations

Example 7, page 426

$$2000 = (r_1 - r_2)\left(4 + \frac{24}{60}\right) \quad \text{Equation 1}$$

$$2000 = (r_1 - r_2)\left(\frac{264}{60}\right) \quad \text{Add.}$$

$$\frac{5000}{11} = (r_1 - r_2) \quad \text{Multiply each side by } \tfrac{60}{264}.$$

$$5000 = 11(r_1 - r_2) \quad \text{Multiply each side by 11.}$$

$$5000 = 11r_1 - 11r_2 \quad \text{Distributive Property}$$

$$2000 = (r_1 + r_2)(4) \quad \text{Equation 2}$$

$$500 = r_1 + r_2 \quad \text{Divide each side by 4.}$$

Example 2 Solving a System of Equations

Example 7, page 438

$$\begin{cases} a - b + c = 3 & \text{Equation 1} \\ a + b + c = 1 & \text{Equation 2} \\ 4a + 2b + c = 6 & \text{Equation 3} \end{cases}$$

$$\begin{cases} a - b + c = 3 \\ 2b = -2 \\ 4a + 2b + c = 6 \end{cases}$$

Adding -1 times the first equation to the second equation produces a new second equation.

$$\begin{cases} a - b + c = 3 \\ b = -1 \\ 4a + 2b + c = 6 \end{cases}$$

Multiplying the second equation by $\frac{1}{2}$ produces a new second equation.

$$\begin{cases} a - b + c = 3 \\ b = -1 \\ 6b - 3c = -6 \end{cases}$$

Adding -4 times the first equation to the third equation produces a new third equation.

$$\begin{cases} a - b + c = 3 \\ b = -1 \\ -3c = 0 \end{cases}$$

Adding -6 times the second equation to the third equation produces a new third equation.

$$\begin{cases} a - b + c = 3 \\ b = -1 \\ c = 0 \end{cases}$$

Multiplying the third equation by $-\frac{1}{3}$ produces a new third equation.

To solve for a, substitute $b = -1$ and $c = 0$ into Equation 1 to obtain

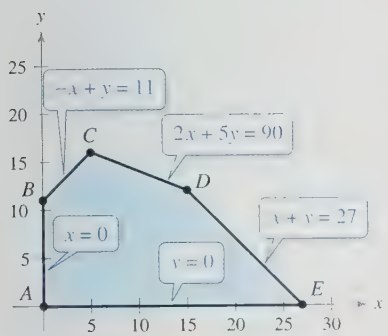
$$a - (-1) + 0 = 3 \qquad a = 2.$$

So, the solution is $a = 2$, $b = -1$, and $c = 0$.

Finding Vertices of a Region

On this page, you can review the algebra used in finding the vertices of the region bounded by the constraints in Example 2 on page 456.

Example 3 Finding Vertices of a Region



Find the vertices of the region bounded by the constraints shown in the figure.

SOLUTION

First, you know that A is at $(0, 0)$ because $x = 0$ and $y = 0$ intersect at the origin. To find vertex B, solve the system

$$\begin{cases} -x + y = 11 & \text{Equation 1} \\ x = 0 & \text{Equation 2} \end{cases}$$

by substituting $x = 0$ into Equation 1 and solving for y.

$$0 + y = 11 \quad \Rightarrow \quad y = 11$$

So, B is at $(0, 11)$. To find vertex C, solve the system

$$\begin{cases} -x + y = 11 & \text{Equation 1} \\ 2x + 5y = 90 & \text{Equation 2} \end{cases}$$

$$\begin{cases} -x + y = 11 \\ 7y = 112 \end{cases}$$

Adding 2 times the first equation to the second equation produces a new second equation.

So, $y = 16$. By back-substituting $y = 16$ into Equation 1, you obtain

$$-x + 16 = 11 \quad \Rightarrow \quad x = 5.$$

So, C is at $(5, 16)$. To find vertex D, solve the system

$$\begin{cases} 2x + 5y = 90 & \text{Equation 1} \\ x + y = 27 & \text{Equation 2} \end{cases}$$

First, solve for y in Equation 2 to get $y = 27 - x$. Then substitute the expression $27 - x$ for y into Equation 1 and solve for x.

$$\begin{aligned} 2x + 5(27 - x) &= 90 && \text{Substitute } 27 - x \text{ for } y. \\ 2x + 135 - 5x &= 90 && \text{Distributive Property} \\ -3x &= -45 && \text{Combine like terms.} \\ x &= 15 && \text{Divide each side by } -3. \end{aligned}$$

By back-substituting $x = 15$ into Equation 2, you obtain

$$15 + y = 27 \quad \Rightarrow \quad y = 12.$$

So, D is at $(15, 12)$. To find vertex E, solve the system

$$\begin{cases} x + y = 27 & \text{Equation 1} \\ y = 0 & \text{Equation 2} \end{cases}$$

by substituting $y = 0$ into Equation 1 and solving for x. You obtain

$$x + 0 = 27 \quad \Rightarrow \quad x = 27.$$

So, E is at $(27, 0)$.

SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 467.
Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 5.1	Review Exercises
■ Solve a system of equations by the method of substitution.	*1–6*
■ Solve a system of equations graphically.	*7, 8, 12*
■ Construct and use a system of equations to solve an application problem.	*9–12*

Section 5.2	
■ Solve a linear system by the method of elimination.	*13–20*
■ Interpret the solution of a linear system graphically.	*21, 22*
■ Construct and use a linear system to solve an application problem.	*23–28*

Section 5.3	
■ Solve a linear system in row-echelon form using back-substitution.	*29, 30*
■ Use Gaussian elimination to solve a linear system.	*31–36*
■ Solve a nonsquare linear system.	*33, 34*
■ Find the equation of a parabola or a circle using a linear system in three or more variables.	*37–40*
■ Construct and use a linear system in three or more variables to solve an application problem.	*41–44*

Section 5.4	
■ Sketch the graph of an inequality in two variables.	*45–50*
■ Solve a system of inequalities.	*51–58*
■ Construct and use a system of inequalities to solve an application problem.	*59–64*

Section 5.5	
■ Use linear programming to minimize or maximize an objective function.	*65–72*
■ Use linear programming to optimize an application.	*73–77*

Study Strategies

■ **Units of Variables in Applied Problems** When using systems of equations to solve real-life applications, be sure to keep track of the unit(s) assigned to each variable. This will allow you to write correctly each equation of the system based on the constraints given in the application.

A wide range of valuable study aids are available to help you master the material in this chapter. The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *www.cengagebrain.com* offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Solving a System by Substitution In Exercises 1–6, solve the system by the method of substitution.

1. $\begin{cases} x + 3y = 10 \\ 4x - 5y = -28 \end{cases}$
2. $\begin{cases} 3x - y - 13 = 0 \\ 4x + 3y - 26 = 0 \end{cases}$
3. $\begin{cases} \frac{1}{2}x + \frac{3}{5}y = -2 \\ 2x + y = 6 \end{cases}$
4. $\begin{cases} 1.3x + 0.9y = 7.5 \\ 0.4x - 0.5y = -0.8 \end{cases}$
5. $\begin{cases} x^2 + y^2 = 100 \\ x + 2y = 20 \end{cases}$
6. $\begin{cases} y = x^3 - 2x^2 - 2x - 3 \\ y = -x^2 + 4x - 3 \end{cases}$

Solving a System of Equations Graphically In Exercises 7 and 8, use a graphing utility to find the point(s) of intersection of the graphs.

7. $\begin{cases} y = x^2 - 3x + 11 \\ y = -x^2 + 2x + 8 \end{cases}$
8. $\begin{cases} y = \sqrt{9 - x^2} \\ y = e^x + 1 \end{cases}$

9. **Break-Even Analysis** You invest \$5000 in a greenhouse. The planter, potting soil, and seed for each plant costs \$6.43, and the selling price is \$12.68. How many of the potted plants do you need to sell to break even?
10. **Break-Even Analysis** You invest \$50,000 to open a temporary employment agency. Each hour of labor that you contract costs you \$13.72 and brings in revenue of \$22.26. How many hours of labor must you contract to break even?
11. **Choice of Newscasts** Television stations A and B are competing for the 6 P.M. newscast audience. Station A is implementing a new newscast format for the 6 P.M. audience. Models that represent the numbers of 6 P.M. viewers each month for the two stations are given by

$$\begin{cases} y = 950x + 10{,}000 & \text{Station A (new format)} \\ y = -875x + 18{,}000 & \text{Station B} \end{cases}$$

where y represents the number of viewers and x represents the month, with $x = 1$ corresponding to the first month of station A's new format. Use the models to estimate when the number of viewers for station A's 6 P.M. newscast will exceed the number of viewers for station B's 6 P.M. newscast.

12. **Comparing Populations** From 2000 through 2009, the population of Kentucky grew more slowly than that of South Carolina. Models that represent the populations of the two states are given by

$$\begin{cases} P = 30.8t + 4035 & \text{Kentucky} \\ P = 61.6t + 3985 & \text{South Carolina} \end{cases}$$

where P represents the population (in thousands) and t represents the year, with $t = 0$ corresponding to 2000. Use a graphing utility to estimate the year when the population of South Carolina first exceeded that of Kentucky. *(Source: U.S. Census Bureau)*

Solving a System by Elimination In Exercises 13–20, solve the system by the method of elimination.

13. $\begin{cases} 2x - 3y = 21 \\ 3x + y = 4 \end{cases}$
14. $\begin{cases} 3u + 5v = 9 \\ 12u + 10v = 22 \end{cases}$
15. $\begin{cases} 4x - 3y = 10 \\ 8x - 6y = 20 \end{cases}$
16. $\begin{cases} 3x + 4y = 18 \\ 6x + 8y = 18 \end{cases}$
17. $\begin{cases} 1.5x + 2.5y = 8.5 \\ 6x + 10y = 24 \end{cases}$
18. $\begin{cases} 1.25x - 2y = 3.5 \\ 5x - 8y = 14 \end{cases}$
19. $\begin{cases} \dfrac{x-2}{3} + \dfrac{y+3}{4} = 5 \\ 2x - y = 7 \end{cases}$
20. $\begin{cases} \frac{3}{5}x + \frac{2}{7}y = 10 \\ x + 2y = 38 \end{cases}$

Interpreting the Solution Graphically In Exercises 21 and 22, describe the graph of the solution of the linear system.

21. $\begin{cases} 2x + y = -1 \\ 3x - 2y = -5 \end{cases}$
22. $\begin{cases} x - 2y = -1 \\ -2x + 4y = 2 \end{cases}$

23. **Acid Mixture** Twelve gallons of a 25% acid solution is obtained by mixing a 10% solution with a 50% solution.
    (a) Write a system of equations that represents the problem and use a graphing utility to graph the equations in the same viewing window.
    (b) How much of each solution is required to obtain the specified concentration of the final mixture?

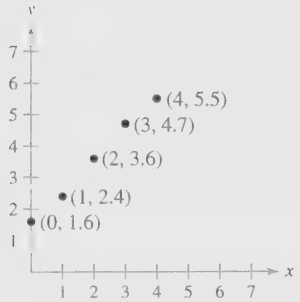
24. Acid Mixture Twenty gallons of a 30% acid solution is obtained by mixing a 12% solution with a 60% solution.

(a) Write a system of equations that represents the problem and use a graphing utility to graph the equations in the same viewing window.

(b) How much of each solution is required to obtain the specified concentration of the final mixture?

25. Fitting a Line to Data Find the least squares regression line $y = ax + b$ for the points shown in the graph below by solving the following system of linear equations for a and b.

$$\begin{cases} 5b + 10a = 17.8 \\ 10b + 30a = 45.7 \end{cases}$$



26. Ethanol Production The table shows the amount of corn y (in billions of bushels) used in the production of ethanol each year from 2005 through 2009. *(Source: U.S. Department of Agriculture)*

Year	2005	2006	2007	2008	2009
t	0	1	2	3	4
Amount, y	1.6	2.1	3.0	3.7	4.3

(a) Solve the system below for a and b to find the least squares regression line $y = at + b$ for the data. Let t represent the year, with $t = 0$ corresponding to 2005.

$$\begin{cases} 5b + 10a = 14.7 \\ 10b + 30a = 36.4 \end{cases}$$

(b) Use a graphing utility to graph the regression line and predict the amount of corn used to produce ethanol in 2010.

(c) Use the *regression* feature of the graphing utility to find a linear model for the data. Compare this model with the one you found in part (a).

Finding the Point of Equilibrium In Exercises 27 and 28, find the point of equilibrium for the demand and supply equations.

	Demand	*Supply*
27.	$p = 37 - 0.0002x$	$p = 22 + 0.00001x$
28.	$p = 120 - 0.0001x$	$p = 45 + 0.0002x$

Using Back-Substitution In Exercises 29 and 30, use back-substitution to solve the system of linear equations.

29. $\begin{cases} x - 3y + 2z = 1 \\ y - 2z = 1 \\ z = 2 \end{cases}$

30. $\begin{cases} x + y - 4z = 6 \\ y + 2z = 2 \\ z = -4 \end{cases}$

Solving a System of Linear Equations In Exercises 31–36, solve the system of linear equations.

31. $\begin{cases} 2x + y + z = 6 \\ x - 4y - z = 3 \\ x + y + z = 4 \end{cases}$

32. $\begin{cases} x + 3y - z = 13 \\ 2x - 5z = 23 \\ 4x - y - 2z = 4 \end{cases}$

33. $\begin{cases} x + y + z = 10 \\ -2x + 3y + 4z = 22 \end{cases}$

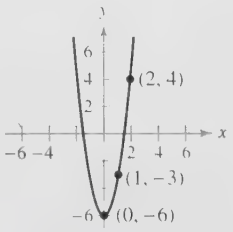
34. $\begin{cases} 5x - 12y + 7z = 16 \\ 3x - 7y + 4z = 9 \end{cases}$

35. $\begin{cases} 2x + 6y - z = 1 \\ x - 3y + z = 2 \\ \frac{3}{2}x + \frac{3}{2}y = 6 \end{cases}$

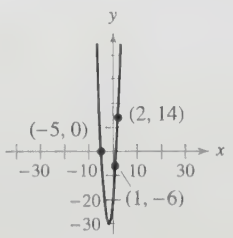
36. $\begin{cases} x + y + z + w = 8 \\ 4y + 5z - 2w = 3 \\ 2x + 3y - z = -2 \\ 3x + 2y - 4w = -20 \end{cases}$

Data Analysis: Curve Fitting In Exercises 37 and 38, find the quadratic equation $y = ax^2 + bx + c$ whose graph passes through the points.

37.

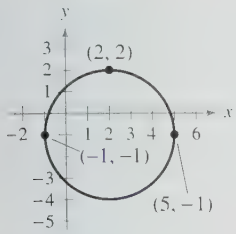


38.

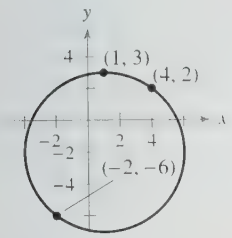


Data Analysis: Curve Fitting In Exercises 39 and 40, find the equation $x^2 + y^2 + Dx + Ey + F = 0$ of the circle that passes through the points.

39.



40.

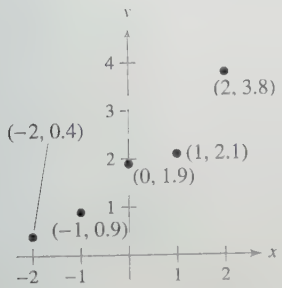


41. Investment Portfolio You are creating an investment portfolio by investing a total of \$500,000 in certificates of deposit, municipal bonds, blue-chip stocks, and growth stocks. The certificates of deposit pay 5% simple annual interest, and the municipal bonds pay 8% simple annual interest. Over a five-year period, you expect the blue-chip stocks to return 10% simple annual interest and the growth stocks to return 15% simple annual interest. You want to have two-fifths of your portfolio in stocks, and twice as much in certificates of deposit as in municipal bonds. How much should you allocate to each type of investment to have a combined return of 9.45%?

42. Investment You receive \$8580 a year in simple annual interest from three investments. The interest rates for the three investments are 6%, 8%, and 10%. The value of the 10% investment is two times that of the 6% investment, and the 8% investment is \$1000 more than the 6% investment. What is the amount of each investment?

43. Fitting a Parabola to Data Find the least squares regression parabola $y = ax^2 + bx + c$ for the points shown in the graph below by solving the following system of linear equations for a, b, and c.

$$\begin{cases} 5c + 10a = 9.1 \\ 10b = 8.0 \\ 10c + 34a = 19.8 \end{cases}$$



44. Sailboat Sales The table shows the number y (in thousands) of new sailboats sold in the United States each year from 2005 through 2009. In the table, t represents the year, with $t = -1$ corresponding to 2005. *(Source: The Sailing Company's Annual Sailing Business Review)*

Year, t	Number, y
−1	14.4
0	12.9
1	11.8
2	9.3
3	5.4

(a) Use a graphing utility to create a scatter plot of the data.

(b) Solve the following system for a and b to find the least squares regression line $y = at + b$ for the data.

$$\begin{cases} 5b + 5a = 53.8 \\ 5b + 15a = 32.2 \end{cases}$$

(c) Solve the following system for a, b, and c to find the least squares regression parabola $y = at^2 + bt + c$ for the data.

$$\begin{cases} 5c + 5b + 15a = 53.8 \\ 5c + 15b + 35a = 32.2 \\ 15c + 35b + 99a = 112 \end{cases}$$

(d) Use the *regression* feature of the graphing utility to find linear and quadratic models for the data. Compare the models with the least squares regression models found in parts (b) and (c).

(e) Use the graphing utility to graph the linear and quadratic models with the scatter plot of the data. Use the models to predict the numbers of sailboats sold in 2010 and 2011. Discuss your results in the context of the situation.

Sketching the Graph of an Inequality In Exercises 45–50, sketch the graph of the inequality.

45. $x \geq -5$

46. $y < -2$

47. $y \leq 5 + \frac{1}{2}x$

48. $3y - x \geq 7$

49. $y - 4x^2 > -1$

50. $y \leq \dfrac{3}{x^2 + 2}$

Solving a System of Inequalities In Exercises 51–58, graph the solution set of the system of inequalities.

51. $\begin{cases} x + 4y < 8 \\ x > 0 \\ y > 0 \end{cases}$

52. $\begin{cases} 2x - y > 6 \\ x < 6 \\ y \geq -2 \end{cases}$

53. $\begin{cases} 3x - y > -4 \\ 2x + y > -1 \\ 7x + y < 4 \end{cases}$

54. $\begin{cases} x + y > 4 \\ 3x + y > 10 \\ x - y \geq 0 \end{cases}$

55. $\begin{cases} x^2 + y^2 \leq 9 \\ x^2 - x - 2 \leq y \end{cases}$

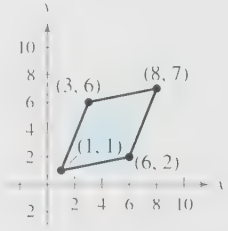
56. $\begin{cases} x^2 + y^2 \leq 4 \\ -2x^2 + 2 < y \end{cases}$

57. $\begin{cases} \ln x < y \\ y > -1 \\ x < 4 \end{cases}$

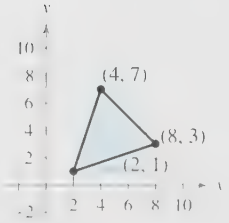
58. $\begin{cases} \ln x \geq y \\ -x + y < -2 \\ x > 2 \end{cases}$

Geometry In Exercises 59 and 60, write a system of inequalities that corresponds to the solution set shown in the graph.

59. Parallelogram



60. Triangle



Consumer and Producer Surpluses In Exercises 61 and 62, find the consumer surplus and producer surplus for the demand and supply equations.

	Demand	*Supply*
61.	$p = 160 - 0.0001x$	$p = 70 + 0.0002x$
62.	$p = 130 - 0.0002x$	$p = 30 + 0.0003x$

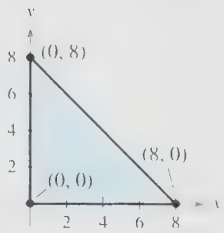
63. Movie Player Inventory A store sells two models of Blu-ray Disc™ players (BDPs). Because of the demand, it is necessary to stock at least twice as many units of model Y as units of model Z. The costs to the store for the two models are \$100 and \$150, respectively. The management does not want more than \$4000 in BDP inventory at any one time, and it wants at least four model Y BDPs and two model Z BDPs in inventory at all times. Find a system of inequalities that describes all possible inventory levels. Sketch the graph of the system.

64. Concert Ticket Sales Two types of tickets are to be sold for a concert. One type costs \$30 per ticket and the other type costs \$50 per ticket. The promoter of the concert can sell up to 10,000 of the \$50 tickets and up to 18,000 of the \$30 tickets, but no more than 24,000 tickets in all. Also, the gross receipts must total at least \$550,000 in order for the concert to be held. Find a system of inequalities describing the numbers of each type of ticket that can be sold so that the concert can be held. Sketch the graph of the system.

Solving a Linear Programming Problem In Exercises 65–68, find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints. (For each exercise, the graph of the region determined by the constraints is provided.)

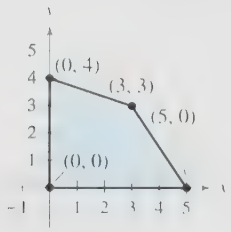
65. *Objective function:* $z = 5x + 6y$

Constraints: $x \geq 0$, $y \geq 0$, $x + y \leq 8$



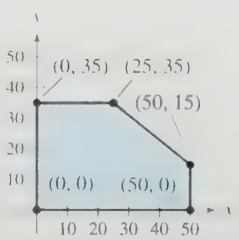
66. *Objective function:* $z = 15x + 12y$

Constraints: $x \geq 0$, $y \geq 0$, $x + 3y \leq 12$, $3x + 2y \leq 15$



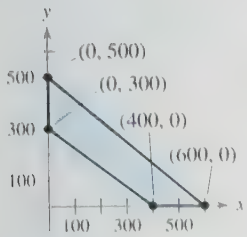
67. *Objective function:* $z = 8x + 10y$

Constraints: $0 \leq x \leq 50$, $0 \leq y \leq 35$, $4x + 5y \leq 275$



68. *Objective function:* $z = 50x + 60y$

Constraints:
$$x \geq 0$$
$$y \geq 0$$
$$3x + 4y \geq 1200$$
$$5x + 6y \leq 3000$$



Solving a Linear Programming Problem In Exercises 69–72, sketch the region determined by the indicated constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the constraints.

69. *Objective function:* $z = 6x + 8y$

Constraints:
$$x \geq 0$$
$$y \geq 0$$
$$x + 4y \leq 16$$
$$3x + 2y \leq 18$$

70. *Objective function:* $z = 5x + 8y$

Constraints:
$$0 \leq x \leq 5$$
$$y \geq 0$$
$$x + 2y \leq 12$$
$$2x + 3y \leq 19$$

71. *Objective function:* $z = 8x + 3y$

Constraints:
$$0 \leq x \leq 5$$
$$0 \leq y \leq 7$$
$$x + y \leq 9$$
$$3x + y \leq 17$$

72. *Objective function:* $z = 10x + 11y$

Constraints:
$$x \geq 0$$
$$y \geq 0$$
$$2x + 5y \leq 30$$
$$x + y \geq 3$$
$$2x + y \leq 14$$

73. Optimal Profit A company makes two models of desks. The times (in hours) required for assembling, finishing, and packaging each model are shown in the table.

Process	Model A	Model B
Assembling	3.5	8
Finishing	2.5	2
Packaging	1.3	0.7

The total times available each month for assembling, finishing, and packaging are 5600 hours, 2000 hours, and 910 hours, respectively. The profits per unit are \$100 for model A and \$150 for model B. What is the optimal monthly production level for each model? What is the optimal monthly profit?

74. Optimal Profit A factory manufactures two television set models: an LCD model that yields \$100 profit and a plasma model that yields a profit of \$180. The times (in hours) required for assembling, testing, and packaging each model are shown in the table.

Process	LCD model	Plasma model
Assembling	2	5
Testing	1	2
Packaging	1	1

The total times available each month for assembling, testing, and packaging are 3000 hours, 1300 hours, and 1000 hours, respectively. What is the optimal monthly production level for each model? What is the optimal monthly profit?

75. Optimal Profit The costs to a merchant for two models of digital camcorders are \$525 and \$675. The \$525 model yields a profit of \$75 and the \$675 model yields a profit of \$125. The merchant estimates that the total monthly demand will not exceed 350 units. There should be no more than \$206,250 in digital camcorder inventory. Find the number of units of each model that should be stocked in order to optimize profit. What is the optimal profit?

76. Optimal Profit The costs to a merchant for two models of home theater systems are \$270 and \$455. The \$270 model yields a profit of \$30 and the \$455 model yields a profit of \$45. The merchant estimates that the total monthly demand will not exceed 100 units. There should be no more than \$36,250 in home theater system inventory. Find the number of units of each model that should be stocked in order to optimize profit. What is the optimal profit?

77. Optimal Revenue An accounting firm has 800 hours of staff time and 90 hours of review time available each week. The firm charges \$2500 for an audit and \$200 for a tax return. Each audit requires 100 hours of staff time and 10 hours of review time. Each tax return requires 10 hours of staff time and 2 hours of review time. What numbers of audits and tax returns will bring in an optimal revenue? What is the optimal revenue?

TEST YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–6, solve the system of equations using the indicated method.

1. *Substitution*

$$\begin{cases} 5x - 7y = -18 \\ 4x + 3y = 20 \end{cases}$$

2. *Substitution*

$$\begin{cases} x + y = 3 \\ x^2 + y = 9 \end{cases}$$

3. *Graphing*

$$\begin{cases} 2x + y = 4 \\ 2x^2 + y = 8 \end{cases}$$

4. *Graphing*

$$\begin{cases} 1.5x - 2.25y = 8 \\ 2.5x + 2y = 5.75 \end{cases}$$

5. *Elimination*

$$\begin{cases} 2x - 4y + z = 11 \\ x + 2y + 3z = 9 \\ 3y + 5z = 12 \end{cases}$$

6. *Elimination*

$$\begin{cases} 3x - 2y + z = 16 \\ 5x - z = 6 \\ 2x - y - z = 3 \end{cases}$$

7. A total of \$80,000 is invested in two funds paying 2.3% and 3.1% simple interest. The total annual interest is \$2000. How much is invested in each fund?

8. Find the point of equilibrium for a system that has a demand equation of $p = 49 - 0.0003x$ and a supply equation of $p = 33 + 0.00002x$.

9. The numbers y of adults (in millions) who participated in baking as a leisure activity in the years 2005 through 2009 are shown in the table at the left. In the table, x represents the year, with $x = 0$ corresponding to 2007. *(Source: Mediamark Research & Intelligence)*

Year, x	Number, y
−2	39.6
−1	41.3
0	47.6
1	52.2
2	58.9

Table for 9

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the system below.

$$\begin{cases} 5c + 10a = 239.6 \\ 10b = 49.5 \\ 10c + 34a = 487.5 \end{cases}$$

(b) Use the model found in part (a) to predict the number of adults who participated in baking as a leisure activity in 2010.

In Exercises 10–13, sketch the graph of the inequality.

10. $x \geq 0$

11. $y \geq 0$

12. $x + 3y \leq 12$

13. $3x + 2y \leq 15$

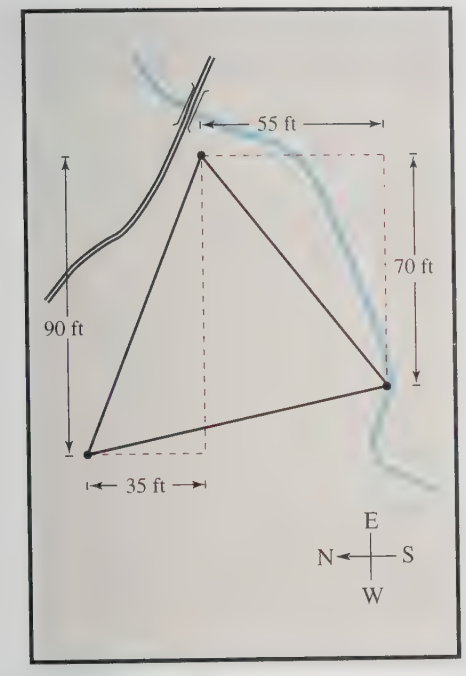
14. Sketch the solution set of the system of inequalities composed of the inequalities in Exercises 10–13.

15. Find the minimum and maximum values of the objective function $z = 6x + 7y$, subject to the constraints given in Exercises 10–13.

16. A manufacturer produces two models of exercise equipment. The times required for assembling, painting, and packaging each model are as follows.

- Assembling: 3.5 hours for model A; 8 hours for model B
- Painting: 2.5 hours for model A; 2 hours for model B
- Packaging: 1.3 hours for model A; 0.9 hour for model B

The total times available for assembling, painting, and packaging are 5600 hours, 2000 hours, and 900 hours, respectively. The profits per unit are \$200 for model A and \$275 for model B. What is the optimal production level for each model? What is the optimal profit? Explain your reasoning.



Exercise 16 on page 528 uses a determinant to find the number of square feet in a triangular tract of land.

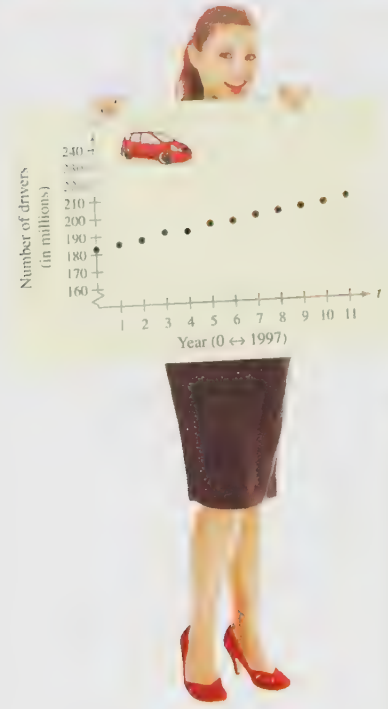
6 Matrices and Determinants

6.1 Matrices and Linear Systems
6.2 Operations with Matrices
6.3 The Inverse of a Square Matrix
6.4 The Determinant of a Square Matrix
6.5 Applications of Matrices and Determinants



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6.1 Matrices and Linear Systems



In Exercise 90 on page 486, you will find the least squares regression line for data shown in a graph by solving a system of equations using matrices.

- Determine the dimension of a matrix.
- Perform elementary row operations on a matrix in order to write the matrix in row-echelon form or reduced row-echelon form.
- Solve a system of linear equations using Gaussian elimination.
- Solve a system of linear equations using Gauss-Jordan elimination.

Matrices

In this section, you will study a streamlined technique for solving systems of linear equations. This technique involves the use of a rectangular array of real numbers called a **matrix.** The plural of matrix is **matrices.**

Definition of a Matrix

If m and n are positive integers, then an $m \times n$ matrix (read "m by n") is a rectangular array

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \Big\} m \text{ rows}$$

in which each **entry,** a_{ij}, of the matrix is a number. An $m \times n$ matrix has m **rows** (horizontal lines) and n **columns** (vertical lines).

The entry in the ith row and jth column is denoted by the *double subscript* notation a_{ij}. That is, a_{21} refers to the entry in row 2, column 1. A matrix having m rows and n columns is said to be of **dimension** $m \times n$. If $m = n$, then the matrix is **square** of dimension $m \times m$ (or $n \times n$). For a square matrix, the entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **main diagonal** entries.

Example 1 Dimensions of Matrices

a. The dimension of the matrix shown at the right is 1×4. $\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$

b. The dimension of the matrix shown at the right is 2×2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

c. The dimension of the matrix shown at the right is 3×2. $\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$

Checkpoint 1

Determine the dimension of the matrix.

$$\begin{bmatrix} 0 & 8 & 5 \\ 3 & 1 & -2 \end{bmatrix}$$

Yuri Arcurs /Shutterstock.com

A matrix that has only one row [such as the matrix in Example 1(a)] is called a **row matrix,** and a matrix that has only one column is called a **column matrix.**

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the **augmented matrix** of the system. Moreover, the matrix derived from the coefficients of the system (but that does not include the constant terms) is the **coefficient matrix** of the system. Note in the matrices below the use of 0 for the coefficient of the missing y-variable in the third equation. Also note that the fourth column (the column of constant terms) in the augmented matrix is separated from the coefficients of the linear system by vertical dots.

System:
$$\begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x - 4z = 6 \end{cases}$$

Augmented Matrix:
$$\left[\begin{array}{rrr:r} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right]$$

Coefficient Matrix:
$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by vertically aligning the variables in the equations and using zeros for the coefficients of any missing variables.

Example 2 **Writing an Augmented Matrix**

Write the augmented matrix for the system of linear equations. What is the dimension of the augmented matrix?

$$\begin{cases} x + 3y = 9 \\ -y + 4z = -2 \\ x - 5z = 0 \end{cases}$$

SOLUTION Begin by writing the linear system and aligning the variables.

$$\begin{cases} x + 3y = 9 \\ -y + 4z = -2 \\ x - 5z = 0 \end{cases}$$

Next, use the coefficients and constant terms as the matrix entries. Include zeros for the coefficients of the missing variables.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left[\begin{array}{rrr:r} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{array}\right]$$

The augmented matrix has three rows and four columns, so it is a 3×4 matrix. The notation R_n is used to designate each row in the matrix. For instance, Row 1 is represented by R_1.

Checkpoint 2

Write the augmented matrix for the system of linear equations. What is the dimension of the augmented matrix?

$$\begin{cases} 4x - 3y = -5 \\ -x + 3y = 12 \end{cases}$$

Elementary Row Operations

In Section 5.3, you studied three operations that can be used on a system of linear equations to produce an equivalent system.

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix of a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a sequence of elementary row operations.

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

STUDY TIP

Elementary row operations involve a lot of arithmetic. So that you can go back and check your work, you should note the elementary row operation performed in each step next to the row you are changing, as shown in Example 3.

Example 3 Elementary Row Operations

a. Interchange the first and second rows.

Original Matrix

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{matrix} R_2 \\ R_1 \end{matrix}\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

b. Multiply the first row by $\frac{1}{2}$.

Original Matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\tfrac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

c. Add -2 times the first row to the third row.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$-2R_1 + R_3 \rightarrow \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

✓Checkpoint 3

Identify the elementary row operation being performed to obtain the new row-equivalent matrix.

Original Matrix *New Row-Equivalent Matrix*

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Gaussian Elimination with Back-Substitution

In Example 2 in Section 5.3, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

TECH TUTOR

Most graphing utilities can perform elementary row operations on matrices. The screen below shows how one graphing utility displays the new row-equivalent matrix from Example 3(c).

```
*row+(-2,[A],1,3
)
  [[1 2  -4 3 ]
   [0 3  -2 -1]
   [0 -3 13 -8]]
```

Example 4 Comparing Linear Systems and Matrix Operations

Linear System

$$\begin{cases} x - 2y + 3z = 9 \\ x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Associated Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Add the first equation to the second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Add the first row to the second row $(R_1 + R_2)$.

$$R_1 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Add -2 times the first equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Add -2 times the first row to the third row $(-2R_1 + R_3)$.

$$-2R_1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

Add the second equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

Add the second row to the third row $(R_2 + R_3)$.

$$R_2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiply the third row by $\frac{1}{2}$.

$$\frac{1}{2}R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

At this point, you can use back-substitution to find that the solution is $x = 1$, $y = -1$, and $z = 2$, as shown in Example 1 in Section 5.3.

Checkpoint 4

Write the system of equations represented by the augmented matrix. Use back-substitution to find the solution. (Use the variables x, y, and z.)

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Remember that you can check a solution by substituting the values of x, y, and z into each equation in the original system. For instance, you can check the solution to Example 4 as shown.

Equation 1: $1 - 2(-1) + 3(2) = 9$ ✓

Equation 2: $-1 + 3(-1) = -4$ ✓

Equation 3: $2(1) - 5(-1) + 5(2) = 17$ ✓

The last matrix in Example 4 is said to be in *row-echelon form*. The term *echelon* refers to the stair-step pattern formed by the nonzero entries of the matrix. To be in this form, a matrix must have the following properties.

TECHNOLOGY



Some graphing utilities can automatically transform a matrix to row-echelon form and reduced row-echelon form. Consult the user's guide for your graphing utility for more information.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below the leading 1.

Note that the row-echelon form of a matrix is not unique. That is, two different sequences of elementary row operations may yield different row-echelon forms. The *reduced* row-echelon form of a given matrix, however, is unique.



Example 5 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

SOLUTION The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in reduced row-echelon form because every column that has a leading 1 has zeros in every position above and below the leading 1. The matrix in (b) is not in row-echelon form because the row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in row 2 is not 1.

✓Checkpoint 5

Determine whether the matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

■

Every matrix can be converted to a row-equivalent matrix that is in row-echelon form. For instance, in Example 5, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$, as shown at the left.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Row-Echelon Form

$$K_2 \rightarrow \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

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Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which the elementary row operations are performed is important. You should operate from *left to right* by columns, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

Example 6 **Gaussian Elimination with Back-Substitution**

Solve the system.

$$\begin{cases} y + z - 2w = -3 \\ x + 2y - z = 2 \\ 2x + 4y + z - 3w = 2 \\ x - 4y - 7z - w = -19 \end{cases}$$

SOLUTION

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & 3 \\ 2 & 4 & 1 & -3 & \vdots & 2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

Interchange R_1 and R_2 so first column has leading 1 in upper left corner.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & 3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & -6 & 6 & -1 & \vdots & -21 \end{bmatrix} \quad \begin{matrix} -2R_1 + R_3 \to \\ -R_1 + R_4 \to \end{matrix}$$

Perform operations on R_3 and R_4 so first column has zeros below its leading 1.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & 3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{bmatrix} \quad 6R_2 + R_4 \to$$

Perform operations on R_4 so second column has zeros below its leading 1.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & 3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{bmatrix} \quad \tfrac{1}{3}R_3 \to$$

Multiply R_3 by $\frac{1}{3}$ so third row has a leading 1.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & 3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \quad -\tfrac{1}{13}R_4 \to$$

Multiply R_4 by $-\frac{1}{13}$ so fourth row has a leading 1.

The matrix is now in row-echelon form, and the corresponding system is

$$\begin{cases} x + 2y - z = 2 \\ y + z - 2w = -3 \\ z - w = -2 \\ w = 3 \end{cases}$$

Using back-substitution, you can determine that the solution is $x = -1$, $y = 2$, $z = 1$, and $w = 3$. Check this in the original system of equations.

Checkpoint 6

Solve the system.

$$\begin{cases} y + 2z - w = 5 \\ x - 3y - z = 0 \\ 2x - 6y + z + 3w = 6 \\ 4x + 4y - 2z + w = 1 \end{cases}$$

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Remember that it is possible for a system to have no solution. In the elimination process, when you obtain a row with zeros except for the last entry, you can conclude that the system has no solution, or is inconsistent.

Example 7 **A System with No Solution**

Solve the system.

$$\begin{cases} x - y + 2z = 4 \\ x + z = 6 \\ 2x - 3y + 5z = 4 \\ 3x + 2y - z = 1 \end{cases}$$

SOLUTION

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 6 \\ 2 & -3 & 5 & \vdots & 4 \\ 3 & 2 & -1 & \vdots & 1 \end{bmatrix}$$

Write augmented matrix.

$$\begin{matrix} \\ -R_1 + R_2 \to \\ 2R_1 + R_3 \to \\ 3R_1 + R_4 \to \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -4 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix}$$

Perform row operations.

$$\begin{matrix} \\ \\ R_2 + R_3 \to \\ \\ \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & -2 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix}$$

Perform row operations.

Note that the third row of this matrix consists of zeros except for the last entry. This means that the original system of linear equations is *inconsistent*. You can see why this is true by converting back to a system of linear equations.

$$\begin{cases} x - y + 2z = 4 \\ y - z = 2 \\ 0 = -2 \\ 5y - 7z = -11 \end{cases}$$

Because $0 = -2$ is a false statement, the system has no solution.

Checkpoint 7

Solve the system.

$$\begin{cases} x + 2y - z = 3 \\ -x - y + 3z = 2 \\ 2x + 3y - 4z = 0 \\ 3x + 2y + z = -4 \end{cases}$$

Gauss-Jordan Elimination

With Gaussian elimination, elementary row operations are applied to a matrix to obtain a (row-equivalent) row-echelon form of the matrix. A second method of elimination, called **Gauss-Jordan elimination** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in Example 8.

STUDY TIP

Either Gaussian elimination or Gauss-Jordan elimination can be used to solve a system of equations. The method you use depends on your preference.

Example 8 Gauss-Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y \quad = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

SOLUTION In Example 4, Gaussian elimination was used to obtain the row-echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Now, rather than using back-substitution, apply additional elementary row operations until you obtain a matrix in *reduced* row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

$$2R_2 + R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Perform operations on R_1 so second column has a zero above its leading 1.

$$\begin{matrix} 9R_3 + R_1 \rightarrow \\ -3R_3 + R_2 \rightarrow \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Perform operations on R_1 and R_2 so third column has zeros above its leading 1

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{cases} x = 1 \\ y = -1. \\ z = 2 \end{cases}$$

An advantage of Gauss-Jordan elimination is that you can read the solution from the matrix in reduced row-echelon form.



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✓Checkpoint 8

Use Gauss-Jordan elimination to solve the system.

$$\begin{cases} x - 3y + 2z = 1 \\ -x - y + 3z = 4 \\ \quad y - 2z = -5 \end{cases}$$

The elimination procedures described in this section sometimes result in fractional coefficients. For instance, in the elimination procedure for the system

$$\begin{cases} 2x - 5y + 5z = 17 \\ 3x - 2y + 3z = 11 \\ -3x + 3y \quad = 6 \end{cases}$$

you may be inclined to multiply the first row by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. You can sometimes avoid fractions by judiciously choosing the order in which you apply elementary row operations.

Example 9 **A System with an Infinite Number of Solutions**

Solve the system.

$$\begin{cases} 2x + 4y - 2z = 0 \\ 3x + 5y \quad\quad = 1 \end{cases}$$

SOLUTION

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$\tfrac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$-3R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 3 & \vdots & 1 \end{bmatrix}$$

$$-R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

$$2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x \quad + 5z = 2 \\ \quad y - 3z = -1 \end{cases}$$

Solving for x and y in terms of z, you have $x = -5z + 2$ and $y = 3z - 1$. To write a solution of the system that does not use any of the three variables of the system, let a represent any real number and let $z = a$. Now, substitute a for z in the equations for x and y.

$$x = -5z + 2 = -5a + 2 \quad \text{and} \quad y = 3z - 1 = 3a - 1$$

So, the solution set has the form $(-5a + 2, 3a - 1, a)$, where a is a real number. Try substituting values for a to obtain a few solutions. Then check each solution in the original system of equations.

STUDY TIP

Remember that the solution set of a system with an infinite number of solutions can be written in several ways. For instance, the solution set in Example 9 can be written as

$$\left(\frac{1 - 5b}{3}, b, \frac{b + 1}{3}\right)$$

where b is a real number.

Checkpoint 9

Solve the system: $\begin{cases} \quad\quad y + 5z = -2 \\ -x + y - 4z = 8 \end{cases}$



SUMMARIZE (Section 6.1)

1. State the definition of a matrix *(page 474)*. For examples of determining the dimensions of matrices, see Examples 1 and 2.
2. List the elementary row operations *(page 476)*. For examples of using elementary row operations, see Examples 3 and 4.
3. State the definitions of row-echelon form and reduced row-echelon form *(page 478)*. For an example of matrices in these forms, see Example 5.
4. Describe the procedure of Gaussian elimination with back-substitution *(page 479)*. For examples of using this procedure, see Examples 6 and 7.
5. Describe the procedure of Gauss-Jordan elimination *(page 481)*. For examples of using this procedure, see Examples 8 and 9.

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SKILLS WARM UP 6.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2. 5.1, and 5.3.

In Exercises 1–4, evaluate the expression.

1. $2(-1) - 3(5) + 7(2)$

2. $-4(-3) + 6(7) + 8(-3)$

3. $11\left(\frac{1}{2}\right) - 7\left(-\frac{3}{2}\right) - 5(2)$

4. $\frac{2}{3}\left(\frac{1}{2}\right) + \frac{4}{3}\left(-\frac{1}{3}\right)$

In Exercises 5 and 6, determine whether $x = 1$, $y = 3$, and $z = -1$ is a solution of the system.

5. $\begin{cases} 4x - 2y + 3z = -5 \\ x + 3y - z = 11 \\ -x + 2y = 5 \end{cases}$

6. $\begin{cases} -x + 2y + z = 4 \\ 2x - 3z = 5 \\ 3x + 5y - 2z = 21 \end{cases}$

In Exercises 7–10, use back-substitution to solve the system of linear equations.

7. $\begin{cases} 2x - 3y = 4 \\ y = 2 \end{cases}$

8. $\begin{cases} 5x + 4y = 0 \\ y = -3 \end{cases}$

9. $\begin{cases} x - 3y + z = 0 \\ y - 3z = 8 \\ z = 2 \end{cases}$

10. $\begin{cases} 2x - 5y + 3z = -2 \\ y - 4z = 0 \\ z = 1 \end{cases}$

Exercises 6.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Dimensions of Matrices In Exercises 1–8, determine the dimension of the matrix. *See Example 1.*

1. $\begin{bmatrix} 0 & -3 & 0 \\ 9 & 2 & -7 \end{bmatrix}$

2. $\begin{bmatrix} -7 & 21 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 4 & 1 \\ 8 & 3 & 0 \\ -1 & 2 & 1 \\ 1 & 5 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \\ 6 \end{bmatrix}$

5. $\begin{bmatrix} 33 & 45 \\ -9 & 20 \\ 12 & 15 \\ 16 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 12 & -2 & 4 \\ -3 & 4 & 0 \\ -8 & 12 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 7 & 11 & -3 \\ -1 & 10 & -5 & 0 \end{bmatrix}$

8. $\begin{bmatrix} -11 \end{bmatrix}$

Writing an Augmented Matrix In Exercises 9–14, write the augmented matrix for the system of linear equations. What is the dimension of the augmented matrix? *See Example 2.*

9. $\begin{cases} 4x - 3y = -5 \\ -x + 3y = 12 \end{cases}$

10. $\begin{cases} 7x + 4y = 22 \\ 5x - 9y = 15 \end{cases}$

11. $\begin{cases} x + 10y - 2z = 2 \\ 5x - 3y + 4z = 0 \\ 2x + y = 6 \end{cases}$

12. $\begin{cases} x - 3y + z = 1 \\ 4y = 0 \\ 7z = -5 \end{cases}$

13. $\begin{cases} 7x - 5y + z = 13 \\ 19x - 8z = 10 \end{cases}$

14. $\begin{cases} 9x + 2y - 3z = 20 \\ -25y + 11z = -5 \end{cases}$

Elementary Row Operations In Exercises 15–18, fill in the blank(s) to form the new row-equivalent matrix. *See Example 3.*

Original Matrix	*New Row-Equivalent Matrix*
15. $\begin{bmatrix} 1 & 1 & 1 \\ 5 & -2 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & & -1 \end{bmatrix}$
16. $\begin{bmatrix} -3 & 3 & 12 \\ 18 & -8 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & \\ 18 & -8 & 4 \end{bmatrix}$
17. $\begin{bmatrix} 1 & 5 & 4 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & & \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$
18. $\begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{bmatrix}$

Elementary Row Operations In Exercises 19–22, identify the elementary row operation(s) being performed to obtain the new row-equivalent matrix. *See Example 3.*

Original Matrix	*New Row-Equivalent Matrix*
19. $\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$	$\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$
20. $\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$	$\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$
21. $\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix}$	$\begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 0 & 7 & -27 & 27 \end{bmatrix}$

Original Matrix *New Row-Equivalent Matrix*

22. $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$ $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 0 & -9 & 7 & -11 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

Writing a System of Linear Equations In Exercises 23–26, write the system of linear equations represented by the augmented matrix. (Use the variables x, y, z, and w.)

23. $\left[\begin{array}{cc|c} 2 & 4 & 6 \\ -1 & 3 & -8 \end{array}\right]$

24. $\left[\begin{array}{cc|c} 7 & -2 & 7 \\ -8 & 3 & -3 \end{array}\right]$

25. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -10 \\ 0 & 3 & -1 & 5 \\ 4 & 2 & 0 & 3 \end{array}\right]$

26. $\left[\begin{array}{cccc|c} 5 & 8 & 2 & 0 & -1 \\ -2 & 15 & 5 & 1 & 9 \\ 1 & 6 & -7 & 0 & -3 \end{array}\right]$

Using Back-Substitution In Exercises 27–30, write the system of linear equations represented by the augmented matrix. Use back-substitution to find the solution. (Use the variables x, y, z, and w.)

27. $\left[\begin{array}{cc|c} 1 & -5 & 6 \\ 0 & 1 & -2 \end{array}\right]$

28. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

29. $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & 1 & 4 & -12 \\ 0 & 0 & 1 & -5 \end{array}\right]$

30. $\left[\begin{array}{cccc|c} 1 & 2 & -2 & 0 & -1 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$

Comparing Linear Systems and Matrix Operations In Exercises 31 and 32, (a) perform the elementary row operations to solve the augmented matrix, (b) write and solve the system of linear equations represented by the augmented matrix, and (c) compare the two solution methods. Which do you prefer? *See Example 4.*

31. $\left[\begin{array}{cc|c} -3 & 4 & 22 \\ 6 & -4 & -28 \end{array}\right]$

(i) Add R_2 to R_1.

(ii) Add -2 times R_1 to R_2.

(iii) Multiply R_2 by $-\frac{1}{4}$.

(iv) Multiply R_1 by $\frac{1}{3}$.

32. $\left[\begin{array}{ccc|c} 7 & 13 & 1 & -4 \\ -3 & -5 & -1 & -4 \\ 3 & 6 & 1 & -2 \end{array}\right]$

(i) Add R_2 to R_1.

(ii) Multiply R_1 by $\frac{1}{4}$.

(iii) Add R_3 to R_2.

(iv) Add -3 times R_1 to R_3.

(v) Add -2 times R_2 to R_1.

33. Using a Graphing Utility Repeat steps (i) through (iv) for the matrix in Exercise 31 using a graphing utility.

34. Using a Graphing Utility Repeat steps (i) through (v) for the matrix in Exercise 32 using a graphing utility.

Row-Echelon Form In Exercises 35–40, determine whether the matrix is in row-echelon form. If it is, determine if it is also in *reduced* row-echelon form. *See Example 5.*

35. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

37. $\begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & -1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

38. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Writing a Matrix in Row-Echelon Form In Exercises 41–44, write the matrix in row-echelon form. (*Note:* Row-echelon forms are not unique.)

41. $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 3 & 2 & 1 & 11 \\ 4 & 8 & 1 & 10 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & -5 & 14 \\ -2 & -1 & -3 & 8 \end{bmatrix}$

43. $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 5 & -4 & 1 & 8 \\ -6 & 8 & 18 & 0 \end{bmatrix}$

44. $\begin{bmatrix} 1 & -3 & 0 & -7 \\ -3 & 10 & 1 & 23 \\ 1 & 0 & 1 & 12 \\ 4 & -10 & 2 & -24 \end{bmatrix}$

Writing a Matrix in Reduced Row-Echelon Form In Exercises 45–50, write the matrix in reduced row-echelon form.

45. $\begin{bmatrix} 4 & 4 & 8 \\ 1 & 2 & 2 \\ -3 & 6 & -9 \end{bmatrix}$

46. $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 15 & 9 \\ 2 & 6 & 10 \end{bmatrix}$

47. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & -6 & 6 \\ 2 & 0 & 5 & -4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

48. $\begin{bmatrix} 1 & 2 & 3 & -5 \\ 1 & 2 & 4 & -9 \\ -2 & -4 & -4 & 3 \\ 4 & 8 & 11 & -14 \end{bmatrix}$

49. $\begin{bmatrix} 2 & 1 \\ 1 & 4 \\ -2 & -1 \end{bmatrix}$

50. $\begin{bmatrix} 1 & -3 \\ -1 & 8 \\ 0 & 4 \\ -2 & 10 \end{bmatrix}$

Interpreting Reduced Row-Echelon Form In Exercises 51–56, an augmented matrix that represents a system of linear equations (in variables x, y, and z) has been reduced using Gauss-Jordan elimination. Write the solution represented by the augmented matrix.

51. $\left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 6 \end{array}\right]$

52. $\left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & -3 \end{array}\right]$

53. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right]$

54. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

55. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

56. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 9 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Gaussian Elimination with Back-Substitution In Exercises 57–74, use matrices to solve the system of equations (if possible). Use Gaussian elimination with back-substitution. *See Examples 6, 7, and 9.*

57. $\begin{cases} x + 2y = 7 \\ 2x + y = 8 \end{cases}$

58. $\begin{cases} 2x + 6y = 16 \\ 2x + 3y = 7 \end{cases}$

59. $\begin{cases} -3x + 5y = -22 \\ 3x + 4y = 4 \\ 4x - 8y = 32 \end{cases}$

60. $\begin{cases} x + 2y = 0 \\ x + y = 6 \\ 3x - 2y = 8 \end{cases}$

61. $\begin{cases} 8x - 4y = 7 \\ 5x + 2y = 1 \end{cases}$

62. $\begin{cases} x - 3y = 5 \\ -2x + 6y = -10 \end{cases}$

63. $\begin{cases} -x + 2y = 1.5 \\ 2x - 4y = 3 \end{cases}$

64. $\begin{cases} 2x - y = -0.1 \\ 3x + 2y = 1.6 \end{cases}$

65. $\begin{cases} x + 2y + z = 8 \\ 3x + 7y + 6z = 26 \end{cases}$

66. $\begin{cases} x + y + 4z = 5 \\ 2x + y - z = 9 \end{cases}$

67. $\begin{cases} 3x + 3y + 12z = 6 \\ x + y + 4z = 2 \\ 2x + 5y + 20z = 10 \\ -x + 2y + 8z = 4 \end{cases}$

68. $\begin{cases} 2x + 10y + 2z = 6 \\ x + 5y + 2z = 6 \\ x + 5y + z = 3 \\ -3x + 15y - 3z = -9 \end{cases}$

69. $\begin{cases} 4x + 12y - 7z - 20w = 22 \\ 3x + 9y - 5z - 28w = 30 \end{cases}$

70. $\begin{cases} x + 2y + 2z + 4w = 11 \\ 3x + 6y + 5z + 12w = 30 \end{cases}$

71. $\begin{cases} x + 2y = 0 \\ -x - y = 0 \end{cases}$

72. $\begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases}$

73. $\begin{cases} x + y + z = 0 \\ 2x + 3y + z = 0 \\ 3x + 5y + z = 0 \end{cases}$

74. $\begin{cases} x - 2y + z + 3w = 0 \\ x - y + w = 0 \\ y - z + 2w = 0 \end{cases}$

Gauss-Jordan Elimination In Exercises 75–80, use matrices to solve the system of equations (if possible). Use Gauss-Jordan elimination. *See Examples 8 and 9.*

75. $\begin{cases} x - 3z = -2 \\ 3x + y - 2z = 5 \\ 2x + 2y + z = 4 \end{cases}$

76. $\begin{cases} 2x - y + 3z = 24 \\ 2y - z = 14 \\ 7x - 5y = 6 \end{cases}$

77. $\begin{cases} x + y - 5z = 3 \\ x - 2z = 1 \\ 2x - y - z = 0 \end{cases}$

78. $\begin{cases} 2x + 3z = 3 \\ 4x - 3y + 7z = 5 \\ 8x - 9y + 15z = 9 \end{cases}$

79. $\begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases}$

80. $\begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = 28 \\ -x + y = 14 \end{cases}$

Comparing Solutions of Two Systems In Exercises 81–84, determine whether the two systems of linear equations yield the same solution. If so, find the solution using matrices.

81. (a) $\begin{cases} x - 2y + z = -6 \\ y - 5z = 16 \\ z = -3 \end{cases}$ (b) $\begin{cases} x + y - 2z = 6 \\ y + 3z = -8 \\ z = -3 \end{cases}$

82. (a) $\begin{cases} x - 3y + 4z = -11 \\ y - z = -4 \\ z = 2 \end{cases}$ (b) $\begin{cases} x + 4y = -11 \\ y + 3z = 4 \\ z = 2 \end{cases}$

83. (a) $\begin{cases} x - 4y + 5z = 27 \\ y - 7z = -54 \\ z = 8 \end{cases}$ (b) $\begin{cases} x - 6y + z = 15 \\ y + 5z = 42 \\ z = 8 \end{cases}$

84. (a) $\begin{cases} x + 3y - z = 19 \\ y + 6z = -18 \\ z = -4 \end{cases}$ (b) $\begin{cases} x - y + 3z = -15 \\ y - 2z = 14 \\ z = -4 \end{cases}$

85. Breeding Facility A city zoo borrowed $2,000,000 at simple annual interest to construct a breeding facility. Some of the money was borrowed at 8%, some at 9%, and some at 12%. Use a system of linear equations to determine how much was borrowed at each rate if the total annual interest was $186,000 and the amount borrowed at 8% was twice the amount borrowed at 12%. Solve the system of linear equations using matrices.

86. Museum A natural history museum borrowed $2,000,000 at simple annual interest to purchase new exhibits. Some of the money was borrowed at 7%, some at 8.5%, and some at 9.5%. Use a system of linear equations to determine how much was borrowed at each rate if the total annual interest was $169,750 and the amount borrowed at 8.5% was four times the amount borrowed at 9.5%. Solve the system of linear equations using matrices.

87. Think About It You and a friend solve the following system of equations independently.

$$\begin{cases} 2x - 4y - 3z = 3 \\ x + 3y + z = -1 \\ 5x + y - 2z = 2 \end{cases}$$

You write your solution set as

$(a, -a, 2a - 1)$

where a is any real number. Your friend's solution set is

$\left(\frac{1}{2}b + \frac{1}{2}, -\frac{1}{2}b \quad \frac{1}{2}, b\right)$

where b is any real number. Are you both correct? Explain. If you let $a = 3$, what value of b must be selected so that you both have the same ordered triple?

88. HOW DO YOU SEE IT? Determine all values of a and b for which the augmented matrix has each given number of solutions.

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & a & b \end{array}\right]$$

(a) Exactly one solution

(b) Infinitely many solutions

(c) No solution

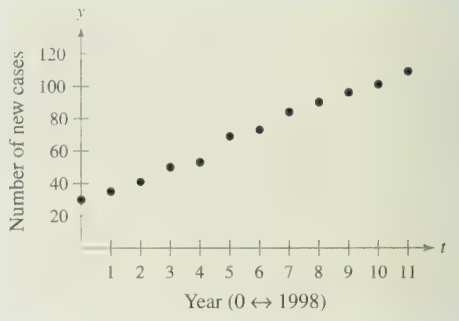
89. Health and Wellness From 1998 through 2009, the numbers of new cases of a waterborne disease in a small city increased in a pattern that was approximately linear (see figure). Find the least squares regression line

$y = at + b$

for the data shown in the figure by solving the following system using matrices. Let t represent the year, with $t = 0$ corresponding to 1998.

$$\begin{cases} 12b + 66a = 831 \\ 66b + 506a = 5643 \end{cases}$$

Use the result to predict the number of new cases of the waterborne disease in 2012. Is the estimate reasonable? Explain.



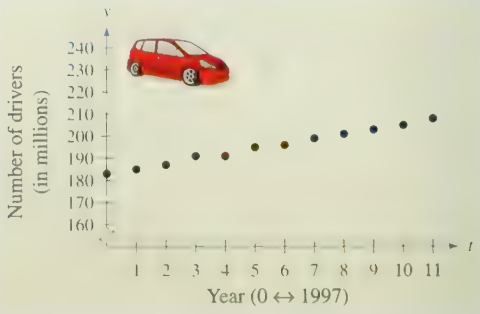
90. Drivers From 1997 through 2008, the numbers y (in millions) of drivers in the United States increased in a pattern that was approximately linear (see figure). Find the least squares regression line

$y = at + b$

for the data shown in the figure by solving the following system using matrices. Let t represent the year, with $t = 0$ corresponding to 1997.

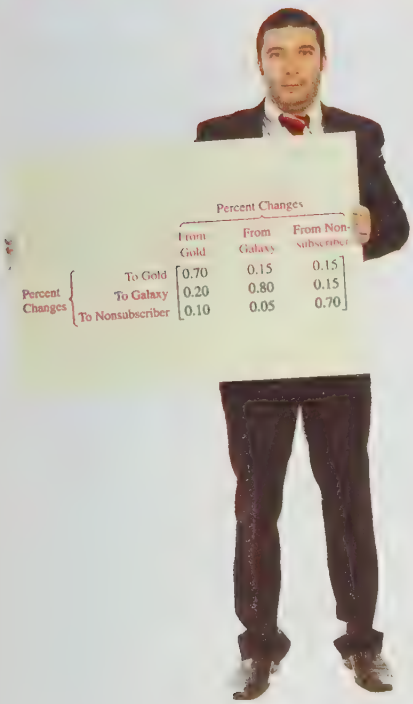
$$\begin{cases} 12b + 66a = 2344 \\ 66b + 506a = 13{,}213 \end{cases}$$

Use the result to predict the number of drivers in 2013. Is the estimate reasonable? Explain. *(Source: Federal Highway Administration)*



6.2 Operations with Matrices

- Determine whether two matrices are equal.
- Add or subtract two matrices and multiply a matrix by a scalar.
- Find the product of two matrices.
- Use matrix operations to solve an application problem.



In Exercise 89 on page 501, you will use matrix multiplication to find the numbers of subscribers that two competing satellite television providers have after a given period of time.

Equality of Matrices

In Section 6.1, you used matrices to solve systems of linear equations. Matrices, however, can do much more than this. There is a rich mathematical theory of matrices, with numerous applications. It is standard mathematical convention to represent matrices in any of the following three ways.

1. A matrix can be denoted by an uppercase letter such as A, B, or C.
2. A matrix can be denoted by a representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.
3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Two matrices

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

are **equal** when they have the same dimension ($m \times n$) and all of their corresponding entries are equal.

Example 1 Equality of Matrices

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

SOLUTION Because two matrices are equal only when they have the same dimension and their corresponding entries are equal, you can conclude that

$$a_{11} = 2, \quad a_{12} = -1, \quad a_{21} = -3, \quad \text{and} \quad a_{22} = 0.$$

Checkpoint 1

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$$

Be sure you see that for two matrices to be equal, they must have the same dimension and their corresponding entries must be equal. So,

$$\begin{bmatrix} 2 & -1 \\ \sqrt{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0.5 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

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Matrix Addition and Scalar Multiplication

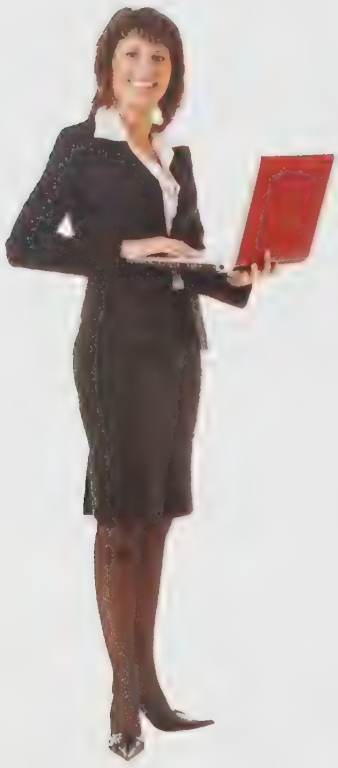
You can **add** two matrices (of the same dimension) by adding their corresponding entries.

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of dimension $m \times n$, then their **sum** is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different dimensions is undefined.



Example 2 Addition of Matrices

a. $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

d. The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined because A is of dimension 3×3 and B is of dimension 3×2.

Checkpoint 2

Evaluate the expression.

$$\begin{bmatrix} 2 & -7 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ -1 & -6 \end{bmatrix}$$

In operations with matrices, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. You can multiply a matrix A by a scalar c by multiplying each entry in A by c, as shown below.

Scalar Matrix

$$3\begin{bmatrix} -1 & 2 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 3(-1) & 3(2) \\ 3(6) & 3(5) \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 18 & 15 \end{bmatrix}$$

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** of A by c is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

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STUDY TIP

The order of operations for matrix expressions is similar to that for real numbers. In particular, you perform scalar multiplication before matrix addition and subtraction, as shown in Example 3(c).

The symbol $-A$ represents the negation of A, or the scalar product $(-1)A$. Moreover, if A and B are of the same dimension, then $A - B$ represents the sum of A and $(-1)B$. That is,

$$A - B = A + (-1)B.$$ Subtraction of matrices

Example 3 **Scalar Multiplication and Matrix Subtraction**

For the following matrices, find (a) $3A$, (b) $-A$, and (c) $3A - B$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

SOLUTION

a. $3A = 3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ Scalar multiplication

$= \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix}$ Multiply each entry by 3.

$= \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$ Simplify.

b. $-A = (-1)\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ Definition of negation

$= \begin{bmatrix} -2 & -2 & -4 \\ 3 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix}$ Multiply each entry by -1.

c. $3A - B = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ Perform scalar multiplication first.

$= \begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$ Subtract corresponding entries.

TECH TUTOR



Most graphing utilities have the capability of performing matrix operations. Consult the user's guide for your graphing utility for more information. Try using a graphing utility to check the solutions to Example 3.

Checkpoint 3

For the following matrices, find (a) $2A$ and (b) $2A - B$.

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 1 & 3 \\ -3 & 2 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 6 & 3 \\ 7 & -4 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the matrix below, the scalar $\frac{1}{2}$ has been factored out of the matrix.

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}$$

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers.

ALGEBRA TUTOR

To review addition and multiplication properties of real numbers, see the *Chapter 6 Algebra Tutor* on page 530.

Properties of Matrix Addition and Scalar Multiplication

If A, B, and C are $m \times n$ matrices and c and d are scalars, then the following properties are true.

1. $A + B = B + A$ — Commutative Property of Matrix Addition
2. $A + (B + C) = (A + B) + C$ — Associative Property of Matrix Addition
3. $(cd)A = c(dA)$ — Associative Property of Scalar Multiplication
4. $1A = A$ — Scalar Identity Property
5. $c(A + B) = cA + cB$ — Distributive Property
6. $(c + d)A = cA + dA$ — Distributive Property

Note that the Associative Property of Matrix Addition allows you to write expressions such as $A + B + C$ without ambiguity, because you obtain the same sum regardless of how the matrices are grouped. In other words, you obtain the same sum whether you group $A + B + C$ as $(A + B) + C$ or as $A + (B + C)$. This same reasoning applies to sums of four or more matrices.

ALGEBRA TUTOR

For more examples of the algebra involved in matrix addition and scalar multiplication, see the *Chapter 6 Algebra Tutor* on page 531.

Example 4 Addition of More than Two Matrices

By adding corresponding entries, you can obtain the sum of four matrices, as shown below.

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

✓Checkpoint 4

Add the matrices $\begin{bmatrix} 2 & -9 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 4 \\ 7 & -3 \end{bmatrix}$, and $\begin{bmatrix} 0 & 6 \\ 5 & -2 \end{bmatrix}$.

STUDY TIP

In Example 5, you could add the two matrices first and then multiply the resulting matrix by 3. The result would be the same.

Example 5 Using the Distributive Property

$$3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right) = 3\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 12 & -6 \\ 9 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix}$$

✓Checkpoint 5

Evaluate the expression using the Distributive Property.

$$4\left(\begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & -6 \end{bmatrix}\right)$$



One important property of addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number c. For matrices, a similar property holds. That is, if A is an $m \times n$ matrix and O is the $m \times n$ **zero matrix** consisting entirely of zeros, then $A + O = A$.

In other words, O is the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrices are the additive identities for the sets of all 2×3 and 2×2 matrices, respectively.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3 \text{ zero matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2 \times 2 \text{ zero matrix}}$$

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the following solutions.

Real Numbers (*Solve for x.*)	*m × n Matrices* (*Solve for X.*)
$x + a = b$	$X + A = B$
$x + a + (-a) = b + (-a)$	$X + A + (-A) = B + (-A)$
$x + 0 = b - a$	$X + O = B - A$
$x = b - a$	$X = B - A$

This means that you can apply some of your knowledge of solving real number equations to solving matrix equations. It is often easier to complete the algebraic steps first, and then substitute the matrices in the equation, as shown in Example 6.

STUDY TIP

Remember that matrices are denoted by capital letters. So, when you solve for X in Example 6, you are solving for a *matrix* that makes the matrix equation true.

Example 6 Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

SOLUTION Begin by solving the equation for X to obtain

$$3X + A = B$$
$$3X = B - A$$
$$X = \frac{1}{3}(B - A).$$

Now, using the matrices A and B, you have

$$X = \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right)$$ Substitute the matrices.

$$= \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$$ Subtract matrix A from matrix B

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$ Multiply the resulting matrix by $\frac{1}{3}$

✓Checkpoint 6

Solve for X in the equation $2X - A = B$, where

$$A = \begin{bmatrix} 7 & 0 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

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Matrix Multiplication

The third basic matrix operation is **matrix multiplication.** At first glance, the definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.

The definition of matrix multiplication indicates a *row-by-column* multiplication, where the entry in the ith row and jth column of the product AB is obtained by multiplying the entries in the ith row of A by the corresponding entries in the jth column of B and then adding the results. The general pattern for matrix multiplication is as follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$$

Example 7 Finding the Product of Two Matrices

Find the product AB using $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

SOLUTION First, note that the product AB is defined because the number of *columns* of A is equal to the number of *rows* of B. Moreover, the product AB has dimension 3×2. To find the entries of the product, multiply each row of A by each column of B.

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

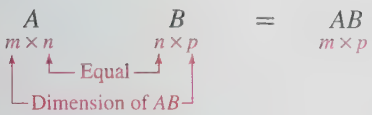
$$= \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

Checkpoint 7

Find the product AB using

$$A = \begin{bmatrix} 2 & 2 \\ 0 & 4 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 \\ 1 & -4 \end{bmatrix}.$$



Be sure you understand that for the product of two matrices to be defined, the number of *columns* of the first matrix must equal the number of *rows* of the second matrix. That is, the middle two indices must be the same and the outside two indices give the dimension of the product, as shown at the left.

Example 8 Finding the Product of Two Matrices

Find the product AB using $A = \begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ -3 & 5 \end{bmatrix}$.

SOLUTION Note that the dimension of A is 3×3 and the dimension of B is 3×2. So, the product AB is defined and is of dimension 3×2.

$$AB = \begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6(1) + 2(2) + 0(-3) & 6(0) + 2(7) + 0(5) \\ 3(1) + (-1)(2) + 2(-3) & 3(0) + (-1)(7) + 2(5) \\ 1(1) + 4(2) + 6(-3) & 1(0) + 4(7) + 6(5) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 \\ -5 & 3 \\ -9 & 58 \end{bmatrix}$$

✓Checkpoint 8

Find the product AB using

$$A = \begin{bmatrix} 0 & 4 & -3 \\ 2 & 1 & 7 \\ 3 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 0 \\ 0 & -4 \\ 1 & 2 \end{bmatrix}.$$

Example 9 Patterns in Matrix Multiplication

a. $\underset{2 \times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}} \underset{3 \times 3}{\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}} = \underset{2 \times 3}{\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}}$

b. $\underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}} \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$

c. The product AB for the following matrices is not defined.

$$A = \underset{3 \times 2}{\begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix}} \quad \text{and} \quad B = \underset{3 \times 4}{\begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}}$$

✓Checkpoint 9

Find AB, if possible, using $A = \begin{bmatrix} 2 & 0 & 1 \\ 6 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 \\ 3 & 0 \end{bmatrix}$.

Example 10 Patterns in Matrix Multiplication

a. $\begin{bmatrix}1 & -2\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}4\end{bmatrix}$

$1 \cdot 2 \quad 2 \cdot -1 \quad 1 \cdot 1$

b. $\begin{bmatrix}2\\-1\end{bmatrix}\begin{bmatrix}1 & -2\end{bmatrix} = \begin{bmatrix}2 & -4\\-1 & 2\end{bmatrix}$

$2 \cdot 1 \quad 1 \cdot 2 \quad 2 \cdot 2$

Checkpoint 10

Find AB and BA using

$$A = \begin{bmatrix}3 & -1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1\\-3\end{bmatrix}.$$

In Example 10, note that the two products are different. Even when AB and BA are defined, matrix multiplication is not, in general, commutative. That is, for most matrices, $AB \neq BA$.



Properties of Matrix Multiplication

Let A, B, and C be matrices and let c be a scalar.

1. $A(BC) = (AB)C$ — Associative Property of Matrix Multiplication
2. $A(B + C) = AB + AC$ — Left Distributive Property
3. $(A + B)C = AC + BC$ — Right Distributive Property
4. $c(AB) = (cA)B = A(cB)$ — Associative Property of Scalar Multiplication

Definition of the Identity Matrix

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of dimension $n \times n$** and is denoted by

$$I_n = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\\0 & 1 & 0 & \cdots & 0\\0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\0 & 0 & 0 & \cdots & 1\end{bmatrix}.$$ Identity matrix

Note that an identity matrix must be *square*. When the dimension is understood to be $n \times n$, you can denote I_n simply by I.

If A is an $n \times n$ matrix, then the identity matrix has the property that $AI_n = A$ and $I_nA = A$. For example,

$$\begin{bmatrix}3 & -2 & 5\\1 & 0 & 4\\-1 & 2 & -3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}3 & -2 & 5\\1 & 0 & 4\\-1 & 2 & -3\end{bmatrix} \qquad AI = A$$

and

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & -2 & 5\\1 & 0 & 4\\-1 & 2 & -3\end{bmatrix} = \begin{bmatrix}3 & -2 & 5\\1 & 0 & 4\\-1 & 2 & -3\end{bmatrix}. \qquad IA = A$$

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Applications

One application of matrix multiplication is the representation of a system of linear equations. Note how the system below can be written as the matrix equation $AX = B$ where A is the *coefficient matrix* of the system, B is the *constant matrix* of the system, and X is a column matrix.

System

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Matrix Equation $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A \quad \times X \quad = \quad B$

In Example 11, note that $[A \vdots B]$ represents the augmented matrix formed when matrix B is *adjoined* to matrix A. Also, $[I \vdots X]$ represents the reduced row-echelon form of the augmented matrix that yields the solution of the system.

Example 11 Solving a System of Linear Equations

Consider the system of linear equations.

$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$$

a. Write the system as a matrix equation $AX = B$.

b. Use Gauss-Jordan elimination on the augmented matrix $[A \vdots B]$ to solve for the matrix X.

SOLUTION

a. In matrix form $AX = B$, the system is written as

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}.$$

Coefficient matrix Constant matrix

b. The augmented matrix is formed by adjoining matrix B to matrix A.

$$[A \vdots B] = \left[\begin{array}{ccc:c} 1 & -2 & 1 & -4 \\ 0 & 1 & 2 & 4 \\ 2 & 3 & -2 & 2 \end{array}\right]$$

Using Gauss-Jordan elimination, you can rewrite this matrix as

$$[I \vdots X] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

So, the solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

✓ Checkpoint 11

For the system of linear equations, (a) write the system as a matrix equation $AX = B$, and (b) use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix X.

$$\begin{cases} -2x_1 - 3x_2 = -4 \\ 6x_1 + x_2 = -36 \end{cases}$$

Example 12 Softball Team Expenses

Two softball teams submit equipment lists to their sponsors, as shown in the table.

Equipment	Women's team	Men's team
Bats	12	15
Balls	45	38
Gloves	15	17

Each bat costs \$90, each ball costs \$6, and each glove costs \$60. Use matrices to find the total cost of equipment for each team.

SOLUTION The equipment lists E and the costs per item C can be written in matrix form as

$$E = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \quad \text{and} \quad C = [90 \quad 6 \quad 60].$$

You can find the total cost of the equipment for each team using the product CE because the number of columns of C (3 columns) equals the number of rows of E (3 rows). The total cost of equipment for each team is given by

$$\begin{aligned} CE &= [90 \quad 6 \quad 60]\begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \\ &= [90(12) + 6(45) + 60(15) \qquad 90(15) + 6(38) + 60(17)] \\ &= [2250 \qquad 2598]. \end{aligned}$$

So, the total cost of equipment for the women's team is \$2250, and the total cost of equipment for the men's team is \$2598.

Checkpoint 12

Repeat Example 12 when each bat costs \$100, each ball costs \$7, and each glove costs \$65.



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SUMMARIZE (Section 6.2)

1. State the conditions under which two matrices are equal *(page 487)*. For an example of the equality of matrices, see Example 1.
2. State the definition of matrix addition *(page 488)*. For an example of matrix addition, see Example 2.
3. State the definition of scalar multiplication *(page 488)*. For an example of scalar multiplication, see Example 3.
4. List the properties of matrix addition and scalar multiplication *(page 490)*. For examples of using these properties, see Examples 4, 5, and 6.
5. State the definition of matrix multiplication *(page 492)*. For examples of matrix multiplication, see Examples 7, 8, 9, and 10.

SKILLS WARM UP 6.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2 and 6.1.

In Exercises 1 and 2, evaluate the expression.

1. $-3\left(-\frac{5}{6}\right) + 10\left(-\frac{3}{4}\right)$

2. $-22\left(\frac{5}{2}\right) + 6(8)$

In Exercises 3 and 4, determine whether the matrix is in reduced row-echelon form.

3. $\begin{bmatrix} 0 & 1 & 0 & -5 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}$

In Exercises 5 and 6, write the augmented matrix for the system of linear equations.

5. $\begin{cases} -5x + 10y = 12 \\ 7x - 3y = 0 \end{cases}$

6. $\begin{cases} 10x + 15y - 9z = 42 \\ 6x - 5y = 0 \end{cases}$

In Exercises 7–10, solve the system of linear equations represented by the augmented matrix.

7. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

10. $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Exercises 6.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Equality of Matrices In Exercises 1–4, find x and y. *See Example 1.*

1. $\begin{bmatrix} 4 & x \\ -1 & y \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} x & -7 \\ 9 & y \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ 9 & -8 \end{bmatrix}$

3. $\begin{bmatrix} -4 & 3 \\ 6 & -1 \\ 8 & 2 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} x-2 & 3 \\ 6 & -1 \\ 8 & -x \\ 5 & 2y-1 \end{bmatrix}$

4. $\begin{bmatrix} x+2 & 8 & -3 \\ 1 & 2y & 2x \\ 7 & -2 & y+2 \end{bmatrix} = \begin{bmatrix} 2x+6 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & 11 \end{bmatrix}$

Operations with Matrices In Exercises 5–14, find (a) $A + B$, (b) $A - B$, (c) $6A$, and (d) $4A - 3B$. *See Examples 2 and 3.*

5. $A = [7 \quad -3]$, $B = [-4 \quad 2]$

6. $A = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$

7. $A = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$

8. $A = \begin{bmatrix} 7 & 4 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 8 & -4 \end{bmatrix}$

9. $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$

10. $A = \begin{bmatrix} 6 & 8 & -3 & 2 & 1 \\ -4 & 2 & 1 & 5 & -2 \end{bmatrix}$,
$B = \begin{bmatrix} 6 & 0 & 4 & -1 & 3 \\ 4 & 5 & -2 & 1 & 2 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 4 & 9 \\ 0 & -7 & 8 \end{bmatrix}$

12. $A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 \\ 6 \\ 2 \end{bmatrix}$

13. $A = [-13 \quad 24 \quad 20 \quad -11]$,
$B = [15 \quad 16 \quad 10 \quad -12]$

14. $A = \begin{bmatrix} 12 & 5 & -24 \\ -10 & 0 & 13 \end{bmatrix}$, $B = \begin{bmatrix} 17 & -6 & 0 \\ -1 & 15 & 18 \end{bmatrix}$

Evaluating an Expression In Exercises 15–24, evaluate the expression. *See Examples 4 and 5.*

15. $\begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$

16. $\begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix}$

17. $[4 \quad -8] + [-3 \quad 1] + [2 \quad 0] + [9 \quad 1]$

18. $\begin{bmatrix} 7 \\ -1 \\ 5 \\ -5 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ -9 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \\ 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -5 \\ -7 \\ -1 \end{bmatrix}$

19. $5\left(\begin{bmatrix} 1 & -3 \\ -2 & -8 \end{bmatrix} + \begin{bmatrix} 10 & -7 \\ 5 & 4 \end{bmatrix}\right)$

20. $-7([-3 \quad -1 \quad 6] - [-5 \quad 4 \quad -4])$

21. $8\left(\begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 3 & 6 & 0 \end{bmatrix}\right)$

22. $\frac{1}{2}([5 \quad -2 \quad 4 \quad 0] + [14 \quad 6 \quad -18 \quad 9])$

23. $-3\left(\begin{bmatrix} 0 & -3 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 8 & 1 \end{bmatrix}\right) - 2\begin{bmatrix} 4 & -4 \\ 7 & -9 \end{bmatrix}$

24. $-1\begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix}\right)$



Operations with Matrices In Exercises 25–28, use the matrix capabilities of a graphing utility to evaluate the expression. Round your results to three decimal places, if necessary.

25. $\frac{3}{7}\begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix} + 6\begin{bmatrix} -3 & 0 \\ 2 & 2 \end{bmatrix}$

26. $55\left(\begin{bmatrix} 14 & -11 \\ -22 & 19 \end{bmatrix} + \begin{bmatrix} -22 & 20 \\ 13 & 6 \end{bmatrix}\right)$

27. $-\begin{bmatrix} 3.211 & 6.829 \\ -1.004 & 4.914 \\ 0.055 & -3.889 \end{bmatrix} - \begin{bmatrix} -1.630 & -3.090 \\ 5.256 & 8.335 \\ -9.768 & 4.251 \end{bmatrix}$

28. $-12\left(\begin{bmatrix} 6 & 20 \\ 1 & -9 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 14 & -15 \\ -8 & -6 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} -31 & -19 \\ 16 & 10 \\ 24 & -10 \end{bmatrix}\right)$

Solving a Matrix Equation In Exercises 29–32, solve for X, where

$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$. *See Example 6.*

29. $X = 3A - 2B$

30. $2X = 2A - B$

31. $2X + 3A = B$

32. $2A + 4B = -2X$

Finding the Product of Two Matrices In Exercises 33–44, find AB, if possible. *See Examples 7, 8, 9, and 10.*

33. $A = [-3 \quad 5]$, $B = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

34. $A = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $B = [8 \quad 2]$

35. $A = \begin{bmatrix} -1 & 2 \\ 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

36. $A = [-5 \quad 0]$, $B = \begin{bmatrix} -3 & 1 \\ -1 & 4 \end{bmatrix}$

37. $A = \begin{bmatrix} 3 & -2 \\ 4 & 5 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 & -2 & 5 \\ 2 & 1 & 3 & -1 \end{bmatrix}$

38. $A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$

39. $A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$

40. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

41. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

42. $A = \begin{bmatrix} 6 \\ -2 \\ 1 \\ 6 \end{bmatrix}$, $B = [10 \quad 12]$

43. $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ 5 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

44. $A = \begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$



Matrix Multiplication In Exercises 45–48, use the matrix capabilities of a graphing utility to find AB, if possible.

45. $A = \begin{bmatrix} 5 & 6 & -3 \\ -2 & 5 & 1 \\ 10 & -5 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 2 \\ 8 & 1 & 4 \\ 4 & -2 & 9 \end{bmatrix}$

46. $A = \begin{bmatrix} 11 & -12 & 4 \\ 14 & 10 & 12 \\ 6 & -2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 12 & 10 \\ -5 & 12 \\ 15 & 16 \end{bmatrix}$

47. $A = \begin{bmatrix} -2 & 4 & 8 \\ 21 & 5 & 6 \\ 13 & 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ -7 & 15 \\ 32 & 14 \\ 0.5 & 1.6 \end{bmatrix}$

48. $A = \begin{bmatrix} 9 & 10 & -38 & 18 \\ 100 & -50 & 250 & 75 \end{bmatrix}$,
$B = \begin{bmatrix} 52 & -85 & 27 & 45 \\ 40 & -35 & 60 & 82 \end{bmatrix}$

Operations with Matrices In Exercises 49–54, find (a) AB, (b) BA, and, if possible, (c) A^2. (*Note:* $A^2 = AA$)

49. $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

50. $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix}$

51. $A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -1 & -2 \\ 2 & 4 \end{bmatrix}$

52. $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

53. $A = [-4 \quad 2 \quad 3]$, $B = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

54. $A = [3 \quad 2 \quad 1 \quad 0]$, $B = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

Think About It In Exercises 55–64, let matrices A, B, C, and D be dimensions 2×3, 2×3, 3×2, and 2×2, respectively. Determine whether it is possible to perform the operation(s). If so, give the dimension of the result.

55. $A + 2C$

56. $B - 3C$

57. AB

58. BC

59. AD

60. CD

61. $BC - D$

62. $CB - D$

63. $D(A - 3B)$

64. $(BC - D)A$

Solving a System of Linear Equations In Exercises 65–72, (a) write the system of linear equations as a matrix equation $AX = B$, and (b) use Gauss-Jordan elimination on the augmented matrix $[A \vdots B]$ to solve for the matrix X. *See Example 11.*

65. $\begin{cases} -x + y = 4 \\ -2x + y = 0 \end{cases}$

66. $\begin{cases} 2x + 3y = 5 \\ x + 4y = 10 \end{cases}$

67. $\begin{cases} x + 2y = 3 \\ 3x - y = 2 \end{cases}$

68. $\begin{cases} -4x - 3y = -2 \\ x + y = -1 \end{cases}$

69. $\begin{cases} x - 4y + 5z = 3 \\ 2x + 5y + z = -24 \\ -x - y - z = 8 \end{cases}$

70. $\begin{cases} 2x - 4y + z = 0 \\ -x + 3y + z = 1 \\ x + y = 3 \end{cases}$

71. $\begin{cases} x - 2y + 3z = 9 \\ -x + 3y - z = -6 \\ 2x - 5y + 5z = 17 \end{cases}$

72. $\begin{cases} x + y - 3z = 1 \\ -x + 2y = 1 \\ x - y + z = 2 \end{cases}$

73. Manufacturing A corporation that makes sunglasses has four factories, each of which manufactures two products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 100 & 120 & 60 & 40 \\ 140 & 160 & 200 & 80 \end{bmatrix}.$$

Find the production levels when production is increased by 10%. (*Hint:* Because an increase of 10% corresponds to 100% + 10%, multiply the matrix by 1.10.)

74. Manufacturing A tire corporation has three factories, each of which manufactures two products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 80 & 120 & 140 \\ 40 & 100 & 80 \end{bmatrix}.$$

Find the production levels when production is decreased by 5%. (*Hint:* Because a decrease of 5% corresponds to 100% − 5%, multiply the matrix by 0.95.)

75. Vacation Packages A vacation service has identified four resort hotels with a special all-inclusive package. The quoted room rates are for double and family (maximum of four people) occupancy for 5 days and 4 nights. The current rates for the two types of rooms at the four hotels are represented by the matrix A.

$$A = \begin{array}{c} \begin{matrix} \text{Hotel } w & \text{Hotel } x & \text{Hotel } y & \text{Hotel } z \end{matrix} \\ \begin{bmatrix} 615 & 670 & 740 & 990 \\ 995 & 1030 & 1180 & 1105 \end{bmatrix} \begin{matrix} \text{Double} \\ \text{Family} \end{matrix} \end{array} \} \text{Occupancy}$$

Room rates are guaranteed not to increase by more than 12% by next season. What is the maximum rate per package per hotel?

76. HOW DO YOU SEE IT? A fruit grower raises apples and peaches, which are shipped to three different outlets. The numbers of units of apples and peaches that are shipped to the three outlets are shown in the matrix A.

$$A = \begin{array}{c} \begin{matrix} \text{Outlet} \\ \overbrace{\qquad\qquad} \\ X \quad Y \quad Z \end{matrix} \\ \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix} \begin{matrix} \text{Apples} \\ \text{Peaches} \end{matrix} \end{array} \} \text{Units shipped}$$

(a) The profit per unit of apples is \$3.50 and the profit per unit of peaches is \$6. Organize the profits per unit in a matrix B.

(b) The product BA is given by the matrix

$$BA = [\$1037.50 \quad \$1400.00 \quad \$1012.50].$$

Explain what this matrix represents.

(c) Can you compute AB to find the profits? Explain.

77. Inventory Levels A company sells five different models of laptop computers through three retail outlets. The inventories of the five models at the three outlets are given by the matrix S.

$$S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \overbrace{\begin{bmatrix} 3 & 2 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 & 3 \\ 4 & 2 & 1 & 3 & 2 \end{bmatrix}}^{\text{Model: A, B, C, D, E}} \} \text{Outlet}$$

The wholesale and retail prices for each model are given by the matrix T.

$$T = \overbrace{\begin{bmatrix} \$250 & \$475 \\ \$400 & \$700 \\ \$575 & \$850 \\ \$650 & \$950 \\ \$760 & \$1200 \end{bmatrix}}^{\text{Price: Wholesale, Retail}} \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array} \} \text{Model}$$

(a) What is the total retail price of the inventory at Outlet 1?

(b) What is the total wholesale price of the inventory at Outlet 3?

(c) Compute the product ST and interpret the result in the context of the problem.

78. Labor/Wage Requirements A company that manufactures boats has the following labor-hour and wage requirements.

Labor-Hour Requirements (per boat)

$$S = \begin{array}{l} \\ \text{Small} \\ \text{Medium} \\ \text{Large} \end{array} \overbrace{\begin{bmatrix} 1.0 \text{ hour} & 0.5 \text{ hour} & 0.2 \text{ hour} \\ 1.6 \text{ hours} & 1.0 \text{ hour} & 0.2 \text{ hour} \\ 2.5 \text{ hours} & 2.0 \text{ hours} & 0.4 \text{ hour} \end{bmatrix}}^{\text{Department: Cutting, Assembly, Packaging}} \} \text{Boat size}$$

Wage Requirements (per hour)

$$T = \overbrace{\begin{bmatrix} \$15 & \$13 \\ \$12 & \$11 \\ \$11 & \$10 \end{bmatrix}}^{\text{Plant: A, B}} \begin{array}{l} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{array} \} \text{Department}$$

(a) What is the labor cost for a medium boat at Plant B?

(b) What is the labor cost for a large boat at Plant A?

(c) Compute ST and interpret the result.

79. Exercise The numbers of calories burned by individuals of different body weights while performing different types of aerobic exercises for a one-hour time period are shown in the matrix A.

$$A = \overbrace{\begin{bmatrix} 472 & 563 \\ 590 & 704 \\ 177 & 211 \end{bmatrix}}^{\text{Calories burned: 130-lb person, 155-lb person}} \begin{array}{l} \text{Basketball} \\ \text{Jumping rope} \\ \text{Weight lifting} \end{array}$$

(a) A 130-pound person and a 155-pound person played basketball for 2 hours, jumped rope for 15 minutes, and lifted weights for 30 minutes. Organize the times spent exercising in a matrix B.

(b) Compute BA and interpret the result.

80. Exercise The numbers of calories burned by individuals of different body weights while performing different types of aerobic exercises for a 30-minute time period are shown in the matrix A.

$$A = \overbrace{\begin{bmatrix} 143 & 163 \\ 286 & 326 \\ 164 & 186 \end{bmatrix}}^{\text{Calories burned: 180-lb person, 205-lb person}} \begin{array}{l} \text{Calisthenics} \\ \text{Tennis} \\ \text{Water aerobics} \end{array}$$

(a) A 180-pound person and a 205-pound person performed calisthenics for 30 minutes, played tennis for 1 hour, and did water aerobics for 45 minutes. Organize the times spent exercising in a matrix B.

(b) Compute BA and interpret the result.

81. Contract Bonuses Professional athletes frequently have bonus or incentive clauses in their contracts. For example, a defensive football player might receive bonuses for defensive plays such as sacks, interceptions, and/or key tackles. In one contract, a sack is worth \$2000, an interception is worth \$1000, and a key tackle is worth \$800. The table shows the numbers of sacks, interceptions, and key tackles for three players.

Player	Sacks	Interceptions	Key tackles
Player X	3	0	4
Player Y	1	2	5
Player Z	2	3	3

(a) Write a matrix D that represents the number of each type of defensive play i made by each player j using the data from the table. State what each entry d_{ij} of the matrix represents.

(b) Write a matrix B that represents the bonus amount received for each type of defensive play. State what each entry b_{ij} of the matrix represents.

(c) Find the product BD of the two matrices and state what each entry of matrix BD represents.

(d) Which player receives the largest bonus?

82. Health Care The health care plans offered this year by a local manufacturing plant are as follows. For individuals, the comprehensive plan costs \$694.32, the HMO standard plan costs \$451.80, and the HMO Plus plan costs \$489.48. For families, the comprehensive plan costs \$1725.36, the HMO standard plan costs \$1187.76, and the HMO Plus plan costs \$1248.12. The plant expects the costs of the plans to change next year as follows. For individuals, the costs for the comprehensive, HMO standard, and HMO Plus plans will be \$683.91, \$463.10, and \$499.27, respectively. For families, the costs for the comprehensive, HMO standard, and HMO Plus plans will be \$1699.48, \$1217.45, and \$1273.08, respectively.

(a) Organize the information using two matrices A and B, where A represents the health care plan costs for this year and B represents the health care plan costs for next year. State what each entry of each matrix represents.

(b) Compute $A - B$ and interpret the result.

(c) The employees receive monthly paychecks from which the health care plan costs are deducted. Use the matrices from part (a) to write matrices that show how much will be deducted from each employee's paycheck this year and next year.

(d) Suppose instead that the costs of the health care plans increase by 4% next year. Write a matrix that shows the new monthly payments.

83. Voting Preference The matrix



$$P = \begin{array}{c} \\ \text{R} \\ \text{D} \\ \text{I} \end{array} \overset{\overset{\text{From}}{\overbrace{}}}{\begin{bmatrix} \text{R} & \text{D} & \text{I} \\ 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix}} \begin{array}{l} \\ \text{R} \\ \text{D} \\ \text{I} \end{array} \Big\} \text{To}$$

is called a *stochastic matrix*. Each entry p_{ij} $(i \neq j)$ represents the proportion of the voting population that changes from Party i to Party j, and p_{ii} represents the proportion that remains loyal to the party from one election to the next. Use a graphing utility to find P^2. (This matrix gives the transition probabilities from the first election to the third.)

84. Voting Preference Use a graphing utility to find P^3, P^4, P^5, P^6, P^7, and P^8 for the matrix given in Exercise 83. Can you detect a pattern?



Finding a Matrix In Exercises 85 and 86, find a matrix B such that AB is the identity matrix. Is there more than one correct result?

85. $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

86. $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$

87. Think About It If a, b, and c are real numbers such that $c \neq 0$ and $ac = bc$, then $a = b$. However, if A, B, and C are matrices such that $AC = BC$, then A is *not* necessarily equal to B. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix},$$

and $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix}$

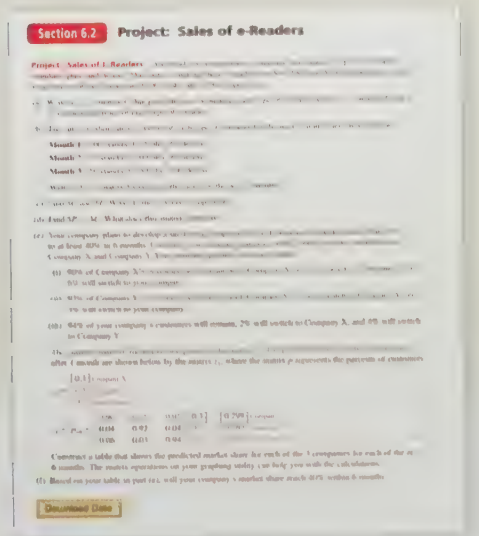
88. Think About It If a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$. However, if A and B are matrices such that $AB = O$, then it is *not* necessarily true that $A = O$ or $B = O$. Illustrate this using the matrices $A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find another example of two nonzero matrices whose product is the zero matrix.

89. Satellite Television Two competing satellite television providers offer service to a city with 100,000 households. Gold Cable Company has 25,000 subscribers and Galaxy Cable Company has 30,000 subscribers. (The other 45,000 households do not subscribe.) The percent changes in subscriptions each year are shown in the matrix below.

$$\begin{array}{c} \\ \text{Percent Changes} \\ \\ \\ \end{array} \begin{array}{l} \\ \text{To Gold} \\ \text{To Galaxy} \\ \text{To Nonsubscriber} \end{array} \overset{\overset{\text{Percent Changes}}{\overbrace{}}}{\begin{bmatrix} \text{From Gold} & \text{From Galaxy} & \text{From Non-subscriber} \\ 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}}$$

Use matrix multiplication to find the number of subscribers each company will have in one year, in two years, and in three years. What is happening to the number of subscribers and nonsubscribers to each company? Explain.

90. Project: Sales of e-Readers For a project involving the sales of e-readers, visit this textbook's website at *www.cengagebrain.com.*



6.3 The Inverse of a Square Matrix

- Verify that a matrix is the inverse of a given matrix.
- Find the inverse of a matrix.
- Find the inverse of a 2 × 2 matrix using a formula.
- Use an inverse matrix to solve a system of linear equations.



In Exercise 82 on page 511, you will use an inverse matrix to find an equation that models the annual revenues for a company.

The Inverse of a Matrix

This section further develops the algebra of matrices. To begin, consider the real number equation $ax = b$. To solve this equation for x, multiply each side of the equation by a^{-1} (provided $a \neq 0$).

$$ax = b$$
$$(a^{-1}a)x = a^{-1}b$$
$$(1)x = a^{-1}b$$
$$x = a^{-1}b$$

The number a^{-1} is called the *multiplicative inverse* of a because $a^{-1}a = 1$. The definition of the multiplicative inverse of a matrix is similar.

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the **inverse** of A. (The symbol A^{-1} is read "A inverse.")

Example 1 The Inverse of a Matrix

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

SOLUTION To show that B is the inverse of A, show that $AB = I = BA$.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Checkpoint 1

Show that B is the inverse of A, where $A = \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$.

Recall that it is not always true that $AB = BA$, even when both products are defined. However, if A and B are both square matrices and $AB = I_n$, then it can be shown that $BA = I_n$. So, in Example 1, you need only check that $AB = I_2$.

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Finding Inverse Matrices

If a matrix A has an inverse, then A is called **invertible** (or **nonsingular**); otherwise, A is called **singular.** A nonsquare matrix cannot have an inverse. To see this, note that when A is of dimension $m \times n$ and B is of dimension $n \times m$ (where $m \neq n$), the products AB and BA are of different dimensions and therefore cannot be equal to each other. Not all square matrices have inverses (see the matrix at the bottom of page 505). When, however, a matrix does have an inverse, that inverse is unique. The next example shows how to use a system of equations to find an inverse.

Example 2 Finding the Inverse of a Matrix

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

SOLUTION To find the inverse of A, try to solve the matrix equation $AX = I$ for X.

$$\begin{matrix} A & X & = & I \end{matrix}$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Write matrix equation.}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Multiply.}$$

Equating corresponding entries, you obtain the following two systems of linear equations.

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \qquad \begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$

You can solve these systems using the methods learned in Chapter 5. From the first system you can determine that $x_{11} = -3$ and $x_{21} = 1$. From the second system you can determine that $x_{12} = -4$ and $x_{22} = 1$. So, the inverse of A is

$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

You can use matrix multiplication to check this result.

CHECK

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

✓Checkpoint 2

Find the inverse of the matrix

$$A = \begin{bmatrix} -4 & -1 \\ 5 & 1 \end{bmatrix}.$$

In Example 2, note that the two systems of linear equations have the *same coefficient matrix A*. Rather than solve the two systems represented by

$$\left[\begin{array}{cc|c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right]$$

separately, you can solve them simultaneously by adjoining the identity matrix to the coefficient matrix to obtain

$$\begin{array}{cc} A & \quad I \\ \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]. \end{array}$$

This "doubly augmented" matrix can be represented as

$$[A \vdots I].$$

Then, by applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process.

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]$$

$$R_1 + R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$-4R_2 + R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

So, from the "doubly augmented" matrix $[A \vdots I]$, you obtained the matrix $[I \vdots A^{-1}]$.

$$\begin{array}{cc} A & \quad I \\ \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \end{array} \Rightarrow \begin{array}{cc} I & \quad A^{-1} \\ \left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

This procedure (or algorithm) works for any square matrix that has an inverse.

TECH TUTOR

Most graphing utilities can find the inverse of a matrix by using the inverse key x^{-1}. The screen below shows how one graphing utility displays the inverse of a matrix. For specific instructions on how to use the inverse key to find the inverse of a matrix, consult the user's guide for your graphing utility.

```
[A]
          [[1   4 ]
           [-1  -3]]
[A]^-1
          [[-3  -4]
           [1   1 ]]
```



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Finding an Inverse Matrix

Let A be a square matrix of dimension $n \times n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain
$$[A \vdots I].$$
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix
$$[A \vdots I].$$
The result will be the matrix
$$[I \vdots A^{-1}].$$
If this is not possible, then A is not invertible.
3. Check your work by multiplying to see that
$$AA^{-1} = I = A^{-1}A.$$

Example 3 **Finding the Inverse of a Matrix**

Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$.

SOLUTION Begin by adjoining the identity matrix to A to form the matrix

$$[A \vdots I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right].$$

Next, use elementary row operations to obtain the matrix $[I \vdots A^{-1}]$.

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -6R_1 + R_3 \rightarrow \end{array} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \rightarrow \\ \\ -4R_2 + R_3 \rightarrow \end{array} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \rightarrow \\ R_3 + R_2 \rightarrow \\ \end{array} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

So, the matrix A is invertible and its inverse is $A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$.

CHECK Confirm this result by multiplying A and A^{-1} to obtain I.

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Checkpoint 3

Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & 2 \\ 1 & -2 & 0 \end{bmatrix}$.

The process shown in Example 3 applies to any $n \times n$ matrix A. So, when A has an inverse, this process will find it. When using this process, if the matrix A does not reduce to the identity matrix, then A does not have an inverse.

For instance, consider the matrix A shown below. To confirm that A has no inverse, begin by adjoining the identity matrix to A.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix} \qquad [A \vdots I] = \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

Then use elementary row operations to obtain

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right].$$

At this point in the elimination process, you can see that it is impossible to obtain the identity matrix I on the left. So, A is not invertible.

The Inverse of a 2 × 2 Matrix (Quick Method)

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of dimensions 3×3 or greater. For 2×2 matrices, however, many people prefer to use a formula for the inverse rather than Gauss-Jordan elimination. This formula works *only* for 2×2 matrices. If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, when $ad - bc \neq 0$, the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ Formula for inverse of a 2×2 matrix

Try verifying this inverse by multiplication.

The denominator $ad - bc$ is called the **determinant** of the 2×2 matrix A. You will study determinants in the next section.

Example 4 Finding the Inverse of a 2×2 Matrix

If possible, find the inverse of each matrix.

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

SOLUTION

a. For matrix A, begin by applying the formula for the determinant of a 2×2 matrix to obtain

$$ad - bc = 3(2) - (-1)(-2) = 4.$$

Because this quantity is not zero, the matrix is invertible. The inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Formula for inverse of a 2×2 matrix

$$= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$ Substitute for a, b, c, d, and the determinant.

$$= \begin{bmatrix} \frac{1}{4}(2) & \frac{1}{4}(1) \\ \frac{1}{4}(2) & \frac{1}{4}(3) \end{bmatrix}$$ Multiply by the scalar $\frac{1}{4}$.

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$ Simplify.

b. For matrix B, you have

$$ad - bc = 3(2) - (-1)(-6)$$
$$= 0.$$

Because $ad - bc = 0$, B is not invertible.

✓Checkpoint 4

Find the inverse of the matrix

$$A = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}.$$

Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution. If the coefficient matrix A of a *square* system (a system that has the same number of equations as variables) is invertible, then the system has a unique solution, which is defined as follows.

A System of Equations with a Unique Solution

If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.

TECH TUTOR

To solve a system of equations with a graphing utility, enter the matrices A and B in the *matrix editor*. Then, using the inverse key, solve for X.

A [x⁻¹] B [ENTER]

The screen will display the solution, matrix X.

Example 5 Solving a System of Equations Using an Inverse Matrix

Use an inverse matrix to solve the system.

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

SOLUTION Begin by writing the system in the matrix form $AX = B$.

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

Next, use Gauss-Jordan elimination to find A^{-1}.

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Finally, multiply B by A^{-1} on the left to obtain the solution.

$$X = A^{-1}B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

So, the solution is $x = 2$, $y = -1$, and $z = -2$.

✓Checkpoint 5

Use an inverse matrix to solve the system.

$$\begin{cases} -x + y + z = 4 \\ 2x - y - 3z = -7 \\ -2x + 3y + 2z = 10 \end{cases}$$



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SUMMARIZE (Section 6.3)

1. State the definition of the inverse of a square matrix *(page 502)*. For an example of showing that a matrix is the inverse of another matrix, see Example 1.
2. Explain how to find an inverse matrix *(page 504)*. For an example of finding an inverse matrix, see Example 3.
3. Give the formula for finding the inverse of a 2×2 matrix *(page 506)*. For an example of using this formula, see Example 4.

SKILLS WARM UP 6.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 6.1 and 6.2.

In Exercises 1–8, perform the indicated matrix operation(s).

1. $4\begin{bmatrix} 1 & 6 \\ 0 & -4 \\ 12 & 2 \end{bmatrix}$

2. $\frac{1}{2}\begin{bmatrix} 11 & 10 & 48 \\ 1 & 0 & 16 \\ 0 & 2 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -10 & 3 \\ 4 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 3 & -4 & 8 \\ 0 & 7 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 5 & 20 \\ -7 & 15 \end{bmatrix} - 3\begin{bmatrix} 6 & 3 \\ 4 & -2 \end{bmatrix}$

5. $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & 5 \\ 3 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & -2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -9 & -8 & 11 \\ 6 & 5 & -7 \end{bmatrix}$

In Exercises 9 and 10, write the matrix in reduced row-echelon form.

9. $\begin{bmatrix} 3 & -2 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{bmatrix}$

Exercises 6.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercis

The Inverse of a Matrix In Exercises 1–10, show that B is the inverse of A. *See Example 1.*

1. $A = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & 1 \\ -9 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 9 & -4 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & -1 \\ 5 & -4 \end{bmatrix}$, $B = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ \frac{5}{3} & -\frac{2}{3} \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -2 \\ 3 & -10 \end{bmatrix}$, $B = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$,

$B = \frac{1}{10}\begin{bmatrix} -6 & 0 & 4 \\ -3 & -5 & 2 \\ 2 & 0 & 2 \end{bmatrix}$

7. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & -1 & 4 \\ 1 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 3 & -1 & 1 & 0 \end{bmatrix}$,

$B = \frac{1}{3}\begin{bmatrix} -1 & 3 & -2 & -2 \\ -2 & 9 & -7 & -10 \\ 1 & 0 & -1 & -1 \\ 3 & -6 & 6 & 6 \end{bmatrix}$

10. $A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 2 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$,

$B = \frac{1}{4}\begin{bmatrix} -4 & 1 & 1 & -4 \\ -4 & -1 & 3 & -4 \\ 0 & 1 & 1 & 0 \\ -4 & -2 & 2 & 0 \end{bmatrix}$

Finding the Inverse of a Matrix In Exercises 11–14, find the inverse of matrix A by solving the matrix equation $AX = I$ for X. *See Example 2.*

11. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 8 & 4 \\ -2 & -2 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Finding the Inverse of a Matrix In Exercises 15–32, find the inverse of the matrix (if it exists). *See Example 3.*

15. $\begin{bmatrix} 11 & 1 \\ -1 & 0 \end{bmatrix}$

16. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 4 \\ -3 & 6 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

19. $\begin{bmatrix} 6 & 15 \\ 4 & 10 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 4 \\ -3 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 7 & 1 \\ -3 & -9 & 2 \end{bmatrix}$

22. $\begin{bmatrix} -2 & 5 \\ 6 & -15 \\ 0 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

26. $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ -4 & 4 & 3 \end{bmatrix}$

27. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

28. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

32. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$



Finding the Inverse of a Matrix In Exercises 33–40, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).

33. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

34. $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

35. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

36. $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

37. $\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & -\frac{3}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

38. $\begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

39. $\begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 4 & -12 & 8 \\ 3 & 0 & 5 & -2 \\ 0 & -3 & 9 & -6 \end{bmatrix}$

40. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

Finding the Inverse of a 2×2 Matrix In Exercises 41–46, use the formula on page 506 to find the inverse of the matrix (if it exists). *See Example 4.*

41. $\begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix}$

42. $\begin{bmatrix} 7 & 12 \\ -8 & -5 \end{bmatrix}$

43. $\begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$

44. $\begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

45. $\begin{bmatrix} \frac{7}{2} & -\frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$

46. $\begin{bmatrix} -\frac{1}{4} & -\frac{2}{3} \\ \frac{1}{3} & \frac{8}{9} \end{bmatrix}$

Solving a System Using an Inverse Matrix In Exercises 47–50, use the inverse matrix found in Exercise 11 to solve the system of linear equations. *See Example 5.*

47. $\begin{cases} x + 2y = 0 \\ 3x + 7y = 1 \end{cases}$

48. $\begin{cases} x + 2y = -5 \\ 3x + 7y = -16 \end{cases}$

49. $\begin{cases} x + 2y = 8 \\ 3x + 7y = 26 \end{cases}$

50. $\begin{cases} x + 2y = -6 \\ 3x + 7y = -21 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 51–54, use the inverse matrix found in Exercise 14 to solve the system of linear equations. *See Example 5.*

51. $\begin{cases} 2x + 3y = 5 \\ x + 4y = 10 \end{cases}$

52. $\begin{cases} 2x + 3y = 0 \\ x + 4y = 3 \end{cases}$

53. $\begin{cases} 2x + 3y = 4 \\ x + 4y = 2 \end{cases}$

54. $\begin{cases} 2x + 3y = 1 \\ x + 4y = -2 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 55 and 56, use the inverse matrix found in Exercise 23 to solve the system of linear equations. *See Example 5.*

55. $\begin{cases} x + y + z = 0 \\ 3x + 5y + 4z = 5 \\ 3x + 6y + 5z = 2 \end{cases}$

56. $\begin{cases} x + y + z = -1 \\ 3x + 5y + 4z = 2 \\ 3x + 6y + 5z = 0 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 57 and 58, use the inverse matrix found in Exercise 40 to solve the system of linear equations. *See Example 5.*

57. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = 2 \end{cases}$

58. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = -2 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = 0 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = -3 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 59–66, use an inverse matrix to solve (if possible) the system of linear equations. *See Example 5.*

59. $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$

60. $\begin{cases} 18x + 12y = 13 \\ 30x + 24y = 23 \end{cases}$

61. $\begin{cases} -0.4x + 0.8y = 1.6 \\ 2x - 4y = 5 \end{cases}$

62. $\begin{cases} 0.2x - 0.6y = 2.4 \\ -x + 1.4y = -8.8 \end{cases}$

63. $\begin{cases} -\frac{1}{4}x + \frac{3}{8}y = -2 \\ \frac{3}{2}x + \frac{3}{4}y = -12 \end{cases}$

64. $\begin{cases} \frac{5}{6}x - y = -20 \\ \frac{4}{3}x - \frac{8}{5}y = -51 \end{cases}$

65. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$

66. $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$



Using a Graphing Utility In Exercises 67 and 68, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.

67. $\begin{cases} 7x - 3y + 2w = 41 \\ 2x + y - w = -13 \\ 4x + z - 2w = 12 \\ x + y - w = -8 \end{cases}$

68. $\begin{cases} 2x + 5y + w = 11 \\ x + 4y + 2z - 2w = -7 \\ 2x - 2y + 5z + w = 3 \\ x - 3w = -1 \end{cases}$

Writing a System of Equations In Exercises 69 and 70, write a system of equations that has the given coefficient matrix and solution. Use an inverse matrix to verify that the system of equations has the given solution.

69. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & -2 \\ 0 & 3 & 2 \end{bmatrix}$ $x = 2$, $y = -3$, $z = 5$

70. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ $x = 5$, $y = -2$, $z = 1$

Bond Investment In Exercises 71–74, you invest in AAA-rated bonds, A-rated bonds, and B-rated bonds. Your average yield is 9% on AAA bonds, 7% on A bonds, and 8% on B bonds. You invest twice as much in B bonds as in A bonds. The desired system of linear equations (where x, y, and z represent the amounts invested in AAA, A, and B bonds, respectively) is as follows.

$$\begin{cases} x + y + z = \text{(total investment)} \\ 0.09x + 0.07y + 0.08z = \text{(annual return)} \\ 2y - z = 0 \end{cases}$$

Use the inverse of the coefficient matrix of this system to find the amount invested in each type of bond for the given total investment and annual return.

71. Total investment = \$35,000; annual return = \$2950

72. Total investment = \$50,000; annual return = \$4180

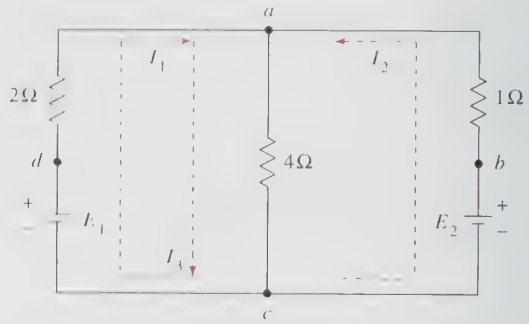
73. Total investment = \$36,000; annual return = \$3040

74. Total investment = \$45,000; annual return = \$3770

Circuit Analysis In Exercises 75 and 76, consider the circuit shown in the figure. The currents I_1, I_2, and I_3, in amperes, are the solution of the system of linear equations

$$\begin{cases} 2I_1 + 4I_3 = E_1 \\ I_2 + 4I_3 = E_2 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$

where E_1 and E_2 are voltages. Use the inverse of the coefficient matrix of this system to find the unknown currents for the given voltages.



75. $E_1 = 28$ volts, $E_2 = 21$ volts

76. $E_1 = 24$ volts, $E_2 = 23$ volts

Raw Materials In Exercises 77–80, find the numbers of bags of potting soil that a company can produce for seedlings, general potting, and hardwood plants with the given amounts of raw materials. The raw materials used in one bag of each type of potting soil are shown below.

	Sand	Loam	Peat moss
Seedlings	2 units	1 unit	1 unit
General	1 unit	2 units	1 unit
Hardwoods	2 units	2 units	2 units

77. 500 units of sand
500 units of loam
400 units of peat moss

78. 500 units of sand
750 units of loam
450 units of peat moss

79. 350 units of sand
445 units of loam
345 units of peat moss

80. 975 units of sand
1050 units of loam
725 units of peat moss

81. Floral Design A florist is creating 10 centerpieces for the tables at a wedding reception. The customer has a budget of \$300 allocated for the centerpieces and wants each centerpiece to contain 12 flowers, with twice as many roses as irises and lilies combined. The florist uses the prices shown to determine the total cost to the customer.

\$2.50 per rose

\$4.00 per lily

\$2.00 per iris

(a) Write a system of linear equations that represents the situation.

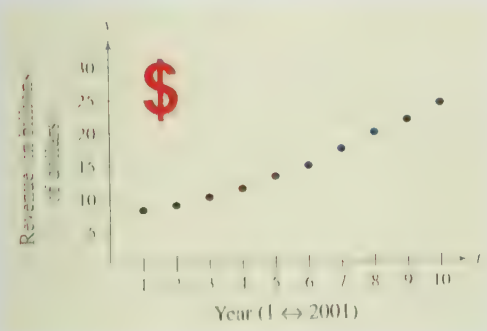
(b) Write a matrix equation that corresponds to your system.

(c) Solve your system of linear equations using an inverse matrix. Find the number of flowers of each type that the florist can use to create the 10 centerpieces.

82. DIRECTV The figure shows the revenue y (in billions of dollars) for DIRECTV each year from 2001 through 2010. The least squares regression parabola $y = at^2 + bt + c$ for these data is found by solving the system

$$\begin{cases} 10c + 55b + 385a = 149.21 \\ 55c + 385b + 3025a = 969.73 \\ 385c + 3025b + 25{,}333a = 7440.01 \end{cases}$$

where t represents the year, with $t = 1$ corresponding to 2001. *(Source: DIRECTV)*



(a) Use a graphing utility to find an inverse matrix to solve this system, and find the equation of the least squares regression parabola.

(b) Use the result from part (a) to predict the revenue for DIRECTV in 2012.

(c) In 2011, DIRECTV projected a revenue of \$28.9 billion for 2012. How does this compare with your prediction in part (b)? Do both amounts seem reasonable?

83. Discovery Let A, B, and C be the 2×2 matrices

$$A = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 13 & 4 \\ 1 & 8 \end{bmatrix}.$$

(a) Find AB and BA. How is each product related to matrix C?

(b) Find C^{-1}, $A^{-1} \cdot B^{-1}$, and $B^{-1} \cdot A^{-1}$. Which of the resulting matrices are equal?

(c) Make a conjecture about the inverse of the product AB of two invertible matrices A and B.

84. HOW DO YOU SEE IT? Let A be the 2×2 matrix

$$A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}.$$

Use the determinant of A to state the conditions for which (a) A^{-1} exists and (b) $A^{-1} = A$.

Think About It In Exercises 85 and 86, find a value of k that makes the matrix invertible. (There are many correct answers.) Then find a value of k that makes the matrix singular.

85. $\begin{bmatrix} 4 & 3 \\ 2 & k \end{bmatrix}$ **86.** $\begin{bmatrix} 2k+1 & 3 \\ -7 & 1 \end{bmatrix}$

True or False? In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.

87. There exists a matrix A such that $A = A^{-1}$.

88. Multiplication of a nonsingular matrix and its inverse is commutative.

89. Verifying a Formula Verify that the inverse of an invertible 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

90. Discovery Consider matrices of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

(a) Write a 2×2 matrix and a 3×3 matrix of the form of A. Find the inverse of each.

(b) Use the result from part (a) to make a conjecture about the inverses of matrices of the form of A.

QUIZ YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, write a matrix of the given dimension.

1. 4×3
2. 3×1

In Exercises 3 and 4, write the augmented matrix for the system of linear equations.

3. $\begin{cases} 3x + 2y = -2 \\ 5x - y = 19 \end{cases}$
4. $\begin{cases} x + 3z = -5 \\ x + 2y - z = 3 \\ 3x + 4z = 0 \end{cases}$
5. Use Gaussian elimination with back-substitution to solve the augmented matrix found in Exercise 3.
6. Use Gauss-Jordan elimination to solve the augmented matrix found in Exercise 4.

In Exercises 7–12, use the following matrices to find the indicated matrix (if possible).

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}$$

7. $2A + 3C$
8. AB
9. $A - 3C$
10. C^2
11. A^{-1}
12. B^{-1}

In Exercises 13 and 14, solve for X using matrices A and C from Exercises 7–12.

13. $X = 3A - 2C$
14. $2X + 4A = 2C$

In Exercises 15–18, a hang glider manufacturer has the labor-hour and wage requirements indicated at the left.

15. What is the labor cost for model A at Plant 1?
16. What is the labor cost for model B at Plant 2?
17. What is the labor cost for model C at Plant 2?
18. Compute LW and interpret the result.

$$L = \begin{matrix} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{matrix} \begin{matrix} \begin{matrix} \text{Assembly} & \text{Finishing} & \text{Packaging} \end{matrix} \\ \begin{bmatrix} 1.0 & 0.6 & 0.2 \\ 2.4 & 1.0 & 0.2 \\ 2.8 & 2.0 & 0.5 \end{bmatrix} \end{matrix}$$

Labor-Hour Requirements
(in hours per hang glider)

$$W = \begin{matrix} \\ \text{Assembly} \\ \text{Finishing} \\ \text{Packaging} \end{matrix} \begin{matrix} \begin{matrix} \text{Plant 1} & \text{Plant 2} \end{matrix} \\ \begin{bmatrix} 15 & 12 \\ 10 & 11 \\ 9 & 8 \end{bmatrix} \end{matrix}$$

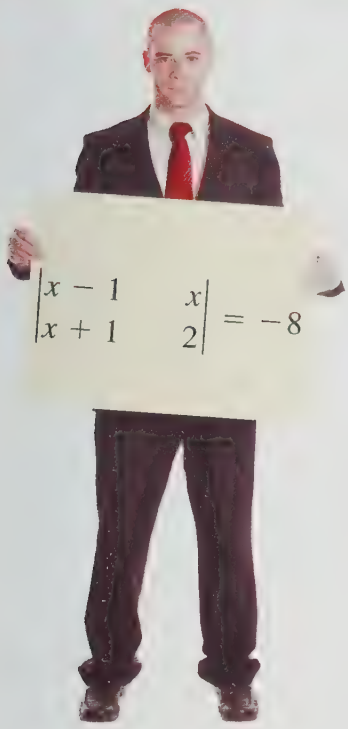
Wage Requirements
(in dollars per hour)

Matrices for 15–18

In Exercises 19 and 20, use an inverse matrix to solve the system of linear equations.

19. $\begin{cases} x - 3y = 10 \\ -2x + y = -10 \end{cases}$
20. $\begin{cases} 2x - y + z = 3 \\ 3x - z = 15 \\ 4y + 3z = -1 \end{cases}$

6.4 The Determinant of a Square Matrix



In Exercises 91–100 on page 520, you will use determinants to solve equations for x.

- Evaluate the determinant of a 2×2 matrix.
- Find the minors and cofactors of a matrix.
- Find the determinant of a square matrix.

The Determinant of a 2×2 Matrix

Every *square* matrix can be associated with a real number called its **determinant.** Determinants have many uses, and several will be discussed in this and the next section. The use of determinants is derived from special number patterns that occur when systems of linear equations are solved. For instance, the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution given by

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Note that the denominator of each fraction is the same. This denominator is called the *determinant* of the coefficient matrix of the system.

Coefficient Matrix	*Determinant*
$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\det(A) = a_1b_2 - a_2b_1$

The determinant of the matrix A can also be denoted by vertical bars on both sides of the matrix, as indicated in the following definition.

Definition of the Determinant of a 2×2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of A. Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended.

A convenient method of remembering the formula for the determinant of a 2×2 matrix is shown in the following diagram.

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Note that the determinant of a 2×2 matrix is the difference of the products of the two diagonals of the matrix.

Benjamin Thorn/www.shutterstock.com

In Example 1 you will see that the determinant of a matrix can be positive, zero, or negative.

Example 1 **The Determinant of a 2 × 2 Matrix**

Find the determinant of each matrix.

a. $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

b. $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

c. $C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$

SOLUTION Use the formula $\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

a. $\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

b. $\det(B) = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

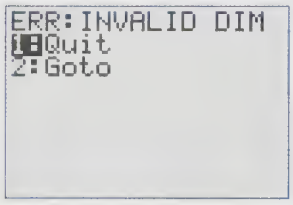
c. $\det(C) = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$

TECH TUTOR

Try evaluating the determinant of A with your graphing utility.

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

You should obtain an error message similar to the one shown below. The graphing utility displays this message because the matrix is not square.



Checkpoint 1

Find the determinant of each matrix.

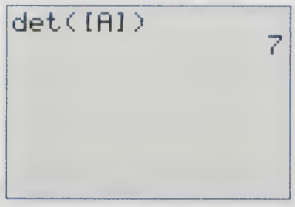
a. $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 5 & 0 \\ -4 & 2 \end{bmatrix}$ **c.** $C = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

The determinant of a matrix of dimension 1×1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, then

$$\det(A) = -2.$$

TECH TUTOR

Most graphing utilities can evaluate the determinant of a matrix. Use a graphing utility to find the determinant of matrix A from Example 1. The result should be 7, as shown below. For specific keystrokes on how to use a graphing utility to evaluate the determinant of a matrix, consult the user's guide for your graphing utility.



Minors and Cofactors

To define the determinant of a square matrix of dimension 3×3 or higher, it is helpful to introduce the concepts of **minors** and **cofactors.**

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

Minors and Cofactors of a Square Matrix

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

In the sign pattern for cofactors shown at the left, notice that *odd* positions (where $i + j$ is odd) have negative signs and *even* positions (where $i + j$ is even) have positive signs.

Example 2 Finding the Minors and Cofactors of a Matrix

Find all the minors and cofactors of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

SOLUTION To find the minor M_{11}, delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find M_{12}, delete the first row and second column.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain the following minors.

$$M_{11} = -1 \qquad M_{12} = -5 \qquad M_{13} = 4$$
$$M_{21} = 2 \qquad M_{22} = -4 \qquad M_{23} = -8$$
$$M_{31} = 5 \qquad M_{32} = -3 \qquad M_{33} = -6$$

Now, to find the cofactors, combine the minors above with the checkerboard pattern of signs for a 3×3 matrix shown at the upper left.

$$C_{11} = -1 \qquad C_{12} = 5 \qquad C_{13} = 4$$
$$C_{21} = -2 \qquad C_{22} = -4 \qquad C_{23} = 8$$
$$C_{31} = 5 \qquad C_{32} = 3 \qquad C_{33} = -6$$

Checkpoint 2

Find all the minors and cofactors of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}.$$

The Determinant of a Square Matrix

The definition below is called **inductive** because it uses determinants of matrices of dimension $(n-1) \times (n-1)$ to define determinants of matrices of dimension $n \times n$.

Determinant of a Square Matrix

If A is a square matrix (of dimension 2×2 or greater), then the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors. For instance, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors.**

Try checking that for a 2×2 matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

this definition of the determinant yields

$$|A| = a_1b_2 - a_2b_1$$

as previously defined.

Example 3 The Determinant of a Matrix of Dimension 3×3

Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

SOLUTION Note that this is the same matrix that was given in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \quad C_{12} = 5, \quad \text{and} \quad C_{13} = 4.$$

So, by the definition of a determinant, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First-row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14. \end{aligned}$$

Checkpoint 3

Find the determinant of

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 2 & -1 \\ 2 & 0 & -4 \end{bmatrix}.$$

In Example 3, the determinant was found by expanding by the cofactors in the first row. You could have used any row or column. For instance, you could have expanded along the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second-row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14. \end{aligned}$$

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero. So, the row (or column) containing the most zeros is usually the best choice for expansion by cofactors.

Example 4 The Determinant of a Matrix of Dimension 4 × 4

Find the determinant of $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & 2 \end{bmatrix}$.

SOLUTION After inspecting this matrix, you can see that three of the entries in the third column are zeros. So, you can eliminate some of the work in the expansion by using the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because C_{23}, C_{33}, and C_{43} have zero coefficients, you only need to find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix}$$ Delete 1st row and 3rd column

$$= \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix}$$ Simplify

Expanding by minors in the second row yields

$$C_{13} = 0(-1)^3\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} + 2(-1)^4\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} + 3(-1)^5\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$$

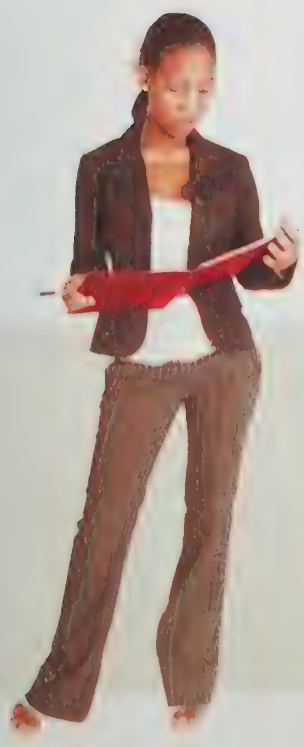
$$= 0 + 2(1)(-8) + 3(-1)(-7)$$

$$= 5.$$

So, you obtain $|A| = 3C_{13} = 3(5) = 15$.

Checkpoint 4

Find the determinant of $A = \begin{bmatrix} 3 & 0 & 7 & 0 \\ 2 & 6 & 0 & 11 \\ 4 & 1 & 0 & 2 \\ 1 & 5 & 0 & 10 \end{bmatrix}$.



SUMMARIZE (Section 6.4)

1. State the definition of the determinant of a 2 × 2 matrix *(page 513)*. For an example of finding the determinant of a 2 × 2 matrix, see Example 1.
2. State the definition of minors and cofactors of a square matrix *(page 515)*. For an example of finding the minors and cofactors of a matrix, see Example 2.
3. State the definition of the determinant of a square matrix using expansion by cofactors *(page 516)*. For examples of finding determinants using this definition, see Examples 3 and 4.

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SKILLS WARM UP 6.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.3, and 6.2.

In Exercises 1–4, perform the indicated matrix operations.

1. $\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 4 & -3 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 5 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$

3. $3\begin{bmatrix} 3 & -4 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$

4. $4\begin{bmatrix} 0 & 2 & 3 \\ -1 & 2 & 3 \\ -2 & 1 & -2 \end{bmatrix}$

In Exercises 5–10, perform the indicated arithmetic operations.

5. $[(1)(3) + (-3)(2)] - [(1)(4) + (3)(5)]$

6. $[(4)(4) + (-1)(-3)] - [(-1)(2) + (-2)(7)]$

7. $\dfrac{4(7) - 1(-2)}{(-5)(-2) - 3(4)}$

8. $\dfrac{3(6) - 2(7)}{6(-5) - 2(1)}$

9. $-5(-1)^2[6(-2) - 7(-3)]$

10. $4(-1)^3[3(6) - 2(7)]$

Exercises 6.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding the Determinant of a Matrix In Exercises 1–16, find the determinant of the matrix. *See Example 1.*

1. $[-5]$

2. $[6]$

3. $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$

6. $\begin{bmatrix} -7 & -4 \\ 8 & 7 \end{bmatrix}$

7. $\begin{bmatrix} 9 & 3 \\ 12 & 4 \end{bmatrix}$

8. $\begin{bmatrix} -5 & -2 \\ 10 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

10. $\begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$

11. $\begin{bmatrix} \frac{2}{3} & 0 \\ -1 & 6 \end{bmatrix}$

12. $\begin{bmatrix} 9 & -\frac{1}{4} \\ 8 & 0 \end{bmatrix}$

13. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ -6 & \frac{1}{3} \end{bmatrix}$

14. $\begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ -1 & -\frac{1}{3} \end{bmatrix}$

15. $\begin{bmatrix} 2.5 & 0.4 \\ -3.5 & -1.6 \end{bmatrix}$

16. $\begin{bmatrix} -0.1 & 7 \\ -0.8 & -5 \end{bmatrix}$

Using a Graphing Utility In Exercises 17–22, use the matrix capabilities of a graphing utility to find the determinant of the matrix.

17. $\begin{bmatrix} 5 & 3 & 2 \\ 7 & 5 & -7 \\ 0 & 6 & -1 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -2 \end{bmatrix}$

19. $\begin{bmatrix} 0.1 & 0.3 & 0.2 \\ -0.3 & -0.2 & 0.1 \\ 1 & 2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 0.2 & -0.1 & -0.3 \\ 0.1 & -0.1 & 0.4 \\ -0.5 & -0.2 & -0.1 \end{bmatrix}$

21. $\begin{bmatrix} 0.9 & 0.7 & 0 \\ -0.1 & 0.3 & 1.3 \\ 2.2 & 4.2 & 6.1 \end{bmatrix}$

22. $\begin{bmatrix} 0.1 & 0.1 & -4.3 \\ 7.5 & 6.2 & 0.7 \\ 0.3 & 0.6 & -1.2 \end{bmatrix}$

Finding the Minors and Cofactors of a Matrix In Exercises 23–30, find all the (a) minors and (b) cofactors of the matrix. *See Example 2.*

23. $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$

24. $\begin{bmatrix} 11 & 0 \\ -3 & 2 \end{bmatrix}$

25. $\begin{bmatrix} 3 & 1 \\ -2 & -4 \end{bmatrix}$

26. $\begin{bmatrix} -6 & 5 \\ 7 & -2 \end{bmatrix}$

27. $\begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

28. $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 5 \\ 4 & -6 & 4 \end{bmatrix}$

29. $\begin{bmatrix} 3 & -2 & 8 \\ 3 & 2 & -6 \\ -1 & 3 & 6 \end{bmatrix}$

30. $\begin{bmatrix} -2 & 9 & 4 \\ 7 & -6 & 0 \\ 6 & 7 & -6 \end{bmatrix}$

Finding the Determinant of a Matrix In Exercises 31–36, find the determinant of the matrix by the method of expansion by cofactors. Expand along the indicated row or column. *See Examples 3 and 4.*

31. $\begin{bmatrix} 4 & 1 & -3 \\ 6 & 5 & -2 \\ -1 & 3 & -4 \end{bmatrix}$

(a) Row 3

(b) Column 2

32. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

(a) Row 2

(b) Column 3

33. $\begin{bmatrix} 7 & 0 & -4 \\ 2 & -3 & 0 \\ 5 & 8 & 1 \end{bmatrix}$

(a) Row 1

(b) Column 3

34. $\begin{bmatrix} 10 & -5 & 5 \\ 30 & 0 & 10 \\ 0 & 10 & 1 \end{bmatrix}$

(a) Row 3

(b) Column 1

35. $\begin{bmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{bmatrix}$

(a) Row 2

(b) Column 2

36. $\begin{bmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{bmatrix}$

(a) Row 3

(b) Column 1

Finding the Determinant of a Matrix In Exercises 37–52, find the determinant of the matrix. Expand by cofactors along the row or column that appears to make the computations easiest. Use a graphing utility to confirm your result. *See Examples 3 and 4.*

37. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

38. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

39. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

40. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

41. $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

42. $\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 6 & -6 \\ -2 & 1 & 4 \end{bmatrix}$

44. $\begin{bmatrix} -1 & 3 & 1 \\ 4 & 2 & 5 \\ -2 & 1 & 6 \end{bmatrix}$

45. $\begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ -0.4 & 0.4 & 0.3 \end{bmatrix}$

46. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

47. $\begin{bmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{bmatrix}$

48. $\begin{bmatrix} 5 & 0 & 3 \\ -4 & 0 & 8 \\ 3 & 0 & -6 \end{bmatrix}$

49. $\begin{bmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{bmatrix}$

50. $\begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix}$

51. $\begin{bmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{bmatrix}$

52. $\begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$



Using a Graphing Utility In Exercises 53–56, use the matrix capabilities of a graphing utility to evaluate the determinant.

53. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix}$

54. $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix}$

55. $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix}$

56. $\begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$

The Determinant of a Matrix Product In Exercises 57–64, find (a) $|A|$, (b) $|B|$, (c) AB, and (d) $|AB|$. What do you notice about $|AB|$?

57. $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

58. $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

59. $A = \begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

60. $A = \begin{bmatrix} 5 & 4 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 6 \\ 1 & -2 \end{bmatrix}$

61. $A = \begin{bmatrix} 0 & 1 & 2 \\ -3 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 0 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

62. $A = \begin{bmatrix} 3 & 2 & 0 \\ -1 & -3 & 4 \\ -2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

63. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

64. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

Diagonal Matrices A **diagonal matrix** is a square matrix with all zero entries above and below its main diagonal. In Exercises 65–68, find the determinant of the diagonal matrix. What do you notice about your result?

65. $\begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix}$

66. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

67. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

68. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Triangular Matrices A **triangular matrix** is a square matrix with all zero entries either above or below its main diagonal. Such a matrix is **upper triangular** when it has all zeros below the main diagonal and **lower triangular** when it has all zeros above the main diagonal. A diagonal matrix is both upper and lower triangular. To find the determinant of a triangular matrix of any dimension, simply find the product of the entries on the main diagonal. In Exercises 69–74, state whether the matrix is *upper triangular*, *lower triangular*, or *diagonal*, and then find the determinant.

69. $\begin{vmatrix} 2 & 0 & 0 \\ 4 & -3 & 0 \\ 6 & 5 & 1 \end{vmatrix}$ **70.** $\begin{vmatrix} 1 & 1 & 5 \\ 0 & -1 & -4 \\ 0 & 0 & 5 \end{vmatrix}$

71. $\begin{vmatrix} 2 & 3 & -1 & -1 \\ 0 & -1 & -3 & 5 \\ 0 & 0 & -2 & 7 \\ 0 & 0 & 0 & -4 \end{vmatrix}$ **72.** $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 6 & -5 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 1 & 2 & 7 & -1 \end{vmatrix}$

73. $\begin{vmatrix} 6 & 7 & 2 & 0 & 5 \\ 0 & 1 & 3 & 4 & 3 \\ 0 & 0 & -7 & 0 & 4 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$

74. $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$

Think About It In Exercises 75–80, find a 4×4 *upper* triangular matrix whose determinant is equal to the given value and a 4×4 *lower* triangular matrix whose determinant is equal to the given value. Use a graphing utility to confirm your results.

75. 6 **76.** -8

77. -18 **78.** -40

79. 28 **80.** 36

Choosing a Method In Exercises 81–86, find the determinant of the matrix. Tell which method you used. (*Hint:* Use information from Exercises 65–74.)

81. $\begin{vmatrix} 2 & 1 & 3 \\ 7 & 3 & -2 \\ 4 & 1 & 1 \end{vmatrix}$ **82.** $\begin{vmatrix} 6 & -5 & 2 \\ 0 & 5 & -3 \\ 0 & 0 & 2 \end{vmatrix}$

83. $\begin{vmatrix} 3 & 0 & 0 \\ 4 & -2 & 0 \\ 5 & 4 & 3 \end{vmatrix}$ **84.** $\begin{vmatrix} 3 & 2 & 4 \\ -1 & 5 & -3 \\ 0 & 1 & 0 \end{vmatrix}$

85. $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ **86.** $\begin{vmatrix} -1 & 0 & 4 & -3 \\ 5 & 2 & 1 & 1 \\ 3 & -6 & 3 & 2 \\ -4 & -2 & 1 & 0 \end{vmatrix}$

Verifying an Equation In Exercises 87–90, evaluate the determinant(s) to verify the equation.

87. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$ **88.** $\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c\begin{vmatrix} w & x \\ y & z \end{vmatrix}$

89. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$

90. $\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$

Solving an Equation In Exercises 91–100, solve for x.

91. $\begin{vmatrix} x & 2 \\ 1 & x \end{vmatrix} = 2$ **92.** $\begin{vmatrix} x & 4 \\ -1 & x \end{vmatrix} = 20$

93. $\begin{vmatrix} 2x & -3 \\ -2 & 2x \end{vmatrix} = 3$ **94.** $\begin{vmatrix} x & 2 \\ 4 & 9x \end{vmatrix} = 8$

95. $\begin{vmatrix} x & 1 \\ 2 & x-2 \end{vmatrix} = -1$ **96.** $\begin{vmatrix} x-2 & -1 \\ -3 & x \end{vmatrix} = 0$

97. $\begin{vmatrix} 2x & 1 \\ -1 & x-1 \end{vmatrix} = x$ **98.** $\begin{vmatrix} x-1 & x \\ x+1 & 2 \end{vmatrix} = -8$

99. $\begin{vmatrix} 1 & 2 & x \\ -1 & 3 & 2 \\ 3 & -2 & 1 \end{vmatrix} = 0$ **100.** $\begin{vmatrix} 1 & x & -2 \\ 1 & 3 & 3 \\ 0 & 2 & -2 \end{vmatrix} = 0$

101. Discovery Consider square matrices in which the entries are consecutive integers. An example of such a matrix is

$$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

Use a graphing utility to find four determinants of matrices of this type. Make a conjecture based on your results. Then verify your conjecture.

102. HOW DO YOU SEE IT? Explain why the determinant of the matrix is equal to zero.

(a) $\begin{bmatrix} 3 & 4 & -2 & 7 \\ 1 & 3 & -1 & 2 \\ 0 & 5 & 7 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 & -1 \\ -6 & -4 & 2 \\ 5 & -7 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -4 & 5 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -4 & 5 & 7 \\ 2 & -2 & 3 & 1 \\ 4 & -4 & 5 & 7 \\ 6 & 1 & -3 & -3 \end{bmatrix}$

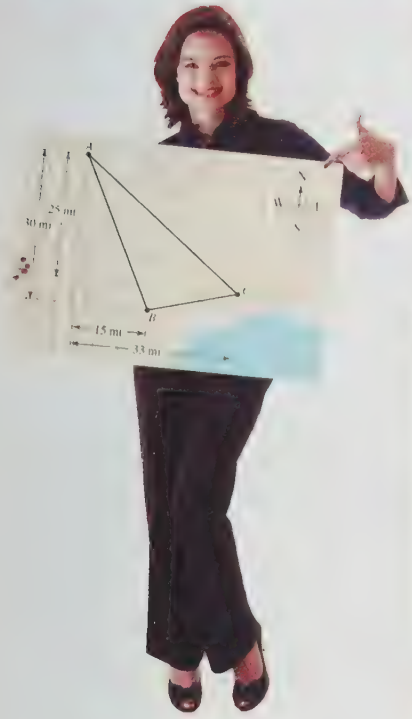
True or False? In Exercises 103 and 104, determine whether the statement is true or false. Justify your answer.

103. If a square matrix has an entire row of zeros, then the determinant will always be zero.

104. If two columns of a square matrix are the same, then the determinant of the matrix will be zero.

6.5 Applications of Matrices and Determinants

- Find the area of a triangle using a determinant.
- Use a determinant to test for collinear points and to find an equation of a line.
- Encode and decode a cryptogram using a matrix.



In Exercise 15 on page 527, you will use a determinant to find the area of a region of forest that has been infested with gypsy moths.

Area of a Triangle

In this section, you will study some additional applications of matrices and determinants. The first application involves a formula for finding the area of a triangle whose vertices are given by three points on a rectangular coordinate system.

Area of a Triangle

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol $(\pm)$ indicates that the appropriate sign should be chosen to yield a positive area.

Example 1 Finding the Area of a Triangle

Find the area of the triangle whose vertices are $(1, 0)$, $(2, 2)$, and $(4, 3)$, as shown in Figure 6.1.

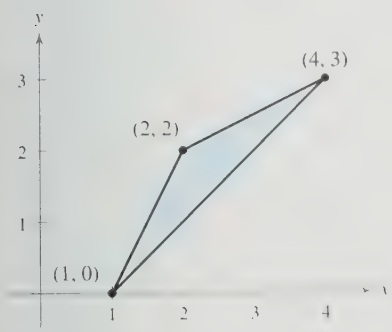


FIGURE 6.1

SOLUTION Let $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 2)$, and $(x_3, y_3) = (4, 3)$. Then, to find the area of the triangle, evaluate the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= 1(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 0(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 1(-1) + 0 + 1(-2)$$

$$= -3.$$

Using this value, you can conclude that the area of the triangle is

$$\text{Area} = -\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$ Choose $(-)$ so the area is positive.

$$= -\frac{1}{2}(-3)$$

$$= \frac{3}{2} \text{ square units.}$$

Checkpoint 1

Find the area of the triangle whose vertices are $(2, 1)$, $(3, 5)$, and $(10, 5)$.

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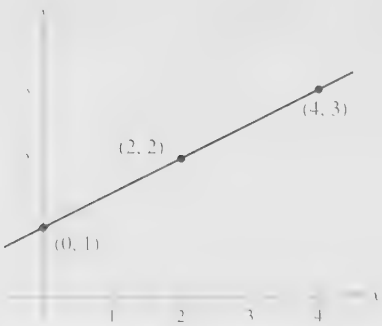


FIGURE 6.2

Lines in the Plane

Suppose the three points in Example 1 had been on the same line. What would have happened had the area formula been applied to three such points? The answer is that the determinant would have been zero. Consider, for instance, the three collinear points (0, 1), (2, 2), and (4, 3), as shown in Figure 6.2. The area of the "triangle" that has these three points as vertices is

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left[0(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 1(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}\right]$$

$$= \frac{1}{2}[0(-1) - 1(-2) + 1(-2)]$$

$$= 0.$$

This result is generalized as follows.

Test for Collinear Points

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Example 2 Testing for Collinear Points

Determine whether the points $(-2, -2)$, $(1, 1)$, and $(7, 5)$ are collinear. (See Figure 6.3.)

SOLUTION Letting $(x_1, y_1) = (-2, -2)$, $(x_2, y_2) = (1, 1)$, and $(x_3, y_3) = (7, 5)$, you have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 1 \\ 1 & 1 & 1 \\ 7 & 5 & 1 \end{vmatrix}$$

$$= -2(-1)^2\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + (-2)(-1)^3\begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix}$$

$$= -2(-4) + 2(-6) + 1(-2)$$

$$= -6.$$

Because the value of this determinant is *not* zero, you can conclude that the three points *do not* lie on the same line.

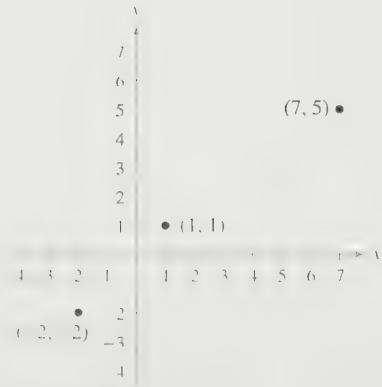


FIGURE 6.3

Checkpoint 2

Determine whether the points

$$(-2, 4), \quad (3, 0), \quad \text{and} \quad (6, -4)$$

are collinear.

Another way to test for collinear points in Example 2 is to find the slope of the line between $(-2, -2)$ and $(1, 1)$ and the slope of the line between $(-2, -2)$ and $(7, 5)$. Try doing this. If the slopes are equal, then the points are collinear. If the slopes are not equal, then the points are not collinear.

The test for collinear points can be adapted for another use. Given two points on a rectangular coordinate system, you can find an equation of the line passing through the two points.

Two-Point Form of the Equation of a Line

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Note that this method of finding the equation of a line works for all lines, including horizontal and vertical lines. (See Exercises 35–38.)

Example 3 **Finding an Equation of a Line**

Find an equation of the line passing through the two points $(2, 4)$ and $(-1, 3)$, as shown in Figure 6.4.

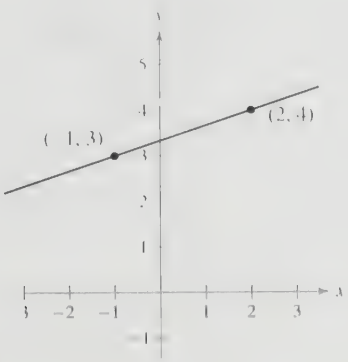


FIGURE 6.4

SOLUTION Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (-1, 3)$. Applying the determinant formula for the equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, you can expand by cofactors along the first row to find an equation of the line.

$$x(-1)^2\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + y(-1)^3\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 0$$

$$x(1) - y(3) + (1)(10) = 0$$

$$x - 3y + 10 = 0$$

Checkpoint 3

Find an equation of the line passing through the two points

$(-3, -1)$ and $(3, 5)$.

Cryptography



During World War II, Navajo soldiers created a code using their native language to send messages between batallions. Native words were assigned to represent characters in the English alphabet, and they created a number of expressions for important military terms, like iron-fish to mean submarine. Without the Navajo Code Talkers, the Second World War might have had a very different outcome.

A **cryptogram** is a message written according to a secret code. (The Greek word *kryptos* means "hidden.") Matrix multiplication can be used to **encode** and **decode** messages. To begin, you need to assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

0 = __	9 = I	18 = R
1 = A	10 = J	19 = S
2 = B	11 = K	20 = T
3 = C	12 = L	21 = U
4 = D	13 = M	22 = V
5 = E	14 = N	23 = W
6 = F	15 = O	24 = X
7 = G	16 = P	25 = Y
8 = H	17 = Q	26 = Z

Then the message is converted to numbers and partitioned into **uncoded row matrices,** each having n entries, as demonstrated in Example 4.

Example 4 Forming Uncoded Row Matrices

Write the uncoded row matrices of dimension 1×3 for the message

MEET ME MONDAY.

SOLUTION Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$$\begin{matrix} [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ M & E & E & T & & M & E & & M & O & N & D & A & Y & \end{matrix}$$

Note that a blank space is used to fill out the last uncoded row matrix.

Checkpoint 4

Write the uncoded row matrices of dimension 1×3 for the message

OWLS ARE NOCTURNAL.

To encode a message, choose an $n \times n$ invertible matrix A such as

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

and multiply the uncoded row matrices by A to obtain **coded row matrices.** The uncoded matrix should be on the left, whereas the encoding matrix A should be on the right. Here is an example.

Uncoded Matrix	*Encoding Matrix A*		*Coded Matrix*
$[13 \quad 5 \quad 5]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[13 \quad -26 \quad 21]$

This technique is further illustrated in Example 5.

Bettman/Corbis

Example 5 **Encoding a Message**

Use the following matrix to encode the message MEET ME MONDAY.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

SOLUTION The coded row matrices are obtained by multiplying each of the uncoded row matrices found in Example 4 by the matrix A, as shown.

Uncoded Matrix	*Encoding Matrix A*		*Coded Matrix*
$[13 \quad 5 \quad 5]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[13 \quad -26 \quad 21]$
$[20 \quad 0 \quad 13]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[33 \quad -53 \quad -12]$
$[5 \quad 0 \quad 13]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[18 \quad -23 \quad -42]$
$[15 \quad 14 \quad 4]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[5 \quad -20 \quad 56]$
$[1 \quad 25 \quad 0]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[-24 \quad 23 \quad 77]$

So, the sequence of coded row matrices is

$$[13 \ -26 \quad 21][33 \ -53 \ -12][18 \ -23 \ -42][5 \ -20 \quad 56][-24 \quad 23 \quad 77].$$

Finally, removing the matrix notation produces the following cryptogram.

$$13 \ -26 \ 21 \ 33 \ -53 \ -12 \ 18 \ -23 \ -42 \ 5 \ -20 \ 56 \ -24 \ 23 \ 77$$

Checkpoint 5

Use the following matrix to encode the message OWLS ARE NOCTURNAL.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$$

For those who do not know the encoding matrix A, decoding the cryptogram found in Example 5 is difficult. But for an authorized receiver who knows the encoding matrix A, decoding is simple. The receiver only needs to multiply the coded row matrices by A^{-1} (on the right) to retrieve the uncoded row matrices. Here is an example.

$$\underbrace{[13 \quad -26 \quad 21]}_{\text{Coded}} \underbrace{\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}}_{A^{-1}} = \underbrace{[13 \quad 5 \quad 5]}_{\text{Uncoded}}$$

The receiver could then easily refer to the number code chart on the previous page and translate $[13 \quad 5 \quad 5]$ into the letters M E E.

Example 6 Decoding a Message

Use the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & 4 \end{bmatrix}$ to decode the cryptogram.

13 −26 21 33 −53 −12 18 −23 −42 5 −20 56 −24 23 77

SOLUTION First, find the decoding matrix A^{-1} by using the techniques demonstrated in Section 6.3. Next, partition the message into groups of three to form the coded row matrices. Then multiply each coded row matrix by A^{-1} on the right to obtain the decoded row matrices.

Coded Matrix	*Decoding Matrix* A^{-1}		*Decoded Matrix*
$[13 \quad -26 \quad 21]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[13 \quad 5 \quad 5]$
$[33 \quad -53 \quad -12]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[20 \quad 0 \quad 13]$
$[18 \quad -23 \quad -42]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[5 \quad 0 \quad 13]$
$[5 \quad -20 \quad 56]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[15 \quad 14 \quad 4]$
$[-24 \quad 23 \quad 77]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[1 \quad 25 \quad 0]$

So, the message is as follows.

[13 5 5]	[20 0 13]	[5 0 13]	[15 14 4]	[1 25 0]
M E E	T	M E	M O N D A Y	



Checkpoint 6

Use the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ to decode the cryptogram.

110, −39, 59, 25, −21, −3, 23, −18, −5,
47, −20, −24, 149, −56, −75, 87, −38, −37

SUMMARIZE (Section 6.5)

1. State the formula for finding the area of a triangle using a determinant *(page 521)*. For an example of finding the area of a triangle, see Example 1.
2. State the test for collinear points *(page 522)*. For an example of testing for collinear points, see Example 2.
3. Explain how to use a determinant to find an equation of a line *(page 523)*. For an example of finding an equation of a line, see Example 3.

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SKILLS WARM UP 6.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 6.2, 6.3, and 6.4.

In Exercises 1–6, evaluate the determinant.

1. $\begin{vmatrix} 4 & 3 \\ -3 & -2 \end{vmatrix}$

2. $\begin{vmatrix} 10 & -20 \\ -1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 4 & 0 \\ -3 & -2 \end{vmatrix}$

4. $\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$

5. $\begin{vmatrix} 4 & 0 & -2 \\ 3 & 1 & 2 \\ -8 & 0 & 6 \end{vmatrix}$

6. $\begin{vmatrix} 3 & 2 & 5 \\ 0 & 0 & -4 \\ -6 & 1 & 1 \end{vmatrix}$

In Exercises 7 and 8, find the inverse of the matrix.

7. $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

8. $\begin{bmatrix} 10 & 5 & -2 \\ -4 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

In Exercises 9 and 10, perform the indicated matrix multiplication.

9. $\begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}$

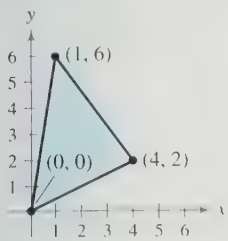
10. $\begin{bmatrix} 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Exercises 6.5

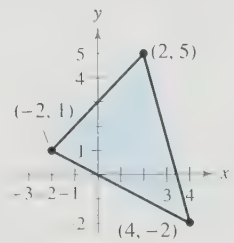
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding an Area In Exercises 1–10, use a determinant to find the area of the triangle with the given vertices. *See Example 1.*

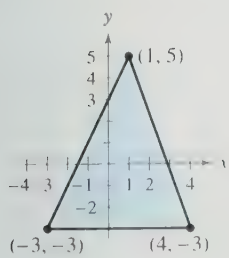
1.



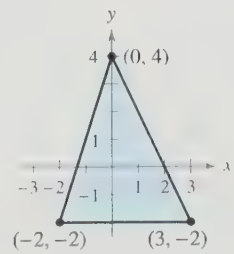
2.



3.



4.



5. $\left(0, \frac{1}{2}\right), (4, 3), \left(\frac{5}{2}, 0\right)$

6. $(-4, -5), (6, 10), (6, -1)$

7. $(-2, 4), (2, 3), (-1, 5)$

8. $(0, -2), (-1, 4), (3, 5)$

9. $(-3, 5), (2, 6), (3, -5)$

10. $(-2, 4), (1, 5), (3, -2)$

Finding a Coordinate In Exercises 11 and 12, find a value of y such that the triangle with the given vertices has an area of 4 square units.

11. $(-5, 1), (0, 2), (-2, y)$

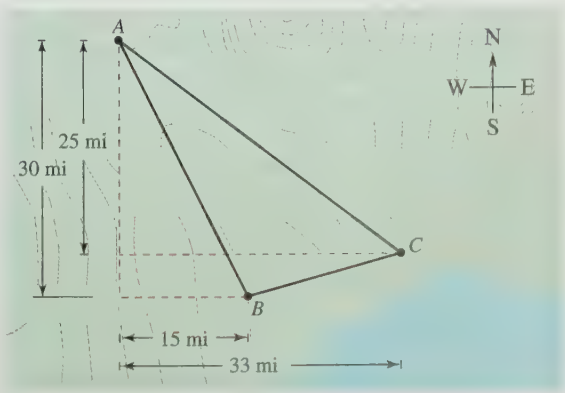
12. $(-4, 2), (-3, 5), (-1, y)$

Finding a Coordinate In Exercises 13 and 14, find a value of y such that the triangle with the given vertices has an area of 6 square units.

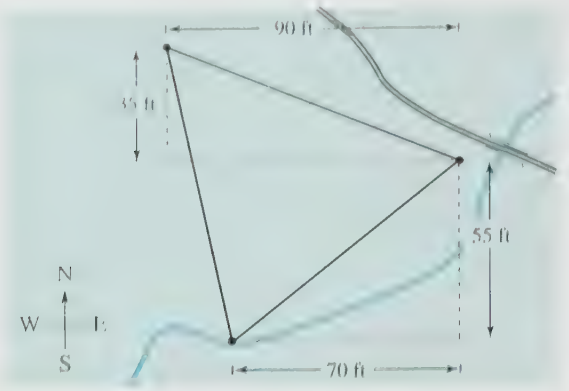
13. $(-2, -3), (1, -1), (-8, y)$

14. $(1, 0), (5, -3), (-3, y)$

15. Gypsy Moths A large region of forest has been infested with gypsy moths. The region is roughly triangular, as shown in the figure. From the northernmost vertex A of the region, the distances to the other vertices are 30 miles south and 15 miles east (for vertex B), and 25 miles south and 33 miles east (for vertex C). Use a graphing utility to approximate the number of square miles in this region.



16. Botany A botanist is studying the plants growing in a triangular tract of land, as shown in the figure. To estimate the number of square feet in the tract, the botanist starts at one vertex, walks 70 feet east and 55 feet north to the second vertex, and then walks 90 feet west and 35 feet north to the third vertex. Use a graphing utility to determine how many square feet there are in the tract.



Testing for Collinear Points In Exercises 17–26, use a determinant to determine whether the points are collinear. *See Example 2.*

17. $(-4, -7), (0, -4), (4, -1)$

18. $(2, 4), (4, 5), (-2, 2)$

19. $(-1, -7), (0, -3), (1, 1)$

20. $(1, 7), (0, 4), (-1, 2)$

21. $(-2, -11), (4, 13), (2, 5)$

22. $(4, 3), (3, 1), (2, -1)$

23. $(-2, 3), (2, -1), (7, -11)$

24. $(-3, -4), (-1, -1), (5, 5)$

25. $\left(\frac{1}{2}, 3\right), \left(-\frac{3}{2}, -5\right), (0, 1)$ **26.** $\left(1, -\frac{5}{2}\right), (0, -3), (6, 0)$

Finding a Coordinate In Exercises 27 and 28, find y such that the points are collinear.

27. $(2, -5), (3, y), (5, -2)$ **28.** $(-6, 2), (-4, y), (-3, 5)$

Finding a Coordinate In Exercises 29 and 30, find x such that the points are collinear.

29. $(-4, -1), (-1, 2), (x, 6)$

30. $(1, 5), (5, -1), (x, 3)$

Finding an Equation of a Line In Exercises 31–40, use a determinant to find an equation of the line passing through the points. *See Example 3.*

31. $(-1, 2), (5, 3)$ **32.** $(3, 1), (-2, -5)$

33. $(-4, 3), (2, 1)$ **34.** $(10, 7), (-2, -7)$

35. $(0, 5), (2, 5)$ **36.** $(3, 3), (6, 3)$

37. $(-4, 5), (-4, -2)$ **38.** $(-1, 3), (-1, 0)$

39. $\left(-\frac{1}{3}, 3\right), \left(\frac{5}{3}, 1\right)$ **40.** $\left(\frac{2}{3}, 4\right), (6, 12)$

In the remaining exercises for this section, use the number code chart on page 524.

Encoding a Message In Exercises 41 and 42, find the uncoded 1×2 row matrices for the message. Then encode the message using the encoding matrix. *See Examples 4 and 5.*

Message	*Encoding Matrix*
41. COME HOME SOON	$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$
42. HELP IS ON THE WAY	$\begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$

Encoding a Message In Exercises 43 and 44, find the uncoded 1×3 row matrices for the message. Then encode the message using the encoding matrix. *See Examples 4 and 5.*

Message	*Encoding Matrix*
43. TEXT ME AT WORK	$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$
44. MISSION ACCOMPLISHED	$\begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

Encoding a Message In Exercises 45–52, write a cryptogram for the message using the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}.$$

45. LANDING SUCCESSFUL

46. DOWNLOAD VIDEO **47.** SEE YOU AT MOVIES

48. GOOD LUCK TOMORROW

49. CONGRATULATIONS

50. HEAD DUE WEST **51.** PICK ME UP AT NOON

52. EVACUATE THE CITY

Decoding a Message In Exercises 53–56, use A^{-1} to decode the cryptogram. *See Example 6.*

53. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

11, 21, 64, 112, 25, 50, 29, 53, 23, 46, 40, 75, 55, 92

54. $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

19, 26, 41, 57, 28, 42, 78, 109, 64, 87, 62, 83, 63, 87, 28, 42, 73, 102, 46, 69

55. $A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

94, 35, 25, 44, 16, 10, 4, −10, 1, 27, 15, 9, 71, 43, 22

56. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

9, −1, −9, 38, −19, −19, 28, −9, −19, −80, 25, 41, −64, 21, 31, −7, −4, 7

Decoding a Message In Exercises 57 and 58, use A^{-1} to decode the cryptogram. *See Example 6.*

$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

57. 20, 17, −15, −9, −44, −83, 64, 136, 157, 24, 31, 12, 4, −37, −102

58. −10, −57, −111, 74, 168, 209, 35, 75, 85, 16, 35, 42, 34, 55, 43

59. Decoding a Message The following cryptogram was encoded with a 2×2 matrix.

8, 21, −15, −10, −13, −13, 5, 10, 5, 25, 5, 19, −1, 6, 20, 40, −18, −18, 1, 16

The last word of the message is __RON. What is the message?

60. Decoding a Message The following cryptogram was encoded with a 2×2 matrix.

5, 2, 25, 11, −2, −7, −15, −15, 32, 14, −8, −13, 38, 19, −19, −19, 37, 16

The last word of the message is __SUE. What is the message?

61. Cryptography A code breaker intercepted the encoded message below.

45, −35, 38, −30, 18, −18, 35, −30, 81, −60, 42, −28, 75, −55, 2, −2, 22, −21, 15, −10

Let $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. You know that $[45 \quad -35]A^{-1} = [10 \quad 15]$ and that $[38 \quad -30]A^{-1} = [8 \quad 14]$, where A^{-1} is the inverse of the encoding matrix A. Explain how you can find the values of w, x, y, and z. Decode the message.

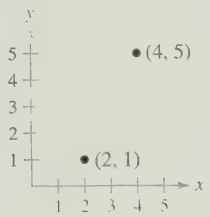
62. Cryptography Your biology professor gives you the encoded message below.

−204, 47, −231, 53, −265, 61, −223, 51, −9, 2, −117, 28, −117, 26, −166, 37, −265, 61, −145, 34, −112, 25, −76, 19

Let $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. You know that $[-204 \quad 47]A^{-1} = [15 \quad 16]$ and that $[-231 \quad 53]A^{-1} = [15 \quad 19]$, where A^{-1} is the inverse of the encoding matrix A. Explain how you can find the values of w, x, y, and z. Decode the message.

63. Reasoning Use a determinant to find the area of the triangle with vertices $(3, -1)$, $(7, -1)$, and $(7, 5)$. Confirm your answer by plotting the points in a coordinate plane and using the formula for the area of a triangle.

64. HOW DO YOU SEE IT? At this point in the text, you have learned several methods for finding an equation of a line that passes through two given points. Briefly describe the methods that can be used to find the equation of the line that passes through the two points shown. Discuss the advantages and disadvantages of each method.





Business Capsule

Voltage Security, Inc. is a leader in secure business communications and data protection. The company provides the most scalable enterprise key management and encryption capabilities for securing data. Invented by Dr. Dan Boneh and Dr. Matt Franklin in 2001, Identity-Based Encryption or IBE is a breakthrough in cryptography. IBE enables users to simply use an identity, such as an e-mail address, to secure business communications.

65. Research Project Use your campus library, the Internet, or some other reference source to find information about a company that generates cryptographic software to secure data. Write a brief paper about such a company.

Kei Uesugi/Taxi/Getty Images

ALGEBRA TUTOR

Basic Rules of Algebra

On this page, you can review some of the basic rules of algebra.

Let a, b, and c be real numbers, variables, or algebraic expressions.

STUDY TIP

Be sure you notice the similarities between the addition and multiplication properties of real numbers and the properties of matrix addition and scalar multiplication.

Property	
Commutative Property of Addition:	$a + b = b + a$
Commutative Property of Multiplication:	$ab = ba$
Associative Property of Addition:	$(a + b) + c = a + (b + c)$
Associative Property of Multiplication:	$(ab)c = a(bc)$
Distributive Property:	$a(b + c) = ab + ac, \quad (a + b)c = ac + bc$
Additive Identity Property:	$a + 0 = a$
Multiplicative Identity Property:	$a \cdot 1 = a$
Additive Inverse Property:	$a + (-a) = 0$
Multiplicative Inverse Property:	$a \cdot \frac{1}{a} = 1, \quad a \neq 0$

Example 1 Identifying the Basic Rules of Algebra

Identify the rule of algebra illustrated by each statement.

a. $(4 + x^3) + 5x^3 = 4 + (x^3 + 5x^3)$

b. $(6x^2)3 = 3(6x^2)$

c. $x^4 + 25 = 25 + x^4$

d. $7y \cdot 1 = 7y$

e. $(x^3 + 1)\left(\frac{1}{x^3 + 1}\right) = 1$

f. $8z^2 + 0 = 8z^2$

g. $(4w^2 + 7w) - (4w^2 + 7w) = 0$

SOLUTION

a. This statement illustrates the Associative Property of Addition. In other words, to form the sum $4 + x^3 + 5x^3$, it does not matter whether 4 and x^3, or x^3 and $5x^3$, are added first.

b. This statement illustrates the Commutative Property of Multiplication. In other words, you get the same result whether you multiply $6x^2$ by 3, or 3 by $6x^2$.

c. This statement illustrates the Commutative Property of Addition. In other words, you get the same result whether you add x^4 and 25, or 25 and x^4.

d. This statement illustrates the Multiplicative Identity Property. In other words, this property simply states that when any expression is multiplied by 1, the result is that expression.

e. This statement illustrates the Multiplicative Inverse Property. In other words, this property states that when any expression is multiplied by its reciprocal, the result is 1.

f. This statement illustrates the Additive Identity Property. In other words, this property simply states that when 0 is added to any expression, the result is that expression.

g. This statement illustrates the Additive Inverse Property. In terms of subtraction, this property simply states that when any expression is subtracted from itself, the result is 0.

Matrix Addition and Scalar Multiplication

Example 2 Addition of Matrices

$$\begin{bmatrix} -4 & 0 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & -1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -4+5+2 & 0+(-1)+2 \\ 3+(-2)+1 & 2+(-3)+(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & -5 \end{bmatrix}$$

Example 3 Scalar Multiplication

Evaluate each expression using two different methods.

a. $(2+5)\begin{bmatrix} 2 & -3 & -1 \\ 4 & 1 & -2 \\ 2 & 5 & 3 \end{bmatrix}$ **b.** $-2\left(4\begin{bmatrix} -3 & -1 \\ 5 & 4 \end{bmatrix}\right)$

SOLUTION

a. Method 1: $(2+5)\begin{bmatrix} 2 & -3 & -1 \\ 4 & 1 & -2 \\ 2 & 5 & 3 \end{bmatrix} = 2\begin{bmatrix} 2 & -3 & -1 \\ 4 & 1 & -2 \\ -2 & 5 & 3 \end{bmatrix} + 5\begin{bmatrix} 2 & -3 & -1 \\ 4 & 1 & -2 \\ -2 & 5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -6 & -2 \\ 8 & 2 & 4 \\ -4 & 10 & 6 \end{bmatrix} + \begin{bmatrix} 10 & -15 & -5 \\ 20 & 5 & -10 \\ -10 & 25 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -21 & -7 \\ 28 & 7 & -14 \\ -14 & 35 & 21 \end{bmatrix}$$

Method 2: $(2+5)\begin{bmatrix} 2 & -3 & -1 \\ 4 & 1 & -2 \\ 2 & 5 & 3 \end{bmatrix} = 7\begin{bmatrix} 2 & -3 & -1 \\ 4 & 1 & -2 \\ -2 & 5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 14 & -21 & -7 \\ 28 & 7 & -14 \\ -14 & 35 & 21 \end{bmatrix}$$

b. Method 1: $-2\left(4\begin{bmatrix} -3 & -1 \\ 5 & 4 \end{bmatrix}\right) = [-2(4)]\begin{bmatrix} -3 & -1 \\ 5 & 4 \end{bmatrix}$

$$= -8\begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 8 \\ -40 & -32 \end{bmatrix}$$

Method 2: $-2\left(4\begin{bmatrix} -3 & -1 \\ 5 & 4 \end{bmatrix}\right) = -2\begin{bmatrix} -12 & -4 \\ 20 & 16 \end{bmatrix}$

$$= \begin{bmatrix} 24 & 8 \\ -40 & -32 \end{bmatrix}$$

Which method do you prefer for each expression in Example 3? Explain your reasoning.

SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 534. Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 6.1

Review Exercises

- Determine the dimension of a matrix. *1–4*

$$\underbrace{\left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array}\right]}_{n \text{ columns}} \Bigg\} m \text{ rows}$$

A matrix having m rows and n columns is of dimension $m \times n$.

- Perform elementary row operations on a matrix in order to write the matrix in row-echelon form or reduced row-echelon form. *5–8*
- Solve a system of linear equations using Gaussian elimination or Gauss-Jordan elimination. *9–18*

Section 6.2

- Determine whether two matrices are equal. *19, 20*
- Add or subtract two matrices and multiply a matrix by a scalar. *21–34*

  If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices and c is a scalar, then $A + B = [a_{ij} + b_{ij}]$ and $cA = [ca_{ij}]$.
- Find the product of two matrices. *35–40*

  If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then AB is an $m \times p$ matrix

  $AB = [c_{ij}]$

  where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.
- Use matrix operations to solve an application problem. *41–44*

Section 6.3

- Verify that a matrix is the inverse of a given matrix. *45, 46*
- Find the inverse of a matrix. *47, 48*
- Find the inverse of a 2×2 matrix using a formula. *49, 50*

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Use an inverse matrix to solve a system of linear equations. *51–60*

*A wide range of valuable study aids are available to help you master the material in this chapter. The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *www.cengagebrain.com* offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

Section 6.4

Review Exercises

- Evaluate the determinant of a 2×2 matrix. *61–64*

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

- Find the minors and cofactors of a matrix. *65–68*
- Find the determinant of a square matrix. *69–76*

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Section 6.5

- Find the area of a triangle using a determinant. *77–80*

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Use a determinant to test for collinear points. *81–84*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

- Use a determinant to find an equation of a line. *85–88*

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

- Encode and decode a cryptogram using a matrix. *89–93*

Study Strategies

- **Variety of Approaches** You can use a variety of approaches when finding the determinant of a square matrix.

  1. For a 2×2 matrix, you can use the definition

$$|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$
$$= a_1b_2 - a_2b_1.$$

  2. For any square matrix (of dimension 2×2 or greater), you can use expansion by cofactors. Be sure you choose the row or column that makes the computations the easiest.

  3. You can always use the matrix capabilities of a graphing utility.

- **Using Technology** Performing operations with matrices can be tedious. You can use a graphing utility to accomplish the following.
  - Perform elementary row operations on matrices.
  - Reduce matrices to row-echelon form and reduced row-echelon form.
  - Add and subtract matrices.
  - Multiply matrices.
  - Multiply matrices by scalars.
  - Find inverses of matrices.
  - Solve systems of equations using matrices.
  - Evaluate determinants of matrices.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Dimensions of Matrices In Exercises 1–4, determine the dimension of the matrix.

1. $\begin{bmatrix} -4 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 9 & -1 & -2 \\ 0 & -5 & 3 \\ -4 & 10 & 6 \\ 5 & 5 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 7 & 4 & -2 \\ 1 & 8 & 6 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 5 \\ -1 \\ 2 \\ 4 \end{bmatrix}$

Writing a Matrix in Row-Echelon Form In Exercises 5 and 6, write the matrix in row-echelon form.

5. $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 3 & 10 & 1 & 8 \\ 2 & 3 & 3 & 10 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & -1 & 0 \\ -2 & -3 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{bmatrix}$

Writing a Matrix in Reduced Row-Echelon Form In Exercises 7 and 8, write the matrix in reduced row-echelon form.

7. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 3 & 1 & 5 \\ 1 & 0 & 5 & 2 \\ -1 & 4 & 3 & 6 \\ 0 & -2 & 6 & -8 \end{bmatrix}$

Solving a System of Equations In Exercises 9–16, use matrices to solve the system of equations (if possible). Use Gaussian elimination with back-substitution or Gauss-Jordan elimination.

9. $\begin{cases} 4x - 3y = 18 \\ x + y = 1 \end{cases}$

10. $\begin{cases} 2x + 4y = 16 \\ -x + 3y = 17 \end{cases}$

11. $\begin{cases} 2x + 3y - z = 13 \\ 3x + z = 8 \\ x - 2y + 3z = -4 \end{cases}$

12. $\begin{cases} 3x + 4y + 2z = 5 \\ 2x + 3y = 7 \\ 2y - 3z = 12 \end{cases}$

13. $\begin{cases} x + 2y + 2z = 10 \\ 2x + 3y + 5z = 20 \end{cases}$

14. $\begin{cases} 3x + 10y + 4z = 20 \\ x + 3y - 2z = 8 \end{cases}$

15. $\begin{cases} 2x + y - 3z = 4 \\ x + 2y + 2z = 10 \\ x - 2z = 12 \\ x + y + z = 6 \end{cases}$

16. $\begin{cases} 2x + 4y + 2z = 10 \\ x + 3z = 9 \\ 3x - 2y = 4 \\ x + y + z = 8 \end{cases}$

17. **Biology** A school district borrowed \$200,000 at simple annual interest to upgrade microbiology equipment. Some of the money was borrowed at 8%, some at 10%, and some at 12%. Use a system of equations to determine how much was borrowed at each rate if the total annual interest was \$20,000 and the amount borrowed at 10% was three times the amount borrowed at 8%. Solve the system using matrices.

18. **Amusement Park** An amusement park borrowed \$650,000 at simple annual interest to renovate a roller coaster. Some of the money was borrowed at 8.5%, some at 9.5%, and some at 10%. Use a system of equations to determine how much was borrowed at each rate if the total annual interest was \$58,250 and the amount borrowed at 8.5% was four times the amount borrowed at 10%. Solve the system using matrices.

Equality of Matrices In Exercises 19 and 20, find x and y.

19. $\begin{bmatrix} -7 & -5 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -7 & x+1 \\ y & -1 \end{bmatrix}$

20. $\begin{bmatrix} 3 & x+4 & -2 \\ 7y & 10 & y+5 \\ -3 & 3x & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2x-1 & -2 \\ -21 & 10 & 2 \\ -3 & 15 & 1 \end{bmatrix}$

Operations with Matrices In Exercises 21–26, find (a) $A + B$, (b) $A - B$, (c) $4A$, and (d) $5A - 2B$.

21. $A = \begin{bmatrix} -1 & 5 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 2 \\ -6 & 3 \end{bmatrix}$

22. $A = \begin{bmatrix} 4 & 6 & -2 \end{bmatrix}, B = \begin{bmatrix} 3 & 7 & -9 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & -4 \\ 0 & -1 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 & 5 \\ 3 & 7 \\ -1 & 4 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 5 \\ 2 & -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -4 & 6 \\ 1 & 2 & -3 \end{bmatrix}$

25. $A = \begin{bmatrix} 1 & 3 & -2 & 6 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 1 & 4 & -5 \\ 3 & -6 & 3 & -2 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$

Evaluating an Expression In Exercises 27–30, evaluate the expression.

27. $\begin{bmatrix} 2 & 0 \\ -5 & -4 \end{bmatrix} + \begin{bmatrix} -10 & 3 \\ 3 & -5 \end{bmatrix} + \begin{bmatrix} -4 & -6 \\ 7 & 4 \end{bmatrix}$

28. $5\left(\begin{bmatrix} -6 & 13 \\ 4 & -8 \\ 7 & 10 \end{bmatrix} - \begin{bmatrix} -5 & 9 \\ -1 & -5 \\ 3 & 11 \end{bmatrix}\right)$

29. $\frac{1}{4}\left(\begin{bmatrix} 3 & -1 & -2 & 18 \end{bmatrix} + \begin{bmatrix} 5 & -3 & -10 & 2 \end{bmatrix}\right)$

30. $-2\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} + 6\begin{bmatrix} -1 & 2 \\ -2 & -3 \end{bmatrix}$

Solving a Matrix Equation In Exercises 31–34, solve for X, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}.$$

31. $X = 4A - 3B$

32. $X = 5B + 2A$

33. $2X - 3A = B$

34. $4X - 8B = 4A$

Find the Product of Two Matrices In Exercises 35–38, find AB, if possible.

35. $A = \begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

36. $A = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}$

37. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 2 & 3 & 6 & -1 \\ 2 & 8 & 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

Operations with Matrices In Exercises 39 and 40, find (a) AB, (b) BA, and, if possible, (c) A^2. (*Note:* $A^2 = AA$.)

39. $A = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

40. $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & 1 \\ 5 & 4 & -2 \end{bmatrix}$

41. Factory Production A window corporation has four factories, each of which manufactures three products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 80 & 120 & 20 & 40 \\ 40 & 60 & 80 & 20 \\ 140 & 60 & 100 & 80 \end{bmatrix}.$$

Find the production levels when production is increased by 20%.

42. Factory Production An electronics manufacturer has three factories, each of which manufactures four products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 120 & 140 & 60 \\ 80 & 100 & 40 \\ 40 & 160 & 80 \\ 20 & 120 & 100 \end{bmatrix}.$$

Find the production levels when production is decreased by 10%.

43. Inventory Levels A company sells four different models of car sound systems through three retail outlets. The inventories of the four models at the three outlets are given by matrix S.

$$S = \begin{array}{c} \\ \text{Outlet} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{bmatrix} 3 & 2 & 1 & 4 \\ 1 & 3 & 4 & 3 \\ 5 & 3 & 2 & 2 \end{bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$$

The wholesale and retail prices of the four models are given by matrix T.

$$T = \begin{array}{c} \\ \text{Model} \end{array} \begin{array}{c} \begin{array}{cc} \text{Wholesale} & \text{Retail} \end{array} \\ \begin{bmatrix} \$350 & \$600 \\ \$425 & \$705 \\ \$300 & \$455 \\ \$750 & \$1150 \end{bmatrix} \end{array} \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}$$

(a) What is the total retail price of the inventory at Outlet 3?

(b) What is the total wholesale price of the inventory at Outlet 1?

(c) Compute ST and interpret the result.

44. Labor/Wage Requirements A company that manufactures racing bicycles has the following labor hour and wage requirements.

Labor-Hour Requirements (per bicycle)

$$S = \begin{array}{c} \\ \text{Models} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Cutting} & \text{Assembly} & \text{Packaging} \end{array} \\ \begin{bmatrix} 0.9 \text{ hour} & 0.8 \text{ hour} & 0.2 \text{ hour} \\ 1.5 \text{ hours} & 1.0 \text{ hour} & 0.4 \text{ hour} \\ 3.5 \text{ hours} & 3.0 \text{ hours} & 0.5 \text{ hour} \end{bmatrix} \end{array} \begin{array}{l} \text{Basic} \\ \text{Light} \\ \text{Ultra-light} \end{array}$$

Wage Requirements (per hour)

$$T = \begin{array}{c} \\ \text{Department} \end{array} \begin{array}{c} \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \begin{bmatrix} \$13.00 & \$14.00 \\ \$11.50 & \$9.50 \\ \$10.00 & \$9.00 \end{bmatrix} \end{array} \begin{array}{l} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{array} \text{ Plant}$$

(a) What is the labor cost for a light racing bicycle at Plant A?

(b) What is the labor cost for an ultra-light racing bicycle at Plant B?

(c) Compute ST and interpret the result.

The Inverse of a Matrix In Exercises 45 and 46, show that B is the inverse of A.

45. $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 4 \\ 0 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -14 & 5 & -2 \\ 9 & -3 & 1 \\ -3 & 1 & 0 \end{bmatrix}$

46. $A = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 2 & 2 \end{bmatrix}$

$$B = \frac{1}{9}\begin{bmatrix} -4 & 6 & 1 & 1 \\ 10 & -6 & 2 & -7 \\ -7 & 6 & 4 & 4 \\ 12 & -9 & -3 & -3 \end{bmatrix}$$

Finding the Inverse of a Matrix In Exercises 47 and 48, find the inverse of the matrix.

47. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ **48.** $\begin{bmatrix} 3 & 2 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Finding the Inverse of a 2×2 Matrix In Exercises 49 and 50, use the formula on page 506 to find the inverse of the matrix.

49. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

50. $\begin{bmatrix} -2 & 1 \\ 4 & 3 \end{bmatrix}$

Solving a System Using an Inverse Matrix In Exercises 51 and 52, use the inverse matrix found in Exercise 49 to solve the system of linear equations.

51. $\begin{cases} x + 3y = 15 \\ 2x + 5y = 26 \end{cases}$ **52.** $\begin{cases} x + 3y = 7 \\ 2x + 5y = 11 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 53 and 54, use the inverse matrix found in Exercise 48 to solve the system of linear equations.

53. $\begin{cases} 3x + 2y + 2z = 13 \\ 2y + z = 4 \\ x + z = 5 \end{cases}$ **54.** $\begin{cases} 3x + 2y + 2z = 12 \\ 2y + z = 13 \\ x + z = 3 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 55 and 56, use an inverse matrix to solve the system of linear equations.

55. $\begin{cases} -3x + 10y = 8 \\ 5x - 17y = -13 \end{cases}$ **56.** $\begin{cases} -x + 4y - 2z = 12 \\ 2x - 9y + 5z = -25 \\ -x + 5y - 4z = 10 \end{cases}$

Raw Materials In Exercises 57 and 58, you are making three types of windshield washer fluid in chemistry class. Fluid X requires 9 cups of water, 1 cup of isopropyl alcohol, and 1 tablespoon of detergent. Fluid Y requires 10 cups of water, 3 cups of isopropyl alcohol, and 1 tablespoon of detergent. Fluid Z requires 14 cups of water, 2 cups of isopropyl alcohol, and 2 tablespoons of detergent. A system of linear equations (where x, y, and z represent fluids X, Y, and Z, respectively) is as follows.

$$\begin{cases} 9x + 10y + 14z = \text{(cups of water)} \\ x + 3y + 2z = \text{(cups of isopropyl alcohol)} \\ x + y + 2z = \text{(tablespoons of detergent)} \end{cases}$$

Use the inverse of the coefficient matrix of this system to find the numbers of units of fluids X, Y, and Z that you can produce with the given amounts of ingredients.

57. 240 cups of water
44 cups of isopropyl alcohol
28 tablespoons of detergent

58. 235 cups of water
41 cups of isopropyl alcohol
29 tablespoons of detergent



59. Operating Stores The numbers of PetSmart stores in operation from 2005 through 2009 increased in a pattern that was approximately parabolic. The least squares regression parabola $y = at^2 + bt + c$ for the data is found by solving the system

$$\begin{cases} 5c + 10b + 30a = 5003 \\ 10c + 30b + 100a = 10{,}856 \\ 30c + 100b + 354a = 33{,}332 \end{cases}.$$

Let t represent the year, with $t = 0$ corresponding to 2005. *(Source: PetSmart, Inc.)*

(a) Use a graphing utility to find an inverse matrix to solve the system, and find the equation of the least squares regression parabola.

(b) Use the result of part (a) to predict the number of stores in operation in 2010.

(c) The actual number of stores in operation in 2010 was about 1170. How does this compare with your prediction in part (b)?



60. Carnivorous Plants A Venus flytrap is grown in a greenhouse, and the size y (in millimeters) of its traps is measured at the end of each year for 5 years. The least squares regression parabola $y = at^2 + bt + c$ for the data is found by solving the system

$$\begin{cases} 5c + 15b + 55a = 53.3 \\ 15c + 55b + 225a = 190.6 \\ 55c + 225b + 979a = 755.8 \end{cases}.$$

Let t represent the year, with $t = 1$ corresponding to the first year.

(a) Use a graphing utility to find an inverse matrix to solve the system, and find the equation of the least squares regression parabola.

(b) Use the result of part (a) to estimate the sizes of the traps after the first and third years.

(c) The actual sizes of the traps were 2.5 millimeters after the first year and 12.8 millimeters after the third year. How do these values compare with your estimates in part (b)?

Finding the Determinant of a Matrix In Exercises 61–64, find the determinant of the matrix.

61. $\begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$ **62.** $\begin{bmatrix} 7 & 2 \\ 9 & -3 \end{bmatrix}$

63. $\begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix}$ **64.** $\begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$

Finding the Minors and Cofactors of a Matrix In Exercises 65–68, find all the (a) minors and (b) cofactors of the matrix.

65. $\begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix}$ **66.** $\begin{bmatrix} 3 & 6 \\ 5 & -4 \end{bmatrix}$

67. $\begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 0 \\ 1 & 8 & 6 \end{bmatrix}$ **68.** $\begin{bmatrix} 8 & 3 & 4 \\ 6 & 5 & -9 \\ -4 & 1 & 2 \end{bmatrix}$

Choosing a Method In Exercises 69–74, find the determinant of the matrix. Tell which method you used. (*Hint:* Use information from Exercises 69–74 on page 520.)

69. $\begin{bmatrix} 1 & 2 & 3 \\ 8 & 6 & 7 \\ 0 & 2 & -1 \end{bmatrix}$ **70.** $\begin{bmatrix} -2 & 3 & 3 \\ -1 & 0 & 5 \\ 1 & 2 & -1 \end{bmatrix}$

71. $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **72.** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 1 & -2 & 0 \\ 6 & 3 & 1 & 1 \end{bmatrix}$

73. $\begin{bmatrix} -2 & 4 & 1 \\ 6 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$ **74.** $\begin{bmatrix} 4 & 7 & -1 \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{bmatrix}$

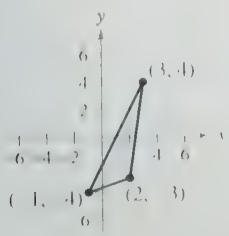
The Determinant of a Matrix Product In Exercises 75 and 76, find (a) $|A|$, (b) $|B|$, (c) AB, and (d) $|AB|$.

75. $A = \begin{bmatrix} -1 & 2 \\ -3 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 1 & -3 \end{bmatrix}$

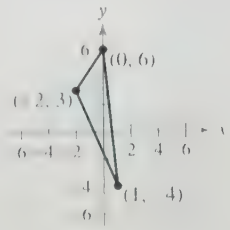
76. $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 3 \\ 2 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -3 & 4 & 0 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

Finding an Area In Exercises 77–80, use a determinant to find the area of the triangle with the given vertices.

77.



78.



79. $(-1, 2), (3, 2), (-2, -3)$

80. $(-6, -1), (1, -3), (-4, -7)$

Testing for Collinear Points In Exercises 81–84, use a determinant to determine whether the points are collinear.

81. $(0, 3), (1, 5), (2, 8)$ **82.** $(2, 6), (-2, 3), (0, 5)$

83. $(-4, 1), (6, 6), (0, 3)$

84. $(-3, -1), (0, 5), (-4, -3)$

Finding an Equation of a Line In Exercises 85–88, use a determinant to find an equation of the line passing through the points.

85. $(-7, 3), (8, 2)$

86. $(5, -4), (-3, 2)$

87. $(2, 4), (2, -7)$

88. $(-5, -1), (1, -1)$

In the remaining exercises for this review, use the number code chart on page 524.

Encoding a Message In Exercises 89 and 90, find the uncoded row matrices for the message. Then encode the message using the encoding matrix.

Message	*Encoding Matrix*
89. TRANSMIT NOW	$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$
90. CALL AT MIDNIGHT	$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

Decoding a Message In Exercises 91 and 92, use A^{-1} to decode the cryptogram.

91. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

14, 53, −17, 96, 5, 10, 12, 64, 5, 10, 3, 11, 25, 50

92. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

−14, −1, 10, −38, 2, 27, −94, 18, 57, 7, −11, −1, −96, 20, 57, −74, 23, 35, 17, −12, −5

93. Cryptography A family sends the encoded message below to a relative overseas.

−57, −13, 91, 26, 97, 29, −76, −19, 5, 5, −84, −21, 55, 16, −28, −7, 97, 28, −8, −2

Let $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

(a) You know that $[-57 \quad -13]A^{-1} = [23 \quad 5]$ and that $[91 \quad 26]A^{-1} = [0 \quad 13]$, where A^{-1} is the inverse of the encoding matrix A. Explain how you can find the values of w, x, y, and z.

(b) Decode the message.

TEST YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, write the augmented matrix for the system of linear equations. What is the dimension of the augmented matrix?

1. $\begin{cases} 2x + y + 4z = 2 \\ x + 4y - z = 0 \\ -x + 3y + 3z = -1 \end{cases}$

2. $\begin{cases} 3x + 4y + 2z = 4 \\ 2x + 3y = -2 \\ 2y - 3z = -13 \end{cases}$

In Exercises 3–5, use matrices to solve the system of equations.

3. $\begin{cases} x + 2y + 3z = 16 \\ 5x + 4y - z = 22 \end{cases}$

4. $\begin{cases} x - 2y + z = 14 \\ y - 3z = 2 \\ z = -6 \end{cases}$

5. $\begin{cases} 2x - 3y + z = 14 \\ x + 2y = -4 \\ y - z = -4 \end{cases}$

In Exercises 6–13, use the matrices to find the indicated matrix.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -2 \\ 3 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad E = \begin{bmatrix} 7 & -8 \\ 6 & 10 \end{bmatrix}$$

6. $2A + C$ **7.** CA **8.** BD **9.** C^2

10. $A + C + E$ **11.** $-4(A + E)$ **12.** $5(C - A)$ **13.** $3C + 2(A - E)$

In Exercises 14–16, find the inverse of the matrix.

14. $A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ **15.** $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **16.** $A = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$

In Exercises 17–19, find the determinant of the matrix.

17. $\begin{bmatrix} -5 & 2 \\ 1 & 3 \end{bmatrix}$ **18.** $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 2 \\ 4 & 5 & 2 \end{bmatrix}$ **19.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

20. Use the inverse matrix found in Exercise 16 to solve the system of linear equations in Exercise 2.

21. Find two nonzero matrices whose product is a zero matrix.

22. Use a determinant to find the area of the triangle whose vertices are $(-3, 1)$, $(0, 4)$, and $(5, 2)$.

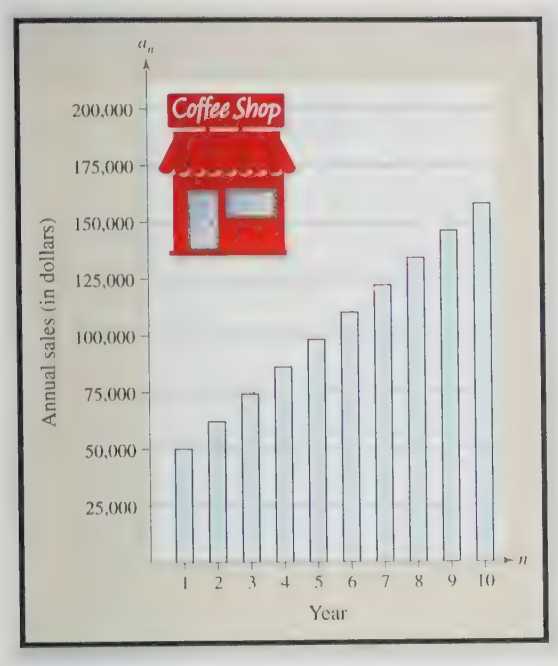
23. Use a determinant to decide whether the points $(2, 1)$, $(-3, -14)$, and $(4, 7)$ are collinear.

24. Use a determinant to find an equation of the line passing through the points $(1, -2)$ and $(5, 2)$.

25. A manufacturer produces three models of a product, which are shipped to two warehouses. The number of units i that are shipped to warehouse j is represented by a_{ij} in matrix A below. The prices per unit for the three models are represented by matrix B. Find the product BA and interpret the result.

$$A = \begin{bmatrix} 1500 & 4000 \\ 3000 & 4500 \\ 5500 & 7000 \end{bmatrix}$$

$$B = [\$55 \quad \$40 \quad \$33]$$



Example 8 on page 554 shows how a sequence can be used to find the projected sales of a coffee shop over a period of time.

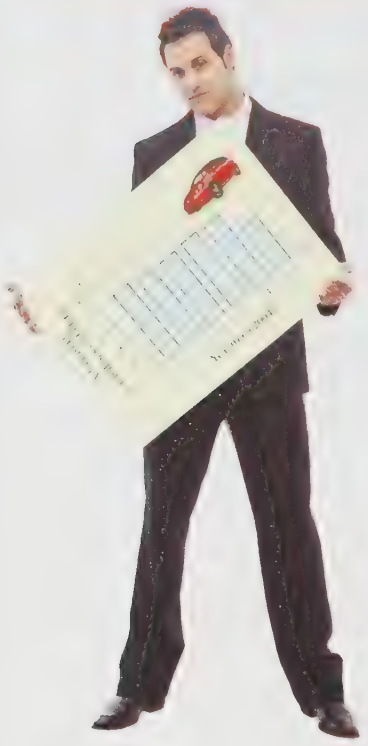
7 Sequences, Series, and Probability

7.1 Sequences and Summation Notation
7.2 Arithmetic Sequences and Partial Sums
7.3 Geometric Sequences and Series
7.4 The Binomial Theorem
7.5 Counting Principles
7.6 Probability



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7.1 Sequences and Summation Notation



In Exercise 92 on page 549, you will evaluate a sequence and a sum to find advertising expenditures of automobile dealerships in the United States.

- Use sequence notation to write the terms of a sequence.
- Evaluate a factorial expression.
- Find the sum of a finite sequence and use summation notation to write the sum of a sequence.
- Use a sequence to solve an application problem.

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers. Rather than using function notation, however, sequences are usually written using subscript notation, as indicated in the definition below.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. If the domain of the function consists of the first n positive integers only, then the sequence is called a **finite sequence.**

On occasion it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become

$$a_0, a_1, a_2, a_3, \ldots$$

In this case, the domain is the set of nonnegative integers.

Example 1 Writing Terms of a Sequence

a. The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

b. The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 3 + 1 = 4$ 2nd term

$a_3 = 3 + (-1)^3 = 3 - 1 = 2$ 3rd term

$a_4 = 3 + (-1)^4 = 3 + 1 = 4.$ 4th term

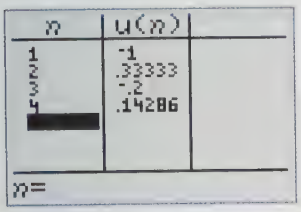
Checkpoint 1

Write the first four terms of the sequence given by $a_n = 2n + 1$.

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TECH TUTOR

You can use the *table* feature of a graphing utility in *sequence* mode to find the terms of the sequence in Example 2 (see the screen below). For instructions on how to use the *table* feature, see Appendix A.



Example 2 **Writing Terms of a Sequence**

The first four terms of the sequence given by $a_n = \frac{(-1)^n}{2n-1}$ are

$$a_1 = \frac{(-1)^1}{2(1)-1} = \frac{-1}{2-1} = -1 \quad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2)-1} = \frac{1}{4-1} = \frac{1}{3} \quad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3)-1} = \frac{-1}{6-1} = -\frac{1}{5} \quad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4)-1} = \frac{1}{8-1} = \frac{1}{7}. \quad \text{4th term}$$

Checkpoint 2

Write the first four terms of the sequence given by $a_n = \frac{(-1)^n}{n+1}$.

It is important to realize that simply listing the first few terms is not sufficient to define a unique sequence—the nth term *must be given.* To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

Example 3 **Writing the nth Term of a Sequence**

Write an expression for the apparent nth term a_n of each sequence.

a. $1, 3, 5, 7, \ldots$

b. $2, 5, 10, 17, \ldots$

SOLUTION

a. n: 1 2 3 4 . . . n

Terms: 1 3 5 7 . . . a_n

Apparent Pattern: Each term is 1 less than twice n. So, the apparent nth term is

$$a_n = 2n - 1.$$

b. n: 1 2 3 4 . . . n

Terms: 2 5 10 17 . . . a_n

Apparent Pattern: Each term is 1 more than the square of n. So, the apparent nth term is

$$a_n = n^2 + 1.$$

Checkpoint 3

Write an expression for the apparent nth term a_n of the sequence $0, 3, 8, 15, 24, \ldots$. (Assume n begins with 1.)

Factorial Notation

Some important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Definition of Factorial

If n is a positive integer, then n **factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots (n-1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

Here are some values of $n!$ for the first several nonnegative integers. Notice that 0! is 1 by definition.

$0! = 1$

$1! = 1$

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

The value of n does not have to be large before the value of $n!$ becomes very large. For instance,

$10! = 3{,}628{,}800.$

TECH TUTOR

Most scientific and graphing calculators have a *factorial* feature built in. Use the *factorial* feature to verify the factorials at the right. Try finding such values as 8!, 15!, and 20!. How does the calculator's display change for increasingly large values of n? (The display for one calculator is shown below.) Consult the user's guide for your calculator for information on how to use the *factorial* feature.

```
8!
                40320
15!
  1.307674368E12
20!
  2.432902008E18
```

Example 4 Writing Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by

$$a_n = \frac{2}{n!}.$$

Begin with $n = 0$.

SOLUTION

$a_0 = \frac{2}{0!} = \frac{2}{1} = 2$ 0th term

$a_1 = \frac{2}{1!} = \frac{2}{1} = 2$ 1st term

$a_2 = \frac{2}{2!} = \frac{2}{2} = 1$ 2nd term

$a_3 = \frac{2}{3!} = \frac{2}{6} = \frac{1}{3}$ 3rd term

$a_4 = \frac{2}{4!} = \frac{2}{24} = \frac{1}{12}$ 4th term

Checkpoint 4

Write the first five terms of the sequence given by $a_n = \frac{6}{n!}$. Begin with $n = 1$. ■

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Factorials follow the same conventions for order of operations as do exponents. For instance,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots n), \quad \text{whereas} \quad (2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots 2n.$$

Example 5 Writing Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by $a_n = \frac{2^n}{n!}$. Begin with $n = 0$.

SOLUTION

$$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1 \qquad \text{0th term}$$

$$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2 \qquad \text{1st term}$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2 \qquad \text{2nd term}$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3} \qquad \text{3rd term}$$

$$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3} \qquad \text{4th term}$$

ALGEBRA TUTOR

For help in simplifying factorial expressions, see the *Chapter 7 Algebra Tutor* on pages 596 and 597.

Checkpoint 5

Write the first five terms of the sequence given by $a_n = \frac{(n+1)!}{4n}$. Begin with $n = 1$.

When working with fractions involving factorials, you will often find that the fractions can be reduced.

Example 6 Simplifying Factorial Expressions

Simplify each factorial expression.

a. $\frac{8!}{2! \cdot 6!}$ **b.** $\frac{2! \cdot 6!}{3! \cdot 5!}$ **c.** $\frac{n!}{(n-1)!}$

SOLUTION

a. $\frac{8!}{2! \cdot 6!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{7 \cdot 8}{2} = 28$

b. $\frac{2! \cdot 6!}{3! \cdot 5!} = \frac{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{6}{3} = 2$

c. $\frac{n!}{(n-1)!} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots (n-1) \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots (n-1)} = n$

STUDY TIP

In Example 6(a), you can also simplify the computation as

$$\frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot 6!}{2! \cdot 6!} = 28.$$

Checkpoint 6

Simplify the factorial expression given by $\frac{4!(n+1)!}{3!n!}$.

Series and Summation Notation

The sum of the terms of an infinite sequence is called an **infinite series.** The sum of the first n terms of a sequence is called a **finite series** or the ***n*th partial sum.** There is a convenient notation for the sum of the terms of a finite sequence. It is called **summation notation** or **sigma notation** because it involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where i is the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

Summation notation is an instruction to add the terms of a sequence. From the definition above, the lower limit of the summation tells you where to begin the sum and the upper limit of the summation tells you where to end the sum.

Example 7 **Using Summation Notation**

Find the sums (a) $\sum_{i=1}^{5} 3i$, (b) $\sum_{k=3}^{6} (1 + k^2)$, and (c) $\sum_{i=0}^{8} \frac{1}{i!}$.

SOLUTION

a. $\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$

$= 3(1 + 2 + 3 + 4 + 5)$

$= 3(15)$

$= 45$

b. $\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$

$= 10 + 17 + 26 + 37$

$= 90$

c. $\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$

≈ 2.71828

Study Tip

Note that the sum in Example 7(c) is very close to the irrational number

$$e \approx 2.718281828.$$

It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum approaches e.

Checkpoint 7

Find the sum $\sum_{i=1}^{4} (4i + 1)$.

In Example 7, note that the lower limit of summation does not have to be 1. Also note that the index of summation does not have to be the letter i. For instance, in part (b), the letter k is the index of summation.

STUDY TIP

Variations in the upper and lower limits of summation can produce quite different-looking summation notations for *the same sum*, as you can see in Example 8. For instance, consider the following two sums.

$$\sum_{i=1}^{3} 3(2^i) = 3\sum_{i=1}^{3} 2^i$$
$$= 3(2^1 + 2^2 + 2^3)$$

$$\sum_{i=0}^{2} 3(2^{i+1}) = 3\sum_{i=0}^{2} 2^{i+1}$$
$$= 3(2^1 + 2^2 + 2^3)$$

Example 8 **Writing a Sum in Summation Notation**

Write each partial sum in summation notation. Decide whether the notation is unique.

a. $\frac{3}{1+2} + \frac{3}{2+2} + \frac{3}{3+2} + \frac{3}{4+2} + \frac{3}{5+2}$ **b.** $2 - 4 + 8 - 16$

SOLUTION

a. Begin by looking for similarities and differences in the terms. Each term has 3 in the numerator. In the denominator, 2 is added to a number that is increasing by 1 for each term. Let i be the index of summation, let 1 be the lower limit of summation, and let 5 be the upper limit of summation. Then, using summation notation, you can write

$$\frac{3}{1+2} + \frac{3}{2+2} + \frac{3}{3+2} + \frac{3}{4+2} + \frac{3}{5+2} = \sum_{i=1}^{5} \frac{3}{i+2}.$$

The sum can also be written in other ways, such as

$$\sum_{i=3}^{7} \frac{3}{i}, \quad \sum_{i=0}^{4} \frac{3}{i+3}, \quad \text{and} \quad \sum_{i=-4}^{8} \frac{3}{i-1}.$$

The summation notation for the sum is not unique.

b. This series has terms with alternating signs, and each term can be written as a power of 2. Let n be the index of summation, let 1 be the lower limit of summation, and let 4 be the upper limit of summation. Then, using summation notation, you can write

$$2 - 4 + 8 - 16 = (-1)^{1+1}(2)^1 + (-1)^{2+1}(2)^2 + (-1)^{3+1}(2)^3 + (-1)^{4+1}(2)^4$$
$$= \sum_{n=1}^{4} (-1)^{n+1}(2)^n.$$

The sum can also be written in other ways, such as

$$\sum_{n=0}^{3} (-1)^n(2)^{n+1} \quad \text{and} \quad \sum_{n=-2}^{5} (-1)^n(2)^{n-1}.$$

The summation notation for the sum is not unique.



Checkpoint 8

Write the partial sum in summation notation. Decide whether the notation is unique.

$$\frac{1-1}{2} + \frac{2-1}{2} + \frac{3-1}{2} + \frac{4-1}{2}$$

The following properties of sums are helpful when adding the terms of a sequence.

Properties of Sums

1. $\sum_{i=1}^{n} c = cn$, c is a constant.
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is a constant.
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

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Example 9 Finding Balances

A deposit of $1000 is made in an account that earns 3% interest, compounded monthly. The balance in the account after n months is given by

$$A_n = 1000\left(1 + \frac{0.03}{12}\right)^n, \quad n = 1, 2, 3, \ldots$$

a. Write the first four terms of the sequence.

b. Find the balance in the account after four years by computing the 48th term of the sequence.

SOLUTION

a. The first four terms of the sequence are shown below.

$$A_1 = 1000\left(1 + \frac{0.03}{12}\right)^1 = \$1002.50 \quad \text{One-month balance}$$

$$A_2 = 1000\left(1 + \frac{0.03}{12}\right)^2 \approx \$1005.01 \quad \text{Two-month balance}$$

$$A_3 = 1000\left(1 + \frac{0.03}{12}\right)^3 \approx \$1007.52 \quad \text{Three-month balance}$$

$$A_4 = 1000\left(1 + \frac{0.03}{12}\right)^4 \approx \$1010.04 \quad \text{Four-month balance}$$

b. The 48th term of the sequence is

$$A_{48} = 1000\left(1 + \frac{0.03}{12}\right)^{48} \approx \$1127.33. \quad \text{Four-year balance}$$

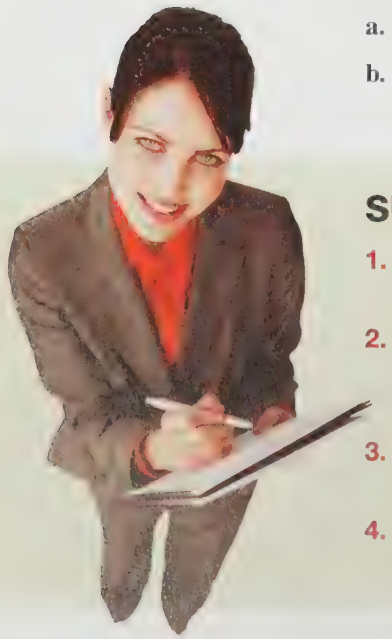
Checkpoint 9

A deposit of $1000 is made in an account that earns 3% interest, compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 1000\left(1 + \frac{0.03}{4}\right)^n, \quad n = 1, 2, 3, \ldots$$

a. Write the first three terms of the sequence.

b. Find the balance in the account after four years by computing the 16th term of the sequence.



SUMMARIZE (Section 7.1)

1. State the definition of sequence *(page 540)*. For examples of writing the terms of a sequence, see Examples 1 and 2.
2. State the definition of factorial *(page 542)*. For examples of writing terms of sequences involving factorials and simplifying factorial expressions, see Examples 4, 5, and 6.
3. State the definition of summation notation *(page 544)*. For examples of using summation notation, see Examples 7 and 8.
4. Describe a real-life example of how a sequence can be used to find the balance in an account *(page 546, Example 9)*.

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SKILLS WARM UP 7.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2 and 2.4.

In Exercises 1 and 2, find the indicated function value.

1. Find $f(2)$ for $f(n) = \dfrac{2n}{n^2 + 1}$.

2. Find $f(3)$ for $f(n) = \dfrac{4}{3(n + 1)}$.

In Exercises 3–6, evaluate the expression when $x = 2$ and $y = -1$.

3. $9(x - 6)$
4. $4(6 - 2x)$
5. $y^3(x^3 - 3)$
6. $xy^5(7 - 3x^2)$

In Exercises 7–10, perform the indicated operations and simplify.

7. $\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\left(\frac{5}{6}\right)$
8. $\dfrac{2 \cdot 4 \cdot 6 \cdot 8}{2^4}$
9. $\dfrac{1}{2 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{2 \cdot 4}$
10. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4}$

Exercises 7.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Writing Terms of a Sequence In Exercises 1–18, write the first five terms of the sequence. (Assume n begins with 1.) *See Examples 1 and 2.*

1. $a_n = 2n - 7$
2. $a_n = 4 - 2n$
3. $a_n = 2^n$
4. $a_n = (-2)^n$
5. $a_n = \left(-\frac{1}{2}\right)^n$
6. $a_n = \left(\frac{1}{2}\right)^n$
7. $a_n = \dfrac{3}{n + 1}$
8. $a_n = \dfrac{n + 1}{2}$
9. $a_n = \dfrac{(-1)^n}{n}$
10. $a_n = \dfrac{(-1)^{n+1}}{n}$
11. $a_n = 2 - \dfrac{1}{3^n}$
12. $a_n = \dfrac{2^n}{3^n}$
13. $a_n = \dfrac{n^2 - 1}{n + 1}$
14. $a_n = \dfrac{n^2 - 1}{n^2 + 2}$
15. $a_n = \dfrac{(-1)^n}{n^2}$
16. $a_n = \dfrac{(-1)^n n}{n + 1}$
17. $a_n = \dfrac{5n^2 - n + 1}{n^2}$
18. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$

Writing the nth Term of a Sequence In Exercises 19–30, write an expression for the apparent nth term a_n of the sequence. (Assume n begins with 1.) *See Example 3.*

19. $1, 4, 7, 10, 13, \ldots$
20. $3, 7, 11, 15, 19, \ldots$
21. $2, -4, 6, -8, 10, \ldots$
22. $-3, 6, -9, 12, -15, \ldots$
23. $2, \frac{1}{2}, \frac{2}{9}, \frac{1}{8}, \frac{2}{25}, \ldots$
24. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$
25. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$
26. $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$
27. $-\frac{1}{3}, \frac{2}{9}, -\frac{4}{27}, \frac{8}{81}, -\frac{16}{243}, \ldots$
28. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, \ldots$
29. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$
30. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

Writing Terms of a Sequence Involving Factorials In Exercises 31–34, write the first five terms of the sequence. *See Examples 4 and 5.*

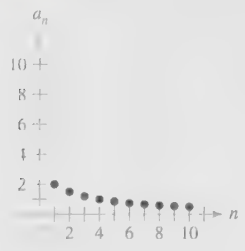
31. $a_n = \dfrac{3^n}{n!}$
32. $a_n = \dfrac{n!}{n}$
33. $a_n = \dfrac{3n!}{(n - 1)!}$
34. $a_n = \dfrac{(n + 1)!}{n!}$

Finding a Term of a Sequence In Exercises 35–40, find the indicated term of the sequence.

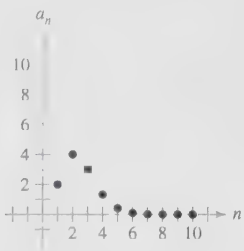
35. $a_n = (-1)^n(3n - 2)$
    $a_{25} =$
36. $a_n = (-1)^{n-1}[n(n - 1)]$
    $a_{16} =$
37. $a_n = \dfrac{4n}{2n^2 - 3}$
    $a_{11} =$
38. $a_n = \dfrac{4n^2 - n + 3}{n(n - 1)(n + 2)}$
    $a_{13} =$
39. $a_n = \dfrac{2^n}{n!}$
    $a_{10} =$
40. $a_n = \dfrac{n!}{2n}$
    $a_8 =$

Matching In Exercises 41–44, match the sequence with the graph of its first 10 terms. [The graphs are labeled (a), (b), (c), and (d).]

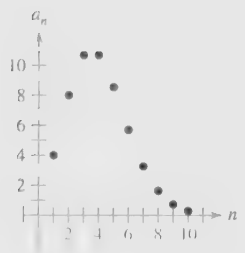
(a)



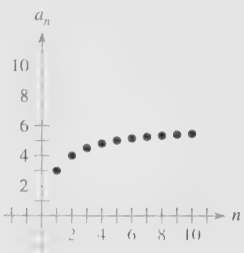
(b)



(c)



(d)



41. $a_n = \dfrac{6}{n+2}$

42. $a_n = \dfrac{6n}{n+1}$

43. $a_n = \dfrac{4^n}{n!}$

44. $a_n = \dfrac{2n}{(n-1)!}$

Simplifying a Factorial Expression In Exercises 45–50, simplify the factorial expression. *See Example 6.*

45. $\dfrac{8!}{7!}$

46. $\dfrac{10!}{9!}$

47. $\dfrac{28!}{25!}$

48. $\dfrac{48!}{50!}$

49. $\dfrac{(n+2)!}{n!}$

50. $\dfrac{(n+3)!}{n!}$

Evaluating a Factorial Expression In Exercises 51–54, simplify the factorial expression. Then evaluate the expression when $n = 0$ and $n = 1$.

51. $\dfrac{(n+1)!}{n!}$

52. $\dfrac{(n+2)!}{n!}$

53. $\dfrac{(2n)!}{(2n+1)!}$

54. $\dfrac{(2n+2)!}{(2n)!}$

Writing the *n*th Term of a Sequence In Exercises 55–58, write an expression for the apparent nth term a_n of the sequence. (Assume n begins with 1.)

55. $1, 2, 6, 24, 120, \ldots$

56. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

57. $1, 2, \dfrac{2^2}{2}, \dfrac{2^3}{6}, \dfrac{2^4}{24}, \dfrac{2^5}{120}, \ldots$

58. $\frac{1}{3}, \frac{2}{9}, \frac{6}{27}, \frac{24}{81}, \frac{120}{243}, \ldots$

Using Summation Notation In Exercises 59–76, find the sum. *See Example 7.*

59. $\displaystyle\sum_{k=1}^{4} 12$

60. $\displaystyle\sum_{k=1}^{5} 5$

61. $\displaystyle\sum_{i=1}^{7} -3i$

62. $\displaystyle\sum_{i=1}^{6} -4i$

63. $\displaystyle\sum_{i=1}^{4} (6i+3)$

64. $\displaystyle\sum_{i=1}^{6} (4i-2)$

65. $\displaystyle\sum_{i=0}^{4} i^2$

66. $\displaystyle\sum_{i=0}^{5} 3i^2$

67. $\displaystyle\sum_{k=2}^{6} \frac{1}{2k}$

68. $\displaystyle\sum_{j=3}^{5} \frac{1}{3j}$

69. $\displaystyle\sum_{i=1}^{4} (i-1)^2$

70. $\displaystyle\sum_{k=2}^{5} (k+1)(k-3)$

71. $\displaystyle\sum_{i=1}^{4} (-1)^i(2i+4)$

72. $\displaystyle\sum_{i=1}^{4} (-2)^i(i+1)$

73. $\displaystyle\sum_{k=0}^{4} \frac{1}{k!}$

74. $\displaystyle\sum_{j=0}^{5} \frac{2}{(j+1)!}$

75. $\displaystyle\sum_{j=0}^{6} \frac{(2j+2)!}{(2j)!}$

76. $\displaystyle\sum_{i=2}^{4} \frac{(3i-2)!}{3i!}$

Writing a Sum in Summation Notation In Exercises 77–86, write the partial sum in summation notation. *See Example 8.*

77. $\dfrac{1}{3(1)} + \dfrac{1}{3(2)} + \dfrac{1}{3(3)} + \cdots + \dfrac{1}{3(9)}$

78. $\dfrac{5}{1+1} + \dfrac{5}{1+2} + \dfrac{5}{1+3} + \cdots + \dfrac{5}{1+15}$

79. $\left[2\left(\frac{1}{8}\right)+3\right] + \left[2\left(\frac{2}{8}\right)+3\right] + \cdots + \left[2\left(\frac{8}{8}\right)+3\right]$

80. $\left[1-\left(\frac{1}{6}\right)^2\right] + \left[1-\left(\frac{2}{6}\right)^2\right] + \cdots + \left[1-\left(\frac{6}{6}\right)^2\right]$

81. $3 - 9 + 27 - 81 + 243 - 729$

82. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{128}$

83. $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots - \dfrac{1}{20^2}$

84. $-\dfrac{1}{1\cdot 3} + \dfrac{1}{2\cdot 4} - \dfrac{1}{3\cdot 5} + \dfrac{1}{4\cdot 6} - \cdots + \dfrac{1}{10\cdot 12}$

85. $\frac{1}{4} + \frac{3}{8} + \frac{7}{16} + \frac{15}{32} + \frac{31}{64}$

86. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64}$

87. **Compound Interest** You deposit \$25,000 in an account that earns 7% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 25{,}000\left(1 + \frac{0.07}{12}\right)^n, \quad n = 1, 2, 3, \ldots$$

(a) Compute the first six terms of the sequence.

(b) Find the balance in the account after 5 years by computing the 60th term of the sequence.

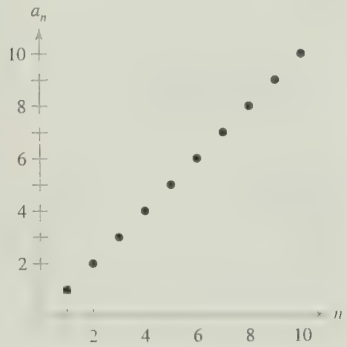
(c) Is the balance after 10 years twice the balance after 5 years? Explain.

88. HOW DO YOU SEE IT? The graph represents the first 10 terms of a sequence. Complete each expression for the apparent nth term a_n of the sequence. Which expressions are appropriate to represent the cost a_n to buy n MP3 songs at a cost of \$1 per song? Explain.

(a) $a_n = 1 \cdot$

(b) $a_n = \dfrac{!}{(n-1)!}$

(c) $a_n = \sum_{k=1}^{n}$



89. Ratio of Males to Females The ratio a_n of males to females in the United States each decade from 1940 through 2010 can be approximated by the model

$$a_n = 1.05 - 0.038n + 0.0036n^2, \quad n = 1, 2, \ldots, 8$$

where n represents the year, with $n = 1, 2, 3, \ldots, 8$ corresponding to 1940, 1950, 1960, . . . , 2010. *(Source: U.S. Census Bureau)*

(a) Use a graphing utility to find the terms of the finite sequence. Interpret the meaning of the terms in the context of the real-life situation.

(b) Construct a bar graph that represents the sequence. Describe any trends you see in your graph.

(c) In 2000, the population of the United States was approximately 281 million. In that year, how many people were females? How many were males?

90. Think About It Evaluate

$$\sum_{j=0}^{6} \frac{(j+1)!}{j!} \quad \text{and} \quad \sum_{k=1}^{7} \frac{k!}{(k-1)!}.$$

What do you observe about the two sums? Is the summation notation for a partial sum unique? Explain.

91. Sales The annual sales a_n (in millions of dollars) for China-Biotics from 2005 through 2009 can be approximated by the model

$$a_n = 22.9 + 3.63n + 2.657n^2, \quad n = 0, 1, 2, 3, 4$$

where n represents the year, with $n = 0$ corresponding to 2005. Find the total sales of China-Biotics from 2005 through 2009 by evaluating the sum

$$\sum_{n=0}^{4} (22.9 + 3.63n + 2.657n^2).$$

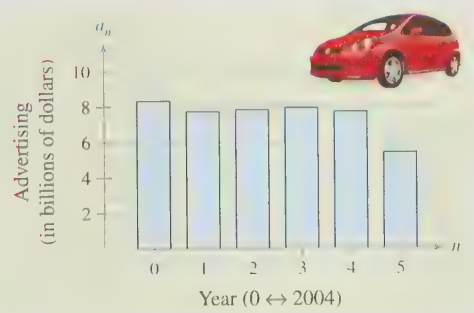
(Source: China-Biotics, Inc.)

92. Advertising The annual advertising expenses a_n (in billions of dollars) of new car dealerships from 2004 through 2009 (see bar graph) can be approximated by the model

$$a_n = -0.1339n^3 + 0.834n^2 - 1.40n + 8.3,$$

$$n = 0, 1, 2, 3, 4, 5$$

where n represents the year, with $n = 0$ corresponding to 2004. *(Source: National Automobile Dealers Association)*



(a) Find the total amount spent on advertising by new car dealerships in 2009 by evaluating the term a_5.

(b) Find the total amount spent on advertising by new car dealerships from 2004 through 2009 by evaluating the sum

$$\sum_{n=0}^{5} (-0.1339n^3 + 0.834n^2 - 1.40n + 8.3).$$

True or False? In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.

93. $\sum_{i=1}^{4} (i^2 + 2i) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4}$

94. $\sum_{i=1}^{4} 2^i = \sum_{i=3}^{6} 2^{i-2}$

Writing Terms of a Sequence Involving x In Exercises 95–98, write the first five terms of the sequence.

95. $a_n = \dfrac{x^n}{n!}$

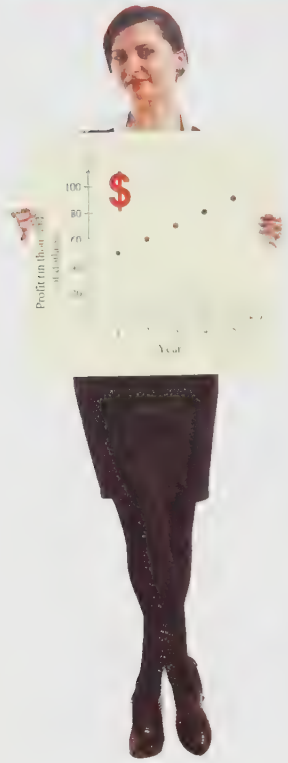
96. $a_n = \dfrac{(-1)^n x^{2n+1}}{2n+1}$

97. $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$

98. $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$

7.2 Arithmetic Sequences and Partial Sums

- Recognize and find the nth term of an arithmetic sequence.
- Find the nth partial sum of an arithmetic sequence.
- Use an arithmetic sequence to solve an application problem.



In Exercise 93 on page 557, you will find the sum of an arithmetic sequence to determine the total profit for a company over a five-year period.

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is called an **arithmetic sequence.**

Definition of Arithmetic Sequence

A sequence is **arithmetic** when the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is arithmetic when there is a number d such that

$$a_2 - a_1 = d, \quad a_3 - a_2 = d, \quad a_4 - a_3 = d, \text{ and so on.}$$

The number d is the **common difference** of the sequence.

Example 1 Examples of Arithmetic Sequences

a. The sequence whose nth term is $4n + 3$ is arithmetic. For this sequence, the common difference between consecutive terms is 4.

$$\underbrace{7, 11}, 15, 19, \ldots, 4n + 3, \ldots \qquad \text{Begin with } n = 1.$$

$11 - 7 = 4$

b. The sequence whose nth term is $100n - 13$ is arithmetic. For this sequence, the common difference between consecutive terms is 100.

$$\underbrace{87, 187}, 287, 387, \ldots, 100n - 13, \ldots \qquad \text{Begin with } n = 1.$$

$187 - 87 = 100$

c. The sequence whose nth term is $7 - 5n$ is arithmetic. For this sequence, the common difference between consecutive terms is -5.

$$\underbrace{2, -3}, -8, -13, \ldots, 7 - 5n, \ldots \qquad \text{Begin with } n = 1.$$

$-3 - 2 = -5$

d. The sequence whose nth term is $\frac{1}{4}(n + 3)$ is arithmetic. For this sequence, the common difference between consecutive terms is $\frac{1}{4}$.

$$\underbrace{1, \frac{5}{4}}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n + 3}{4}, \ldots \qquad \text{Begin with } n = 1.$$

$\frac{5}{4} - 1 = \frac{1}{4}$

Checkpoint 1

Determine whether each sequence is arithmetic. If it is, find the common difference.

a. $5, 7, 9, 11, \ldots$ **b.** $-2, 1, 6, 13, \ldots$ **c.** $\frac{7}{2}, 3, \frac{5}{2}, 2, \ldots$

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The *n*th Term of an Arithmetic Sequence

The nth term of an arithmetic sequence has the form

$$a_n = a_1 + (n - 1)d$$

where d is the common difference between consecutive terms of the sequence and a_1 is the first term of the sequence.

Example 2 Finding the nth Term of an Arithmetic Sequence

Find a formula for the nth term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

SOLUTION Because the sequence is arithmetic, you know that the formula for the nth term is $a_n = a_1 + (n - 1)d$. Because the common difference is $d = 3$ and the first term is $a_1 = 2$, the formula must have the form

$$a_n = 2 + 3(n - 1).$$ Substitute 2 for a_1 and 3 for d.

So, the formula for the nth term is

$$a_n = 3n - 1.$$

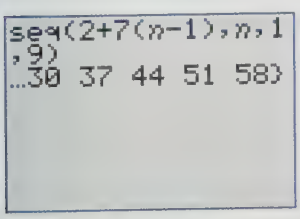
The sequence is $2, 5, 8, 11, 14, \ldots, 3n - 1, \ldots$.

Checkpoint 2

Find a formula for the nth term of the arithmetic sequence whose common difference is 2 and whose first term is 5.

TECH TUTOR

Most graphing utilities have a built-in function that will display the terms of an arithmetic sequence, as shown in the screen below. Consult the user's guide for your graphing utility for more information.



Example 3 Finding a Term of an Arithmetic Sequence

Find the ninth term of the arithmetic sequence whose first two terms are 2 and 9.

SOLUTION You know the sequence is arithmetic. Also, $a_1 = 2$ and $a_2 = 9$. So, the common difference for this sequence is

$$d = 9 - 2 = 7.$$

There are two ways to find the ninth term. One way is simply to write out the first nine terms by repeatedly adding 7.

$$2, \quad 9, \quad 16, \quad 23, \quad 30, \quad 37, \quad 44, \quad 51, \quad 58$$

Another way to find the ninth term is to first find a formula for the nth term. Because the common difference is $d = 7$ and the first term is $a_1 = 2$, the formula must have the form

$$a_n = 2 + 7(n - 1).$$ Substitute 2 for a_1 and 7 for d.

So, a formula for the nth term is

$$a_n = 7n - 5$$

which implies that the ninth term is

$$a_9 = 7(9) - 5 = 58.$$

Checkpoint 3

Find the 10th term of the arithmetic sequence whose first two terms are 1 and 4.

When you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the **recursion formula**

$$a_{n+1} = a_n + d. \qquad \text{Recursion formula}$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the previous term. For instance, when you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

Example 4 Using a Recursion Formula

The 14th term of an arithmetic sequence is 136 and the common difference is 11.

a. What is the 15th term of the sequence?

b. What is the first term?

SOLUTION

a. You know that $a_{14} = 136$ and $d = 11$. So, using the recursion formula $a_{15} = a_{14} + d$, you can determine that the 15th term of the sequence is

$$a_{15} = 136 + 11 = 147.$$

b. Using $n = 14$, $d = 11$, and $a_{14} = 136$ in the formula $a_n = a_1 + (n - 1)d$ yields

$$136 = a_1 + (14 - 1)(11)$$

$$136 = a_1 + 143$$

$$-7 = a_1.$$

✓Checkpoint 4

The 11th term of an arithmetic sequence is 37 and the common difference is 6.

a. What is the 12th term of the sequence?

b. What is the first term?

Example 5 Writing the First Several Terms of an Arithmetic Sequence

The 4th term of an arithmetic sequence is 20, and the 13th term is 65. Write the first several terms of this sequence.

SOLUTION You know that $a_4 = 20$ and $a_{13} = 65$. So, you must add the common difference d to the 4th term nine times to obtain the 13th term. Therefore, the 4th and 13th terms of the sequence are related by

$$a_{13} = a_4 + 9d. \qquad a_4 \text{ and } a_{13} \text{ are nine terms apart.}$$

Using $a_4 = 20$ and $a_{13} = 65$, you can conclude that $d = 5$, which implies that the sequence is as follows.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	
5,	10,	15,	20,	25,	30,	35,	40,	45,	50,	55,	. . .

✓Checkpoint 5

The 3rd term of an arithmetic sequence is 12, and the 14th term is 45. Write the first several terms of this sequence.

The Sum of a Finite Arithmetic Sequence

There is a formula for the sum of a finite arithmetic sequence. Be sure you see that this formula works only for *arithmetic* sequences.

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Example 6 Finding the Sum of a Finite Arithmetic Sequence

Find the sum: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$.

SOLUTION The sequence is arithmetic (with a common difference of 2). Moreover, the sequence has 10 terms. So, the sum of the sequence is

$$S_{10} = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$$

$= \frac{n}{2}(a_1 + a_n)$ Formula for the sum of a finite arithmetic sequence

$= \frac{10}{2}(1 + 19)$ Substitute 10 for n, 1 for a_1, and 19 for a_n.

$= 5(20)$ Simplify.

$= 100.$ Multiply.

Checkpoint 6

Find the sum: $4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31$.

The sum of the first n terms of an infinite sequence is the ***n*th partial sum.** The nth partial sum can be found by using the formula for the sum of a finite arithmetic sequence.

Example 7 Finding a Partial Sum of an Arithmetic Sequence

Find the 150th partial sum of the arithmetic sequence $5, 16, 27, 38, 49, \ldots$.

SOLUTION For this arithmetic sequence, you have $a_1 = 5$ and $d = 16 - 5 = 11$. So, $a_n = 5 + 11(n - 1)$ and the nth term is $a_n = 11n - 6$. Therefore, $a_{150} = 11(150) - 6 = 1644$, and the sum of the first 150 terms is

$$S_n = \frac{n}{2}(a_1 + a_n)$$ nth partial sum formula

$$= \frac{150}{2}(5 + 1644)$$ Substitute 150 for n, 5 for a_1, and 1644 for a_n.

$$= 75(1649)$$ Simplify.

$$= 123{,}675.$$ Multiply.

Checkpoint 7

Find the 100th partial sum of the arithmetic sequence $4, 10, 16, 22, 28, \ldots$.

Application

Example 8 Total Sales

A small coffee shop sells $50,000 worth of beverages during its first year. The owner of the business has set a goal of increasing annual sales by $12,000 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years this business is in operation. What kinds of economic factors could prevent the shop owner from meeting the sales goal?

SOLUTION The annual sales form an arithmetic sequence in which $a_1 = 50{,}000$ and $d = 12{,}000$. So, $a_n = 50{,}000 + 12{,}000(n - 1)$ and the nth term of the sequence is

$$a_n = 12{,}000n + 38{,}000.$$

This implies that the 10th term of the sequence is

$$a_{10} = 12{,}000(10) + 38{,}000 = 158{,}000.$$

The sum of the first 10 terms of the sequence is

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad n\text{th partial sum formula}$$

$$= \frac{10}{2}(50{,}000 + 158{,}000) \qquad \text{Substitute 10 for } n, 50{,}000 \text{ for } a_1, \text{ and } 158{,}000 \text{ for } a_n.$$

$$= 5(208{,}000) \qquad \text{Simplify.}$$

$$= 1{,}040{,}000. \qquad \text{Multiply.}$$

So, the total sales for the first 10 years are $1,040,000. Many economic factors, such as the location of the shop, demand by consumers, and marketing techniques, could prevent the shop owner from meeting the sales goal.

Checkpoint 8

A company sells $160,000 worth of printing paper during its first year. The sales manager has set a goal of increasing annual sales of printing paper by $20,000 each year for 9 years. Assuming that this goal is met, find the total sales of printing paper during the first 10 years this business is in operation.



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SUMMARIZE (Section 7.2)

1. State the definition of arithmetic sequence *(page 550)*. For an example of an arithmetic sequence, see Example 1.
2. State the formula for the nth term of an arithmetic sequence *(page 551)*. For examples of using this formula, see Examples 2 and 3.
3. State the recursion formula for the $(n + 1)$th term of an arithmetic sequence *(page 552)*. For examples of using this formula, see Examples 4 and 5.
4. State the formula for the sum of a finite arithmetic sequence and explain how to use it to find an nth partial sum *(page 553)*. For examples of using this formula, see Examples 6 and 7.
5. Describe a real-life example of how an arithmetic sequence can be used to analyze the total sales of a coffee shop *(page 554, Example 8)*.

SKILLS WARM UP 7.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.1, 0.2, 2.4, and 7.1.

In Exercises 1 and 2, find the sum.

1. $\sum_{i=1}^{6}(2i-1)$ **2.** $\sum_{i=1}^{10}(4i+2)$

In Exercises 3 and 4, find the distance between the two real numbers.

3. $\frac{5}{2}, 8$ **4.** $\frac{4}{3}, \frac{14}{3}$

In Exercises 5 and 6, evaluate the function as indicated.

5. Find $f(3)$ for $f(n) = 10 + (n-1)4$. **6.** Find $f(10)$ for $f(n) = 1 + (n-1)\frac{1}{3}$.

In Exercises 7–10, evaluate the expression.

7. $\frac{11}{2}(1+25)$ **8.** $\frac{16}{2}(4+16)$

9. $\frac{20}{2}[2(5)+(12-1)3]$ **10.** $\frac{8}{2}[2(-3)+(15-1)5]$

Exercises 7.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Identifying an Arithmetic Sequence In Exercises 1–12, determine whether the sequence is arithmetic. If it is, find the common difference. *See Example 1.*

1. 3, 7, 11, 15, 19, . . . **2.** 20, 17, 14, 11, 8, . . .

3. −12, −8, −4, 0, 4, 8, . . . **4.** −3, −2, 0, 3, 7, . . .

5. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \ldots$ **6.** $\frac{7}{5}, \frac{4}{5}, \frac{1}{5}, \frac{-2}{5}, \ldots$

7. $3, \frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}, \ldots$ **8.** $\frac{9}{4}, 2, \frac{7}{4}, \frac{3}{2}, \frac{5}{4}, 1, \ldots$

9. $\log_{10} 10, \log_{10} 100, \log_{10} 1000, \log_{10} 10{,}000, \ldots$

10. $\ln 1, \ln 2, \ln 3, \ln 4, \ln 5, \ldots$

11. 5.3, 5.7, 6.2, 6.8, 7.5, . . .

12. 1.2, 2.4, 3.6, 4.8, 6.0, . . .

Writing the Terms of a Sequence In Exercises 13–22, write the first five terms of the sequence. Determine whether the sequence is arithmetic. If it is, find the common difference.

13. $a_n = 5 + 3n$ **14.** $a_n = 100 - 3n$

15. $a_n = (2^n)n$ **16.** $a_n = 2^{n-1}$

17. $a_n = (2+n) - (1+n)$

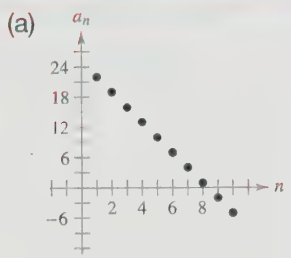
18. $a_n = 1 + (n-1)4$

19. $a_n = \frac{2}{n-1}$ **20.** $a_n = \frac{2n+1}{4n}$

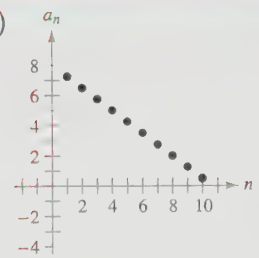
21. $a_n = (-1)^n$ **22.** $a_n = (-1)^{2n+1}$

Matching In Exercises 23–26, match the arithmetic sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

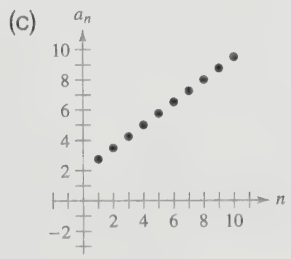
(a)



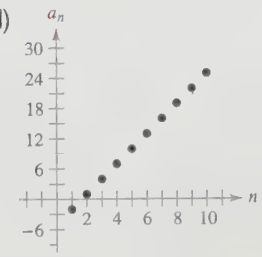
(b)



(c)



(d)



23. $a_n = 3n - 5$ **24.** $a_n = 25 - 3n$

25. $a_n = -\frac{3}{4}n + 8$ **26.** $a_n = 2 + \frac{3}{4}n$



Graphing the Terms of a Sequence In Exercises 27–30, use a graphing utility to graph the first 10 terms of the sequence.

27. $a_n = 15 - \frac{3}{2}n$ **28.** $a_n = -5 + 2n$

29. $a_n = 0.4n - 2$ **30.** $a_n = -1.3n + 7$

Finding the *n*th Term of an Arithmetic Sequence In Exercises 31–42, find a formula for the *n*th term of the arithmetic sequence. *See Examples 2 and 3.*

31. $a_1 = 4, d = 5$ **32.** $a_1 = 3, d = 8$

33. $a_1 = 50, d = -12$ **34.** $a_1 = 102, d = -6$

35. $a_1 = 9, a_2 = 17$ **36.** $a_1 = -16, a_2 = -6$

37. $a_1 = -7, a_2 = -11$ **38.** $a_1 = 5, a_2 = -4$

39. $2, 8, 14, 20, 26, \ldots$

40. $3, -7, -17, -27, -37, \ldots$

41. $2, \frac{11}{4}, \frac{7}{2}, \frac{17}{4}, 5, \ldots$ **42.** $\frac{16}{3}, \frac{14}{3}, 4, \frac{10}{3}, \frac{8}{3}, \ldots$

Using a Recursion Formula In Exercises 43–48, use the given term and common difference of an arithmetic sequence to find (a) the next term and (b) the first term a_1 of the sequence. *See Example 4.*

43. $a_7 = 29, d = 4$ **44.** $a_9 = 40, d = 6$

45. $a_{12} = -13, d = -3$ **46.** $a_{15} = -4, d = -7$

47. $a_{14} = 15\frac{1}{3}, d = \frac{1}{3}$ **48.** $a_{11} = 12\frac{1}{4}, d = \frac{3}{4}$

Writing the Terms of an Arithmetic Sequence In Exercises 49–54, write the first five terms of the arithmetic sequence. *See Example 5.*

49. $a_1 = 2, a_{12} = 46$ **50.** $a_5 = 28, a_{10} = 3$

51. $a_8 = 26, a_{12} = 10$ **52.** $a_4 = 16, a_{10} = 46$

53. $a_7 = 3.7, a_{13} = 7.3$ **54.** $a_6 = 10.3, a_{13} = 1.2$

Finding the Sum of a Finite Arithmetic Sequence In Exercises 55–62, find the sum of the finite arithmetic sequence. *See Example 6.*

55. $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$

56. $1 + 4 + 7 + 10 + 13 + 16 + 19$

57. $-1 + (-3) + (-5) + (-7) + (-9)$

58. $-5 + (-3) + (-1) + 1 + 3 + 5$

59. Sum of the integers from 1 to 100

60. Sum of the first 100 odd natural numbers

61. Sum of the integers from -20 to 80

62. Sum of the integers from -100 to 30

Finding a Partial Sum of an Arithmetic Sequence In Exercises 63–70, find the indicated nth partial sum of the arithmetic sequence. *See Example 7.*

63. $6, 12, 18, 24, \ldots, n = 8$

64. $4, 11, 18, 25, \ldots, n = 10$

65. $-10, -8, -6, -4, \ldots, n = 12$

66. $-2, -9, -16, -23, \ldots, n = 10$

67. $0.5, 0.9, 1.3, 1.7, \ldots, n = 10$

68. $1.50, 1.45, 1.40, 1.35, \ldots, n = 20$

69. $a_1 = 100, a_{25} = 220, n = 25$

70. $a_1 = 15, a_{100} = 307, n = 100$

Finding a Partial Sum In Exercises 71–80, find the partial sum.

71. $\sum_{n=1}^{4} n$ **72.** $\sum_{n=1}^{100} 2n$

73. $\sum_{n=32}^{100} 5n$ **74.** $\sum_{n=51}^{100} 7n$

75. $\sum_{n=0}^{50} (1000 - 5n)$ **76.** $\sum_{n=1}^{250} (1000 - n)$

77. $\sum_{n=1}^{20} (3n + 2)$ **78.** $\sum_{n=1}^{500} (n + 6)$

79. $\sum_{n=1}^{100} \frac{n+4}{2}$ **80.** $\sum_{n=0}^{100} \frac{8-3n}{16}$

Using Summation Notation In Exercises 81–84, use summation notation to write the sum.

81. $3 + 6 + 9 + 12 + 15 + \cdots + 300$

82. $5 + 7 + 9 + 11 + 13 + \cdots + 43$

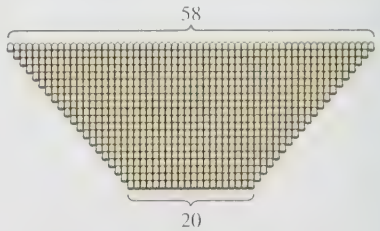
83. $\frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \cdots + \frac{11}{4}$

84. $4 + \frac{13}{3} + \frac{14}{3} + 5 + \frac{16}{3} + \cdots + 12$

Job Offer In Exercises 85 and 86, consider a job offer with the given starting salary and the given annual raise. (a) Determine the salary during the sixth year of employment. (b) Determine the total compensation from the company through six full years of employment.

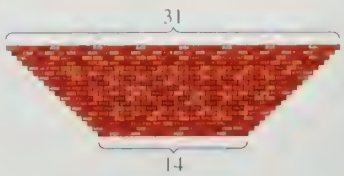
	Starting Salary	*Annual Raise*
85.	\$35,000	\$1750
86.	\$39,400	\$2125

87. Seating Capacity An auditorium has 20 rows of seats. There are 20 seats in the first row, 22 seats in the second row, 24 seats in the third row, and so on (see figure). How many seats are there in all 20 rows?

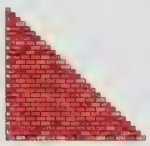


88. Seating Capacity An auditorium has 36 rows of seats. There are 15 seats in the first row, 18 seats in the second row, 21 seats in the third row, and so on. How many seats are there in all 36 rows?

89. Brick Pattern A brick patio has the approximate shape of a trapezoid (see figure). The patio has 18 rows of bricks. The first row has 14 bricks and the 18th row has 31 bricks. How many bricks are there in the patio?



90. Brick Pattern A triangular brick wall is made by cutting some bricks in half to use in the first column of every other row. The wall has 28 rows. The top row is one-half brick wide and the bottom row is 14 bricks wide. How many bricks are used in the finished wall?



91. Sales The annual sales a_n (in billions of dollars) for Nike from 2004 through 2009 can be approximated by the model

$$a_n = 12.229 + 1.4471n, \quad n = 1, 2, \ldots, 6$$

where n represents the year, with $n = 1$ corresponding to 2004. *(Source: Nike, Inc.)*

(a) Construct a bar graph showing the annual sales for Nike from 2004 through 2009.

(b) Find the total sales from 2004 through 2009.

92. Revenue The annual revenues a_n (in millions of dollars) for Panera Bread Company from 2005 through 2010 can be modeled by

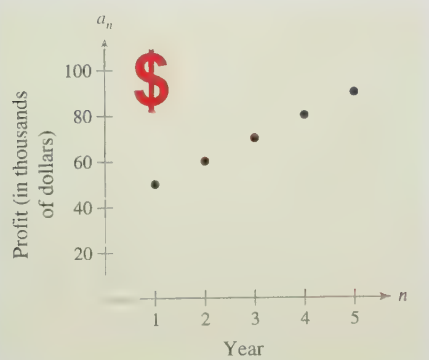
$$a_n = 670.6 + 180.48n, \quad n = 1, 2, \ldots, 6$$

where n represents the year, with $n = 1$ corresponding to 2005. *(Source: Panera Bread Company)*

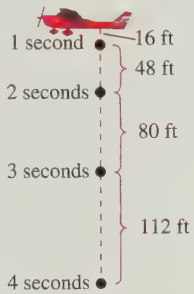
(a) Construct a bar graph showing the annual revenues for Panera Bread Company from 2005 through 2010.

(b) Find the total revenues from 2005 through 2010.

93. Profit A small landscaping company makes a profit of \$50,000 during its first year. The company president sets a goal of increasing profit by \$10,000 each year for 4 years (see figure). Assuming that this goal is met, find the total profit during the first 5 years of this business. What kinds of economic factors could prevent the company from meeting the profit goal?



94. HOW DO YOU SEE IT? A steel ball (with negligible air resistance) is dropped from a plane. The figure shows the distance that the ball falls during each of the first four seconds after it is dropped.



(a) Describe a pattern of the distances shown. Explain why the distances form a finite arithmetic sequence.

(b) Assume the pattern described in part (a) continues. Describe the steps and formulas for using the sum of a finite sequence to find the total distance the ball falls in a given whole number of seconds n.

95. Falling Object A heavy object (with negligible air resistance) is dropped from a plane. During the 1st second of fall, the object falls 4.9 meters; during the 2nd second, it falls 14.7 meters; during the 3rd second, it falls 24.5 meters; and during the 4th second, it falls 34.3 meters. Assume this pattern continues. How many meters will the object fall in 10 seconds?

96. Profit The annual profits a_n (in billions of dollars) for IBM from 2005 through 2010 are shown in the table, where n represents the year, with $n = 1$ corresponding to 2005. *(Source: International Business Machines Corporation)*

Year, n	1	2	3
Profit, a_n	8.519	9.416	10.418

Year, n	4	5	6
Profit, a_n	12.334	13.425	14.833

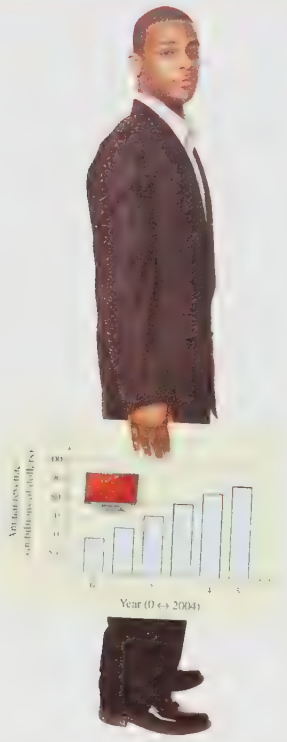
(a) Construct a bar graph showing the annual profits from 2005 through 2010.

(b) Use the *linear regression* feature of a graphing utility to find an arithmetic sequence that approximates the annual profits from 2005 through 2010.

(c) Use summation notation to represent the *total* profit from 2005 through 2010. Find the total profit.

7.3 Geometric Sequences and Series

- Recognize and find the nth term of a geometric sequence.
- Find the sum of a finite geometric sequence.
- Find the sum of an infinite geometric series.
- Use a geometric sequence to solve an application problem.



In Exercise 98 on page 566, you will use a geometric sequence to approximate the total revenue earned by the cable television industry during a six-year period.

Geometric Sequences

In Section 7.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio*.

Definition of Geometric Sequence

A sequence is **geometric** when the ratios of consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is geometric when there is a number r such that

$$\frac{a_2}{a_1} = r, \quad \frac{a_3}{a_2} = r, \quad \frac{a_4}{a_3} = r, \quad r \neq 0$$

and so on. The number r is the **common ratio** of the sequence.

Example 1 Examples of Geometric Sequences

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio between consecutive terms is 2.

$$\underbrace{2, 4}, 8, 16, \ldots, 2^n, \ldots \qquad \text{Begin with } n = 1.$$

$$\frac{4}{2} = 2$$

b. The sequence whose nth term is $4(3^n)$ is geometric. For this sequence, the common ratio between consecutive terms is 3.

$$\underbrace{12, 36}, 108, 324, \ldots, 4(3^n), \ldots \qquad \text{Begin with } n = 1.$$

$$\frac{36}{12} = 3$$

c. The sequence whose nth term is $\left(-\frac{1}{3}\right)^n$ is geometric. For this sequence, the common ratio between consecutive terms is $-\frac{1}{3}$.

$$\underbrace{-\frac{1}{3}, \frac{1}{9}}, -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots \qquad \text{Begin with } n = 1.$$

$$\frac{\frac{1}{9}}{-\frac{1}{3}} = -\frac{1}{3}$$

Checkpoint 1

Determine whether the sequence

$$2, 8, 32, 128, \ldots$$

is geometric. If it is, find its common ratio.

David Gilder/Shutterstock.com

In Example 1, notice that each of the geometric sequences has an nth term that is of the form ar^n, where the common ratio of the sequence is r.

The nth Term of a Geometric Sequence

The nth term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\underset{\downarrow}{a_1},\ \underset{\downarrow}{a_2},\ \underset{\downarrow}{a_3},\ \underset{\downarrow}{a_4},\ \underset{\downarrow}{a_5},\ \ldots,\ \underset{\downarrow}{a_n},\ \ldots$$
$$a_1,\ a_1 r,\ a_1 r^2,\ a_1 r^3,\ a_1 r^4,\ \ldots,\ a_1 r^{n-1},\ \ldots$$

When you know the nth term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is,

$$a_{n+1} = ra_n.$$



TECH TUTOR

You can generate the geometric sequence in Example 2 with a graphing utility using the following steps.

3 ENTER

2 × 2nd [ANS]

Now press ENTER repeatedly to generate the terms of the sequence.

Example 2 Writing the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$.

SOLUTION Starting with 3, repeatedly multiply by 2 to obtain the following.

$a_1 = 3$ 1st term

$a_2 = 3(2^1) = 6$ 2nd term

$a_3 = 3(2^2) = 12$ 3rd term

$a_4 = 3(2^3) = 24$ 4th term

$a_5 = 3(2^4) = 48$ 5th term

Checkpoint 2

Write the first five terms of the geometric sequence whose first term is $a_1 = 2$ and whose common ratio is $r = 3$.

Example 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

SOLUTION

$a_n = a_1 r^{n-1}$ Formula for the nth term of a geometric sequence

$a_{15} = 20(1.05)^{15-1}$ Substitute 20 for a_1, 1.05 for r, and 15 for n.

≈ 39.599 Use a calculator.

Checkpoint 3

Find the 12th term of the geometric sequence whose first term is 14 and whose common ratio is 1.2.

Example 4 Finding the nth Term of a Geometric Sequence

Find a formula for the nth term of the geometric sequence 5, 15, 45, ... Then find the 12th term of the sequence.

SOLUTION The common ratio of this sequence is $r = 15/5 = 3$. Because the first term is $a_1 = 5$, the formula for the nth term of the sequence must have the form

$$a_n = a_1 r^{n-1} \qquad \text{Formula for the } n\text{th term}$$
$$= 5(3)^{n-1}. \qquad \text{Substitute 5 for } a_1 \text{ and 3 for } r.$$

You can determine the 12th term ($n = 12$) to be

$$a_{12} = 5(3)^{12-1} \qquad \text{Substitute 12 for } n.$$
$$= 5(177{,}147) \qquad \text{Use a calculator.}$$
$$= 885{,}735. \qquad \text{Multiply.}$$

Checkpoint 4

Find a formula for the nth term of the geometric sequence 4, 20, 100, ... Then find the 12th term of the sequence.

STUDY TIP

When you know any two terms of a geometric sequence, you can use that information to find a formula for the nth term of the sequence.

Example 5 Finding a Term of a Geometric Sequence

The 4th term of a geometric sequence is 125, and the 10th term is $\frac{125}{64}$. Find the 14th term. (Assume that the terms of the sequence are positive.)

SOLUTION Because r is the common ratio of *consecutive* terms of a geometric sequence, you have to find the relationship between the 10th term and the 4th term.

$$a_{10} = a_1 r^9$$
$$= a_1 \cdot r \cdot r \cdot r \cdot r^6$$
$$= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6$$
$$a_{10} = a_4 r^6$$

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\frac{125}{64} = 125r^6$$
$$\frac{1}{64} = r^6$$
$$\pm\frac{1}{2} = r$$

Because the terms of the sequence are assumed to be positive, choose the positive solution, $r = \frac{1}{2}$. You can obtain the 14th term by multiplying the 10th term by r^4.

$$a_{14} = a_{10} r^4 = \frac{125}{64}\left(\frac{1}{2}\right)^4 = \frac{125}{1024}$$

Checkpoint 5

The second term of a geometric sequence is 6, and the fifth term is $\frac{81}{4}$. Find the eighth term.

The Sum of a Finite Geometric Sequence

The Sum of a Finite Geometric Sequence

The sum of the finite geometric sequence

$$a_1, a_1r, a_1r^2, a_1r^3, a_1r^4, \ldots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

Example 6 Finding the Sum of a Finite Geometric Sequence

Find the sum $\sum_{n=1}^{12} 4(0.3)^{n-1}$.

SOLUTION By writing out a few terms, you have

$$\sum_{n=1}^{12} 4(0.3)^{n-1} = 4(0.3)^0 + 4(0.3)^1 + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{11}.$$

Now, because $a_1 = 4$, $r = 0.3$, and $n = 12$, you can apply the formula for the sum of a finite geometric sequence to obtain

$$\begin{aligned}\sum_{n=1}^{12} 4(0.3)^{n-1} &= a_1\left(\frac{1 - r^n}{1 - r}\right) \\ &= 4\left[\frac{1 - (0.3)^{12}}{1 - 0.3}\right] \\ &\approx 5.714.\end{aligned}$$

Checkpoint 6

Find the sum $\sum_{n=1}^{10} 2(3^{n-1})$.

When using the formula for the sum of a finite geometric sequence, be careful to check that the sum is of the form

$$\sum_{i=1}^{n} a_1 r^{i-1}.$$

For a sum that is not of this form, you must adjust the formula for the nth partial sum. For instance, if the sum in Example 6 was $\sum_{n=1}^{12} 4(0.3)^n$, then you would evaluate the sum as

$$\begin{aligned}\sum_{n=1}^{12} 4(0.3)^n &= 4(0.3) + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{12} \\ &= 4(0.3) + [4(0.3)](0.3) + [4(0.3)](0.3)^2 + \cdots + [4(0.3)](0.3)^{11}.\end{aligned}$$

So, $a_1 = 4(0.3)$, $r = 0.3$, and $n = 12$, and the sum is

$$\begin{aligned}S_{12} &= 4(0.3)\left(\frac{1 - (0.3)^{12}}{1 - 0.3}\right) \\ &\approx 1.714.\end{aligned}$$

Geometric Series

The summation of the terms of an infinite geometric *sequence* is called an **infinite geometric series,** or simply a **geometric series.**

The formula for the sum of a *finite* geometric *sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite* geometric *series*. Specifically, if the common ratio r has the property that $|r| < 1$, then it can be shown that r^n becomes arbitrarily close to zero as n increases without bound. Consequently,

$$a_1\left(\frac{1 - r^n}{1 - r}\right) \to a_1\left(\frac{1 - 0}{1 - r}\right) \quad \text{as} \quad n \to \infty.$$

The Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1 - r}.$$

Note that when $|r| \geq 1$, the series

$$\sum_{i=0}^{\infty} a_1r^i$$

does *not* have a sum.

Example 7 Finding the Sum of an Infinite Geometric Series

Find each sum.

a. $\sum_{n=0}^{\infty} 4(0.6)^n$ **b.** $\sum_{n=1}^{\infty} 3(0.1)^{n-1}$

SOLUTION

a. $\sum_{n=0}^{\infty} 4(0.6)^n = 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots + 4(0.6)^n + \cdots$

$$= \frac{4}{1 - 0.6} \qquad \frac{a_1}{1 - r}$$

$$= 10$$

b. $\sum_{n=1}^{\infty} 3(0.1)^{n-1} = 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots + 3(0.1)^{n-1} + \cdots$

$$= \frac{3}{1 - 0.1} \qquad \frac{a_1}{1 - r}$$

$$= \frac{10}{3}$$

$$\approx 3.33$$

Checkpoint 7

Find the sum $\sum_{n=1}^{\infty} 5(0.4)^{n-1}$.

Application

Example 8 Increasing Annuity

A deposit of \$50 is made on the first day of each month in an account that pays 3% compounded monthly. What is the balance at the end of 2 years? (This type of savings plan is called an **increasing annuity.**)

SOLUTION To find the balance in the account after 24 months, consider each of the 24 deposits separately. The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.03}{12}\right)^{24}$$

$$= 50(1.0025)^{24}.$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.03}{12}\right)^{23}$$

$$= 50(1.0025)^{23}.$$

The last deposit will gain interest for only 1 month, and its balance will be

$$A_1 = 50\left(1 + \frac{0.03}{12}\right)^{1}$$

$$= 50(1.0025)^{1}.$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.0025)$, $r = 1.0025$, and $n = 24$, you have

$$S_n = A_1\left(\frac{1 - r^n}{1 - r}\right)$$ Formula for sum of a finite geometric sequence

$$S_{24} = 50(1.0025)\left[\frac{1 - (1.0025)^{24}}{1 - 1.0025}\right]$$ Substitute $50(1.0025)$ for A_1, 1.0025 for r, and 24 for n.

$$\approx \$1238.23.$$ Simplify.

Checkpoint 8

A deposit of \$70 is made on the first day of each month in an account that pays 2% compounded quarterly. What is the balance at the end of 4 years?



SUMMARIZE (Section 7.3)

1. State the definition of geometric sequence *(page 558)*. For an example of a geometric sequence, see Example 1.
2. State the formula for the nth term of a geometric sequence *(page 559)*. For examples of using this formula, see Examples 3 and 4.
3. State the formula for the sum of a finite geometric sequence *(page 561)*. For an example of using this formula, see Example 6.
4. State the formula for the sum of an infinite geometric series *(page 562)*. For an example of using this formula, see Example 7.

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SKILLS WARM UP 7.3 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 7.1.

In Exercises 1–4, evaluate the expression.

1. $\left(\frac{4}{5}\right)^3$ **2.** $\left(\frac{3}{4}\right)^2$ **3.** 2^{-4} **4.** $4(3^4)$

In Exercises 5–8, simplify the expression.

5. $(2n)(3n^2)$ **6.** $n(3n)^3$ **7.** $\frac{4n^5}{n^2}$ **8.** $\frac{(2n)^3}{8n}$

In Exercises 9 and 10, use summation notation to write the sum.

9. $2 + 2(3) + 2(3^2) + 2(3^3)$ **10.** $3 + 3(2) + 3(2^2) + 3(2^3)$

Exercises 7.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises

Finding a Common Ratio In Exercises 1–8, find the common ratio of the geometric sequence. *See Example 1.*

1. 3, 15, 75, 375, 1875, . . .

2. 4, 32, 256, 2048, 16,384, . . .

3. 10, −20, 40, −80, 160, . . .

4. −2, 8, −32, 128, −512, . . .

5. 216, 36, 6, 1, $\frac{1}{6}$, . . .

6. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \ldots$

7. 0.6, 1.8, 5.4, 16.2, 48.6, . . .

8. −320, 80, −20, 5, −1.25, . . .

Identifying a Geometric Sequence In Exercises 9–20, determine whether the sequence is geometric. If it is, find the common ratio and a formula for the *n*th term. *See Example 1.*

9. 1, 4, 16, 64, . . .

10. 2, 10, 50, 250, . . .

11. 4, −12, 36, −108, . . .

12. 1, −2, 4, −8, . . .

13. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

14. $\frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}, \ldots$

15. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

16. $-8, \frac{2}{3}, 4, -2, \ldots$

17. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

18. $\frac{1}{5}, \frac{2}{3}, \frac{3}{9}, \frac{4}{11}, \ldots$

19. 4, 6, 9, 13.5, . . .

20. −25, 5, −1, 0.2, . . .

Writing the Terms of a Geometric Sequence In Exercises 21–30, write the first five terms of the geometric sequence. *See Example 2.*

21. $a_1 = 4, r = 2$

22. $a_1 = 6, r = 3$

23. $a_1 = 1, r = \frac{1}{3}$

24. $a_1 = 1, r = \frac{2}{3}$

25. $a_1 = \frac{1}{2}, r = -\frac{1}{3}$

26. $a_1 = \frac{5}{7}, r = -\frac{1}{5}$

27. $a_1 = 200, r = 0.5$

28. $a_1 = 50, r = -0.1$

29. $a_1 = 1, r = e$

30. $a_1 = 2, r = e^{0.1}$

Finding a Term of a Geometric Sequence In Exercises 31–38, find the indicated term of the geometric sequence. *See Example 3.*

31. $a_1 = 16, r = \frac{1}{4}$, 5th term

32. $a_1 = 9, r = \frac{2}{3}$, 7th term

33. $a_1 = 6, r = -\frac{1}{3}$, 12th term

34. $a_1 = 1, r = -\frac{2}{3}$, 10th term

35. $a_1 = 100, r = e$, 9th term

36. $a_1 = 6, r = e^{0.1}$, 9th term

37. $a_1 = 500, r = 1.02$, 40th term

38. $a_1 = 1000, r = 1.005$, 60th term

Finding the *n*th Term of a Geometric Sequence In Exercises 39–46, find a formula for the *n*th term of the geometric sequence. Then find the indicated term of the sequence. *See Example 4.*

39. 7th term: 7, 14, 28, 56, . . .

40. 9th term: 5, 20, 80, 320, . . .

41. 15th term: 4, 12, 36, 108, . . .

42. 9th term: 2, 24, 288, 3456, . . .

43. 20th term: 6, 30, 150, 750, . . .

44. 22nd term: 3, −6, 12, −24, . . .

45. 8th term: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

46. 7th term: $\frac{3}{2}, -1, \frac{2}{3}, -\frac{4}{9}, \ldots$

Finding a Term of a Geometric Sequence In Exercises 47–50, find the indicated term of the geometric sequence. *See Example 5.*

47. $a_2 = -40, a_4 = -10$, 3rd term

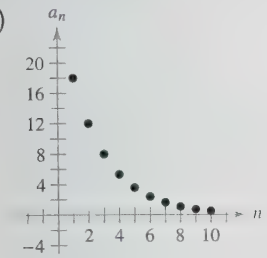
48. $a_2 = 3, a_5 = \frac{3}{64}$, 3rd term

49. $a_2 = -18, a_5 = \frac{2}{3}$, 6th term

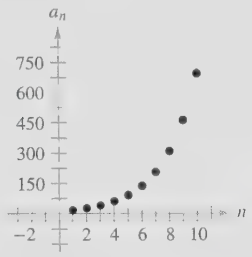
50. $a_3 = \frac{16}{3}, a_5 = \frac{64}{27}$, 7th term

Matching In Exercises 51–54, match the geometric sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

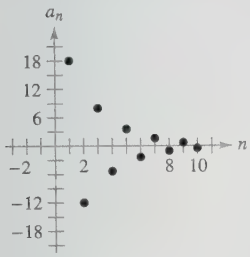
(a)



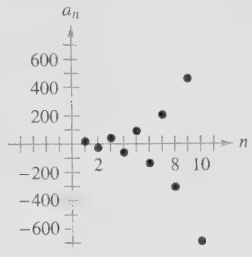
(b)



(c)



(d)



51. $a_n = 18\left(\frac{2}{3}\right)^{n-1}$

52. $a_n = 18\left(-\frac{2}{3}\right)^{n-1}$

53. $a_n = 18\left(\frac{3}{2}\right)^{n-1}$

54. $a_n = 18\left(-\frac{3}{2}\right)^{n-1}$



Using a Graphing Utility In Exercises 55–60, use a graphing utility to graph the first 10 terms of the sequence.

55. $a_n = 12(-0.75)^{n-1}$

56. $a_n = 10(-1.5)^{n-1}$

57. $a_n = 12(-0.4)^{n-1}$

58. $a_n = 20(-1.25)^{n-1}$

59. $a_n = 2(1.3)^{n-1}$

60. $a_n = 10(1.2)^{n-1}$

Finding the Sum of a Finite Geometric Sequence In Exercises 61–74, find the sum of the finite geometric sequence. *See Example 6.*

61. $\sum_{n=1}^{10} 8(2^n)$

62. $\sum_{n=1}^{6} 3(4^n)$

63. $\sum_{n=0}^{8} 2^n$

64. $\sum_{n=0}^{8} (-2)^n$

65. $\sum_{i=1}^{10} 8\left(\frac{1}{4}\right)^{i-1}$

66. $\sum_{i=1}^{10} 5\left(\frac{1}{3}\right)^{i-1}$

67. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$

68. $\sum_{i=1}^{6} 30\left(\frac{1}{5}\right)^{i-1}$

69. $\sum_{n=0}^{40} 2\left(-\frac{1}{4}\right)^n$

70. $\sum_{n=0}^{50} 10\left(\frac{2}{3}\right)^{n-1}$

71. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^n$

72. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^n$

73. $\sum_{i=1}^{40} 4\left(-\frac{5}{2}\right)^{i-1}$

74. $\sum_{i=1}^{100} 8\left(\frac{7}{4}\right)^{i-1}$

Using Summation Notation In Exercises 75–80, use summation notation to write the sum.

75. $5 + 15 + 45 + \cdots + 3645$

76. $7 + 14 + 28 + \cdots + 896$

77. $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$

78. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$

79. $0.1 + 0.4 + 1.6 + \cdots + 102.4$

80. $32 + 24 + 18 + \cdots + 10.125$

Finding the Sum of an Infinite Geometric Series In Exercises 81–88, find the sum of the infinite geometric series, if it exists. *See Example 7.*

81. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

82. $\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^n = 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$

83. $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

84. $\sum_{n=1}^{\infty} 2\left(-\frac{2}{3}\right)^{n-1} = 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \cdots$

85. $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$

86. $\sum_{n=0}^{\infty} 4\left(\frac{1}{4}\right)^n = 4 + 1 + \frac{1}{4} + \frac{1}{16} + \cdots$

87. $\sum_{n=0}^{\infty} 3\left(\frac{1}{10}\right)^n = 3 + 0.3 + 0.03 + 0.003 + \cdots$

88. $\sum_{n=0}^{\infty} 5\left(\frac{4}{3}\right)^n = 5 + \frac{20}{3} + \frac{80}{9} + \frac{320}{27} + \cdots$

Rewriting a Repeating Decimal In Exercises 89–92, find the rational number representation of the repeating decimal.

89. $0.\overline{36}$

90. $0.\overline{297}$

91. $0.3\overline{18}$

92. $1.3\overline{8}$

93. Compound Interest You are saving for a motorcycle. You deposit \$300 at the beginning of each month for 4 years in an account that pays 6% compounded monthly. The balance A in the account at the end of 4 years is

$$A = 300\left(1 + \frac{0.06}{12}\right)^1 + \cdots + 300\left(1 + \frac{0.06}{12}\right)^{48}.$$

(a) Is there enough money in the account after 4 years to buy a \$17,000 motorcycle?

(b) Repeat part (a) for an interest rate of 9%.

94. Compound Interest You are saving to pay for an addition to your house. You deposit \$200 at the beginning of each month for 3 years in an account that pays 5% compounded monthly. The balance A in the account at the end of 3 years is

$$A = 200\left(1 + \frac{0.05}{12}\right)^1 + \cdots + 200\left(1 + \frac{0.05}{12}\right)^{36}.$$

(a) Is there enough money in the account after 3 years to pay for an \$8000 addition to your house?

(b) Repeat part (a) for an interest rate of 7%.

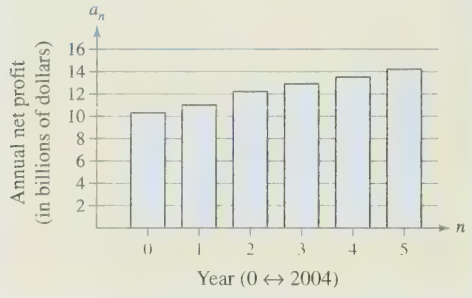
95. Compound Interest You deposit $100 in an account at the beginning of each month for 10 years. The account pays 8% compounded monthly. Use a graphing utility to find your balance at the end of 10 years. Then find the balance if the interest is compounded continuously.

96. Compound Interest You deposit $150 in an account at the beginning of each month for 20 years. The account pays 6% compounded monthly. Use a graphing utility to find your balance at the end of 20 years. Then find the balance if the interest is compounded continuously.

97. Profit The annual net profits a_n (in billions of dollars) for Wal-Mart Stores, Inc. from 2004 through 2009 can be approximated by the model

$$a_n = 10.46e^{0.064n}, \quad n = 0, 1, \ldots, 5$$

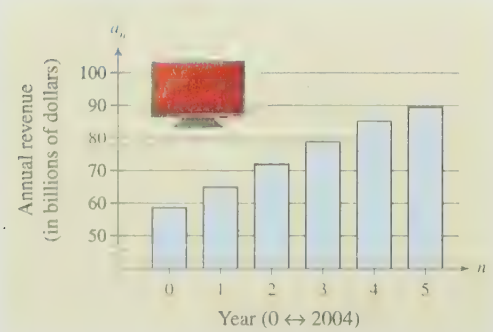
where n represents the year, with $n = 0$ corresponding to 2004 (see figure). Use the formula for the sum of a finite geometric sequence to approximate the total net profit earned during this six-year period. *(Source: Wal-Mart Stores, Inc.)*



98. Revenue The annual revenues a_n (in billions of dollars) for the cable television industry from 2004 through 2009 can be approximated by the model

$$a_n = 60.04e^{0.084n}, \quad n = 0, 1, \ldots, 5$$

where n represents the year, with $n = 0$ corresponding to 2004 (see figure). Use the formula for the sum of a finite geometric sequence to approximate the total revenue earned during this six-year period. *(Source: SNL Kagan)*



99. Cellular Telecommunications The numbers of text messages a_n (in billions) sent in the United States from 2003 through 2009 can be approximated by the model

$$a_n = 5.09e^{0.575n}, \quad n = 0, 1, \ldots, 6$$

where n represents the year, with $n = 0$ corresponding to 2003. *(Source: CTIA-The Wireless Association)*

(a) Construct a bar graph showing the annual numbers of text messages from 2003 through 2009.

(b) Use the formula for the sum of a finite geometric sequence to approximate the total number of text messages sent during this seven-year period.

100. Sales The annual sales a_n (in millions of dollars) for La-Z-Boy Inc. from 2004 through 2009 can be approximated by the model

$$a_n = 2082.76e^{-0.121n}, \quad n = 0, 1, \ldots, 5$$

where n represents the year, with $n = 0$ corresponding to 2004. *(Source: La-Z-Boy Inc.)*

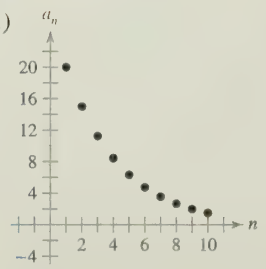
(a) Construct a bar graph showing the annual sales from 2004 through 2009.

(b) Use the formula for the sum of a finite geometric sequence to approximate the total sales during this six-year period.

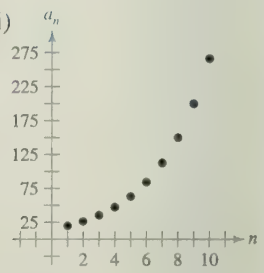
101. Daily Wage You decide to work at a company that pays $0.01 for the first day, $0.02 for the second day, $0.04 for the third day, and so on. If the daily wage keeps doubling, what will be your total income for working 29 days? 30 days? 31 days? Which job pays more the first year: the job that pays $0.01 the first day and twice the previous day's wage for every day thereafter, or a job that pays $30,000 per year?

102. HOW DO YOU SEE IT? Use the figures shown below.

(i)



(ii)



(a) Without doing any calculations, determine which figure shows the terms of the sequence $a_n = 20\left(\frac{4}{3}\right)^{n-1}$ and which one shows the terms of $a_n = 20\left(\frac{3}{4}\right)^{n-1}$. Explain your reasoning.

(b) Which sequence has terms that can be summed? Explain your reasoning.

QUIZ YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, write the first five terms of the sequence. (Assume n begins with 1.)

1. $a_n = 2n - 3$

2. $a_n = \dfrac{n}{(n+1)^2}$

In Exercises 3 and 4, simplify the factorial expression.

3. $\dfrac{5!}{3!}$

4. $\dfrac{30!}{27!}$

In Exercises 5 and 6, write an expression for the apparent nth term a_n of the sequence. (Assume n begins with 1.)

5. $3, \frac{3}{4}, \frac{1}{3}, \frac{3}{16}, \frac{3}{25}, \ldots$

6. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

In Exercises 7–10, find a formula for the nth term of the sequence.

7. Arithmetic: $a_1 = 2, d = 5$

8. Arithmetic: $a_1 = 7, d = 3$

9. Geometric: $a_1 = 10, r = 3$

10. Geometric: $a_1 = 4, r = -\frac{1}{2}$

11. Find the 13th term of the arithmetic sequence whose first two terms are 3 and 11.

12. The seventh term of an arithmetic sequence is 175 and the common difference is 28. What is the eighth term of the sequence? What is the first term of the sequence?

13. Find the 20th term of the geometric sequence whose 1st term is 16 and whose common ratio is 1.04.

14. The 5th term of a geometric sequence is 100, and the 10th term is $\frac{100}{243}$. Find the 12th term. (Assume that the terms of the sequence are positive.)

In Exercises 15–18, determine whether the sequence is arithmetic, geometric, or neither. Explain your reasoning.

15. $a_n = 2 + 3(n - 1)$

16. $1, 1, 2, 3, 5, 8, 13, \ldots$

17. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

18. $a_n = 3(2)^n$

In Exercises 19–24, find the sum.

19. $\displaystyle\sum_{i=1}^{5} 9$

20. $\displaystyle\sum_{i=1}^{4} 2i^2$

21. $\displaystyle\sum_{n=1}^{25} (4n - 3)$

22. $\displaystyle\sum_{n=1}^{6} 2^n$

23. $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$

24. $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{1}{5}\right)^n$

25. A person accepts a position with a company and will receive a salary of \$48,500 for the first year and a guaranteed raise of \$1940 per year.

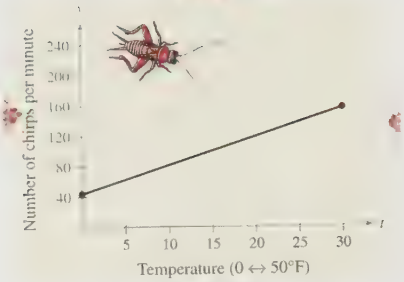
(a) What is the employee's salary during the sixth year of employment?

(b) What is the employee's total compensation from the company during this six-year period?

26. You deposit \$200 in an account at the beginning of each month for 20 years. The account pays 4% compounded monthly. What is your balance at the end of the 20 years?

7.4 The Binomial Theorem







In Exercise 72 on page 574, you will use the Binomial Theorem to rewrite a function that approximates the numbers of chirps a cricket makes each minute at various temperatures.

- Find a binomial coefficient.
- Use Pascal's Triangle to determine a binomial coefficient.
- Expand a binomial using binomial coefficients and find a specific term in a binomial expansion.

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that gives a quick method of raising a binomial to a power. To begin, look at the expansion of

$$(x + y)^n$$

for several values of n.

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.
2. In each expansion, x and y have symmetrical roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.
3. The degree of each term is n. For instance, in the expansion of

   $$(x + y)^5$$

   the degree of each term is 5.

   $$(x + y)^5 = x^5 + \underbrace{5x^4y^1}_{4+1=5} + \underbrace{10x^3y^2}_{3+2=5} + 10x^2y^3 + 5xy^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.** To find them, you can use the **Binomial Theorem.**

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$, the $(r + 1)$th term, is given by

$${}_nC_r = \frac{n!}{(n - r)!r!}.$$

The notation $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote a binomial coefficient.

Yuri Arcurs / Shutterstock.com

Example 1 Finding Binomial Coefficients

a. $${}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

Coefficient of third term in expansion of $(x + y)^8$

b. $${}_{10}C_3 = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

Coefficient of fourth term in expansion of $(x + y)^{10}$

c. $${}_7C_0 = \frac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$$

Coefficient of first term in expansion of $(x + y)^7$

d. $${}_8C_8 = \frac{\cancel{8!}}{0! \cdot \cancel{8!}} = 1$$

Coefficient of ninth term in expansion of $(x + y)^8$

Checkpoint 1

Find each binomial coefficient.

a. ${}_5C_2$ **b.** ${}_6C_6$

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) of Example 1, there is a simple pattern for evaluating binomial coefficients.

$${}_8C_2 = \overbrace{\frac{8 \cdot 7}{2 \cdot 1}}^{\text{2 factors}} \quad \text{and} \quad {}_{10}C_3 = \overbrace{\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}}^{\text{3 factors}}$$

$$\underbrace{}_{\text{2 factorial}} \qquad \underbrace{}_{\text{3 factorial}}$$

Example 2 Finding Binomial Coefficients

Find each binomial coefficient using the pattern shown above.

a. ${}_7C_3$ **b.** $\binom{7}{4}$ **c.** ${}_{12}C_1$ **d.** $\binom{12}{11}$

SOLUTION

a. $${}_7C_3 = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

Coefficient of fourth term in expansion of $(x + y)^7$

b. $$\binom{7}{4} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

Coefficient of fifth term in expansion of $(x + y)^7$

c. $${}_{12}C_1 = \frac{12}{1} = 12$$

Coefficient of second term in expansion of $(x + y)^{12}$

d. $$\binom{12}{11} = \frac{12!}{1! \cdot 11!} = \frac{(12) \cdot \cancel{11!}}{1! \cdot \cancel{11!}} = \frac{12}{1} = 12$$

Coefficient of twelfth term in expansion of $(x + y)^{12}$

Checkpoint 2

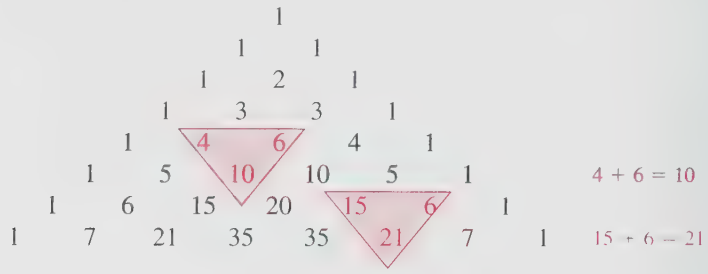
Find each binomial coefficient using the pattern shown before Example 2.

a. ${}_9C_2$ **b.** $\binom{6}{4}$

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, ${}_nC_r = {}_nC_{n-r}$.

Pascal's Triangle

There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).





The first and last numbers in each row of Pascal's Triangle are 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers that are the coefficients of binomial expansions, as follows.

$$(x + y)^0 = 1 \qquad \text{0th row}$$
$$(x + y)^1 = 1x + 1y \qquad \text{1st row}$$
$$(x + y)^2 = 1x^2 + 2xy + 1y^2 \qquad \text{2nd row}$$
$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \qquad \text{3rd row}$$
$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \qquad \vdots$$
$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$
$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$
$$(x + y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

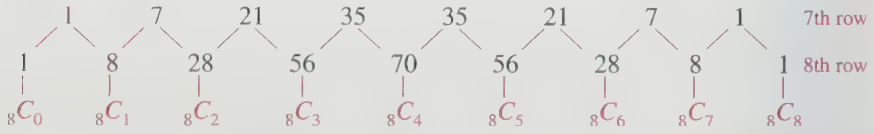
The top row in Pascal's Triangle is called the *zero row* because it corresponds to the binomial expansion $(x + y)^0 = 1$. Similarly, the next row is called the *first row* because it corresponds to the binomial expansion $(x + y)^1 = 1x + 1y$. In general, the *nth row* in Pascal's Triangle gives the coefficients of $(x + y)^n$.

Example 3 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

$${}_8C_0 \quad {}_8C_1 \quad {}_8C_2 \quad {}_8C_3 \quad {}_8C_4 \quad {}_8C_5 \quad {}_8C_6 \quad {}_8C_7 \quad {}_8C_8$$

SOLUTION



Checkpoint 3

Use the eighth row of Pascal's Triangle to find the binomial coefficients.

$${}_9C_3 \quad {}_9C_5 \quad {}_9C_7$$

■

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Binomial Expansions

As mentioned at the beginning of this section, when you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial.** Pascal's Triangle and the formulas for binomial coefficients give you a way to expand binomials, as demonstrated in the next two examples.

Example 4 Expanding a Binomial

Write the expansion of the expression

$$(x + 1)^3.$$

SOLUTION The binomial coefficients from the third row of Pascal's Triangle are

1, 3, 3, 1.

So, the expansion is

$$\begin{aligned}(x + 1)^3 &= (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)\\ &= x^3 + 3x^2 + 3x + 1.\end{aligned}$$

✓Checkpoint 4

Write the expansion of the expression

$$(x + 2)^5.$$

To expand binomials representing *differences,* rather than sums, you alternate signs. Here are two examples.

$$\begin{aligned}(x - 1)^3 &= [x + (-1)]^3\\ &= x^3 - 3x^2 + 3x - 1\\ (x - 1)^4 &= [x + (-1)]^4\\ &= x^4 - 4x^3 + 6x^2 - 4x + 1\end{aligned}$$

Example 5 Expanding a Binomial

Write the expansion of each expression.

a. $(2x - 3)^4$ **b.** $(x - 2y)^4$

SOLUTION The binomial coefficients from the fourth row of Pascal's Triangle are

1, 4, 6, 4, 1.

The expansions are shown below.

a. $\begin{aligned}(2x - 3)^4 &= (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4)\\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$

b. $\begin{aligned}(x - 2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4\\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$

✓Checkpoint 5

Write the expansion of the expression

$$(3x + 1)^5.$$

Sometimes you will need to find a specific term in a binomial expansion. Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$th term is

$${}_nC_r x^{n-r}y^r.$$

Note that this formula is for the $(r + 1)$th term. So, r is one less than the number of the term you need. For instance, to find the third term of the expression in Example 5(a), you could use this formula with $r = 2$ and $n = 4$ to obtain

$${}_4C_2(2x)^{4-2}(-3)^2 = 6(4x^2)(9) = 216x^2.$$

Example 6 Finding a Term or Coefficient in a Binomial Expansion

a. Find the sixth term of $(a + 2b)^8$.

b. Find the coefficient of the term a^6b^5 in the expansion of $(3a - 2b)^{11}$.

SOLUTION

a. Because the formula ${}_nC_r x^{n-r}y^r$ is for the $(r + 1)$th term, r is one less than the number of the term you need. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$, and $y = 2b$.

$$\begin{aligned} {}_nC_r x^{n-r}y^r &= {}_8C_5 a^{8-5}(2b)^5 \\ &= 56 \cdot a^3 \cdot (32b^5) \\ &= 1792a^3b^5 \end{aligned}$$

b. Note that $(3a - 2b)^{11} = [3a + (-2b)]^{11}$. So, $n = 11$, $r = 5$, $x = 3a$, and $y = -2b$. Substitute these values to obtain

$$\begin{aligned} {}_nC_r x^{n-r}y^r &= {}_{11}C_5(3a)^6(-2b)^5 \\ &= (462)(729a^6)(-32b^5) \\ &= -10{,}777{,}536a^6b^5. \end{aligned}$$

So, the coefficient is $-10{,}777{,}536$.

✓Checkpoint 6

Find the specified term in the expansion of each binomial.

a. $(2a + b)^{10}$, sixth term

b. $(3a + 5b)^6$, fourth term



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SUMMARIZE (Section 7.4)

1. State the Binomial Theorem *(page 568)*. For examples of using the Binomial Theorem to find binomial coefficients, see Examples 1 and 2.
2. Explain how Pascal's Triangle can be used to find binomial coefficients *(page 570)*. For an example of using Pascal's Triangle, see Example 3.
3. Explain what is meant by expanding a binomial *(page 571)*. For examples of expanding a binomial, see Examples 4 and 5.
4. Explain how to find a specific term in a binomial expansion *(page 572)*. For an example of finding a specific term in a binomial expansion, see Example 6.

SKILLS WARM UP 7.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, 0.5, and 7.1.

In Exercises 1–6, perform the indicated operations and simplify.

1. $5x^2(x^3 + 3)$
2. $(x + 5)(x^2 - 3)$
3. $(x + 4)^2$
4. $(2x - 3)^2$
5. $x^2y(3xy^{-2})$
6. $(-2z)^5$

In Exercises 7–10, evaluate the expression.

7. $5!$
8. $\frac{8!}{5!}$
9. $\frac{10!}{7!}$
10. $\frac{6!}{3!3!}$

Exercises 7.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding Binomial Coefficients In Exercises 1–12, find the binomial coefficient. *See Examples 1 and 2.*

1. ${}_6C_2$
2. ${}_9C_4$
3. ${}_{10}C_4$
4. ${}_{10}C_6$
5. ${}_{12}C_0$
6. ${}_{12}C_{12}$
7. ${}_{20}C_{15}$
8. ${}_{20}C_5$
9. $\binom{7}{7}$
10. $\binom{20}{0}$
11. $\binom{100}{98}$
12. $\binom{100}{2}$

Using Pascal's Triangle In Exercises 13–20, evaluate using Pascal's Triangle. *See Example 3.*

13. ${}_4C_2$
14. ${}_4C_3$
15. ${}_7C_4$
16. ${}_6C_3$
17. $\binom{8}{5}$
18. $\binom{9}{8}$
19. $\binom{10}{8}$
20. $\binom{7}{5}$

Expanding a Binomial In Exercises 21–40, use the Binomial Theorem to expand the expression. Simplify your answer. *See Examples 4 and 5.*

21. $(x + 1)^4$
22. $(x + 1)^6$
23. $(x + 6)^4$
24. $(x + 5)^5$
25. $(x - 2)^4$
26. $(x - 2)^5$
27. $(x + y)^5$
28. $(x + y)^6$
29. $(3s + 2)^4$
30. $(2s + 3)^5$
31. $(3 - 2s)^6$
32. $(2 - 3s)^4$
33. $(x - y)^5$
34. $(2x - y)^5$
35. $(1 - 2x)^3$
36. $(1 - 3x)^4$
37. $(x^2 - y^2)^5$
38. $(x^2 + y^2)^6$
39. $\left(\frac{1}{x} + y\right)^5$
40. $\left(x - \frac{3}{y}\right)^3$

Expanding Complex Numbers In Exercises 41–46, use the Binomial Theorem to expand the complex number. Simplify your answer by using the fact that $i^2 = -1$.

41. $(1 + i)^4$
42. $(2 - i)^5$
43. $\left(3 - \sqrt{-4}\right)^6$
44. $\left(5 + \sqrt{-16}\right)^3$
45. $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)^3$
46. $\left(\frac{3}{4} - \frac{\sqrt{2}}{4}i\right)^4$

Using Pascal's Triangle In Exercises 47–50, expand the expression by using Pascal's Triangle to determine the coefficients.

47. $(2t - 1)^5$
48. $(x + 4)^5$
49. $(3 + 2z)^4$
50. $(3y + 2)^6$

Finding a Term in a Binomial Expansion In Exercises 51–58, find the specified nth term in the expansion of the binomial. *See Example 6.*

Binomial	*Term*
51. $(x + y)^{10}$	4th term
52. $(x - y)^6$	7th term
53. $(x - 6y)^5$	3rd term
54. $(x - 10z)^7$	4th term
55. $(4x + 3y)^9$	8th term
56. $(5a + 6b)^5$	5th term
57. $(10x - 3y)^{12}$	10th term
58. $(7x + 2y)^{15}$	7th term

Finding a Coefficient in a Binomial Expansion In Exercises 59–66, find the coefficient a of the term in the expansion of the binomial. *See Example 6.*

Binomial	*Term*
59. $(x + 2)^{10}$	ax^5
60. $(x + 3)^{12}$	ax^8
61. $(x - 2y)^{10}$	ax^8y^2

Binomial	*Term*
62. $(4x - y)^{10}$	ax^3y^7
63. $(3x - 2y)^9$	ax^5y^4
64. $(2x - 3y)^{11}$	ax^3y^8
65. $(x^2 - 1)^8$	ax^6
66. $(x^2 + 2)^{12}$	ax^{10}

Expanding a Binomial In Exercises 67–70, use the Binomial Theorem to expand the expression.

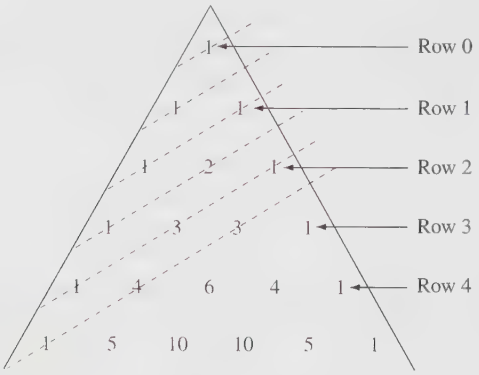
67. $\left(\frac{1}{2} + \frac{1}{2}\right)^7$

68. $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$

69. $(0.6 + 0.4)^5$

70. $(0.35 + 0.65)^6$

71. Finding a Pattern Describe the pattern formed by the sums of the numbers along the diagonal lines shown in Pascal's Triangle (see figure).



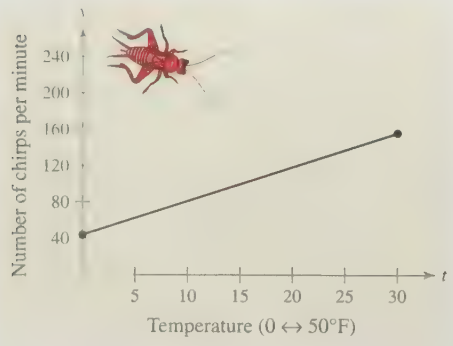
72. Crickets You conduct an experiment and determine that the numbers of chirps $f(t)$ that a cricket makes each minute at various temperatures can be approximated by

$$f(t) = 0.004t^2 + 3.73t + 44, \quad 0 \le t \le 30$$

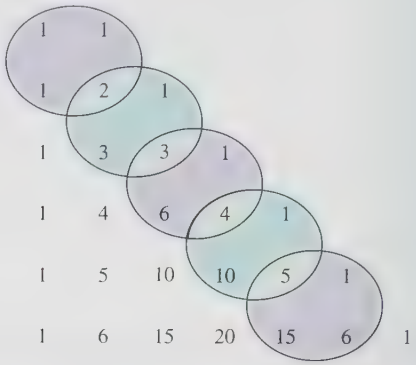
where $t = 0$ represents 50°F (see figure).

(a) You want to adjust this model so that $t = 0$ corresponds to 60°F rather than 50°F. To do this, you shift the graph of f 10 units *to the left* and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.



73. Finding a Pattern Use each of the encircled groups of numbers in the figure to form a 2×2 matrix. Find the determinant of each matrix. Describe the pattern.



74. HOW DO YOU SEE IT? The expansions of $(x + y)^4$, $(x + y)^5$, and $(x + y)^6$ are shown below.

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

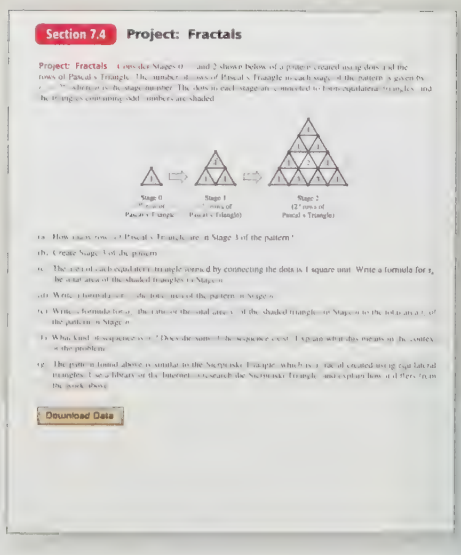
$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

(a) Explain how the exponent of a binomial is related to the number of terms in its expansion.

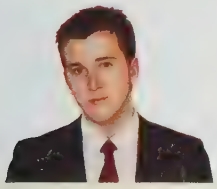
(b) How many terms are in the expansion of $(x + y)^n$?

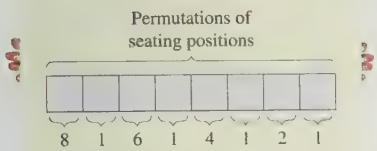
75. Project: Fractals To work an extended application involving fractals, visit this text's website at *www.cengagebrain.com.*

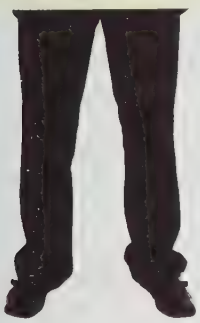


7.5 Counting Principles

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.







In Exercise 25 on page 583, you will use counting principles to find the number of ways four couples can sit at a concert.

Simple Counting Problems

This section and Section 7.6 offer a brief introduction to some of the basic counting principles and their application to probability.

Example 1 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *replaced in the box*. Then, a second piece of paper is drawn from the box, and its number is written down. Finally, the two numbers are added together. In how many different ways can a sum of 12 be obtained?

SOLUTION To solve this problem, count the different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

From this list, you can see that a sum of 12 can occur in five different ways.

✓Checkpoint 1

In Example 1, in how many different ways can a sum of 14 be obtained? ■

Example 2 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box and, *without replacing the paper*, a second piece of paper is drawn. The numbers on the pieces of paper are written down and totaled. In how many different ways can a sum of 12 be obtained?

SOLUTION To solve this problem, count the different ways that a sum of 12 can be obtained using two *different* numbers from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can be obtained in four different ways.

✓Checkpoint 2

In Example 2, in how many different ways can a sum of 14 be obtained? ■

The counting problems in Examples 1 and 2 can be distinguished by saying that the random selection in Example 1 occurs **with replacement,** whereas the random selection in Example 2 occurs **without replacement,** which eliminates the possibility of choosing two 6's.

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Counting Principles

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is the best way to solve a counting problem. Some events, however, can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle.**

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1, E_2, and E_3 can occur is $m_1 \cdot m_2 \cdot m_3$.

Example 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

SOLUTION This experiment has two events. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of pairs of letters is $26 \cdot 26 = 676$.

Checkpoint 3

The extension for a file name on a computer typically consists of three letters from the English alphabet. How many different extensions are possible?

Example 4 Using the Fundamental Counting Principle

Telephone numbers in the United States currently have 10 digits. The first three are the *area code* and the next seven are the *local telephone number*. How many different local telephone numbers are possible within each area code? (Note that at this time, a local telephone number cannot begin with 0 or 1.)

SOLUTION Because the first digit cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of local telephone numbers that are possible within each area code is $8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000$.

Checkpoint 4

The catalog number for a particular product consists of one letter from the English alphabet followed by a five-digit number. How many different catalog numbers are possible?

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 5 Finding the Number of Permutations of n Elements

How many permutations of the letters

A, B, C, D, E, and F

are possible?

SOLUTION Consider the following reasoning.

First position: Any of the *six* letters.

Second position: Any of the remaining *five* letters.

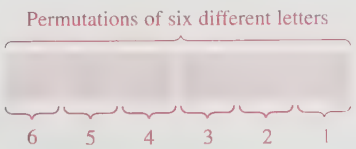
Third position: Any of the remaining *four* letters.

Fourth position: Any of the remaining *three* letters.

Fifth position: Either of the remaining *two* letters.

Sixth position: The *one* remaining letter.

So, the numbers of choices for the six positions are shown below.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 720.$$

Checkpoint 5

How many permutations of the letters

W, X, Y, and Z

are possible?

Number of Permutations of n Elements

The number of permutations of n distinct elements is given by

$$n \cdot (n - 1) \cdot \cdots 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n distinct elements can be ordered.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you might want to choose (and order) r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time.**



Example 6 Counting Kidney Donors

A patient with end-stage kidney (renal) disease has eight family members who are potential kidney donors. How many *possible* orders are there for a best match, a second-best match, and a third-best match?

SOLUTION Here are the different possibilities.

Best match (first position):	*Eight* choices
Second-best match (second position):	*Seven* choices
Third-best match (third position):	*Six* choices

The numbers of choices for the three positions are shown below.

Different orders of donors

8 7 6

So, using the Fundamental Counting Principle, you can determine that the number of different orders is $8 \cdot 7 \cdot 6 = 336$.

Checkpoint 6

In how many different ways can the top three potential donors in Example 6 be ordered when six family members are potential donors?

Permutations of n Elements Taken r at a Time

The number of permutations of n distinct elements taken r at a time is

$$
\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!} \\
&= n(n-1)(n-2) \cdot \cdots \cdot (n-r+1).
\end{aligned}
$$

Using this formula, you can rework Example 6 to find the number of permutations of the eight potential donors taken three at a time.

$$
\begin{aligned}
{}_8P_3 &= \frac{8!}{(8-3)!} \\
&= \frac{8!}{5!} \\
&= \frac{(8 \cdot 7 \cdot 6) \cdot 5!}{5!} \\
&= 336
\end{aligned}
$$

The answer is 336 different orders, the same answer obtained in Example 6.

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Remember that for permutations, order is important. So, when finding the possible permutations of the letters A, B, C, and D taken three at a time, the permutations

ABD and BAD

would be different because the *order* of the elements is different.

Consider, however, the possible permutations of the letters A, A, B, and C. The total number of permutations of these four letters would be

$${}_4P_4 = 4!.$$

Not all of these arrangements, however, would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdots + n_k.$$

Then the number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!}.$$

Example 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

SOLUTION This word has six letters, so $n = 6$, but you have to take into account that there are three A's, two N's, and one B. So, the number of distinguishable ways the letters can be written is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3!} = \frac{6!}{3! \cdot 2! \cdot 1!}$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 2!}$$

$$= 60.$$

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN	AABNAN	AABNNA
AANABN	AANANB	AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN	ABNANA	ABNNAA
ANAABN	ANAANB	ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB	ANNABA	ANNBAA
BAAANN	BAANAN	BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA	NAAABN	NAAANB
NAABAN	NAABNA	NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA	NBAAAN	NBAANA
NBANAA	NBNAAA	NNAAAB	NNAABA	NNABAA	NNBAAA

Checkpoint 7

In how many distinguishable ways can the letters in MITOSIS be written?

Combinations

When you count the number of possible permutations of a set of elements, *order* is important. As a final topic in this section, you will look at a method of selecting subsets of a larger set in which order is *not important*. Such subsets are called **combinations of n elements taken r at a time.** For instance, the combinations

$$\{A, B, C\} \quad \text{and} \quad \{B, A, C\}$$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is *not important*. So, you would count only one of the two sets. A common example of how a combination occurs is a card game in which the player is free to reorder the cards after they have been dealt.

Example 8 Combinations of n Elements Taken r at a Time

In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

SOLUTION The following subsets represent the different combinations of three letters that can be chosen from the five letters.

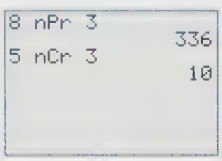
$\{A, B, C\}$ $\{A, B, D\}$
$\{A, B, E\}$ $\{A, C, D\}$
$\{A, C, E\}$ $\{A, D, E\}$
$\{B, C, D\}$ $\{B, C, E\}$
$\{B, D, E\}$ $\{C, D, E\}$

From this list, you can conclude that there are 10 different ways that three letters can be chosen from the five letters.

TECH TUTOR



Most graphing utilities have keys that will evaluate the formulas for the number of permutations or combinations of n elements taken r at a time, as shown in the screen below. Consult the user's guide for your graphing utility for specific instructions on how to evaluate permutations and combinations.



Checkpoint 8

In how many different ways can two letters be chosen from the letters J, K, L, M, and N? (The order of the two letters is not important.)

Combinations of n Elements Taken r at a Time

The number of combinations of n elements taken r at a time is

$$_nC_r = \frac{n!}{(n-r)!r!}.$$

Note that the formula for $_nC_r$ is the same one given for binomial coefficients. To see how this formula is used, solve the counting problem in Example 8. In that problem, you are asked to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$$\begin{aligned} {}_5C_3 &= \frac{5!}{2!3!} \\ &= \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3} \cdot 2!}{2! \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \\ &= 10 \end{aligned}$$

which is the same answer obtained in Example 8.

Example 9 Counting Card Hands

A standard poker hand consists of five cards dealt from a deck of 52. How many different poker hands are possible? (After the cards are dealt the player may reorder them, and therefore order is not important.)

SOLUTION Using the formula for the number of combinations of 52 elements taken five at a time, the number of different poker hands is

$$_{52}C_5 = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960 \text{ different hands.}$$

✓Checkpoint 9

In three-card poker, each player is dealt three cards from a deck of 52. How many different three-card poker hands are possible? (Order is not important.)

Example 10 Choosing Between Permutations and Combinations

Decide whether each scenario should be counted using permutations or combinations.

a. Number of different arrangements of three types of flowers from an array of 20 types

b. Number of different three-digit pin numbers for a debit card

SOLUTION To determine which counting principle is necessary to solve each problem, ask yourself two questions: (1) Is the order of the elements important? If yes, then you should solve the problem using permutations. (2) Are the chosen elements a subset of a larger set in which order is not important? If yes, then you should solve the problem using combinations.

a. The order of the flowers is not important. So, the number of possible flower arrangements should be counted using combinations.

b. The order of the digits in the pin number matters. So, the number of possible pin numbers should be counted using permutations.



✓Checkpoint 10

Should the number of different nine-digit social security numbers be counted using permutations or combinations?

SUMMARIZE (Section 7.5)

1. State the Fundamental Counting Principle *(page 576)*. For examples of using the Fundamental Counting Principle, see Examples 3 and 4.
2. State the definition of permutation and explain how to find the number of permutations of n elements *(page 577)*, the number of permutations of n elements taken r at a time *(page 578)*, and the number of distinguishable permutations of n elements *(page 579)*. For examples of finding numbers of permutations, see Examples 5, 6, and 7.
3. Explain how to find the number of combinations of n elements taken r at a time *(page 580)*. For examples of finding numbers of combinations, see Examples 8 and 9.



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SKILLS WARM UP 7.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 7.1, and 7.4.

In Exercises 1–4, evaluate the expression.

1. $13 \cdot 8^2 \cdot 2^3$
2. $10^2 \cdot 9^3 \cdot 4$
3. $\dfrac{12!}{2!(7!)(3!)}$
4. $\dfrac{25!}{22!}$

In Exercises 5 and 6, find the binomial coefficient.

5. ${}_{12}C_7$
6. ${}_{25}C_{22}$

In Exercises 7–10, simplify the expression.

7. $\dfrac{n!}{(n-4)!}$
8. $\dfrac{(2n)!}{4(2n-3)!}$
9. $\dfrac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2n)}{2^n}$
10. $\dfrac{3 \cdot 6 \cdot 9 \cdot 12 \cdot \cdots \cdot (3n)}{3^n}$

Exercises 7.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Random Selection In Exercises 1–10, determine the number of ways a computer can randomly generate the specified integer(s) from 1 through 15. *See Examples 1 and 2.*

1. An odd integer
2. An even integer
3. A prime integer
4. An integer that is greater than 9
5. An integer that is divisible by 4
6. An integer that is divisible by 5
7. Two integers whose sum is 20
8. Two integers whose sum is 16
9. Two *distinct* integers whose sum is 20
10. Two *distinct* integers whose sum is 16

11. **Job Applicants** A small college needs two additional faculty members: a chemist and a statistician. There are five applicants for the chemistry position and six applicants for the statistics position. In how many ways can these positions be filled?
12. **Dissections** In a physiology class, a student must dissect three different organisms. The student can select one of nine earthworms, one of four frogs, and one of seven fetal pigs. In how many ways can the student select the specimens?
13. **Toboggan Ride** Six people are lining up for a ride on a toboggan, but only two of the six are willing to take the first position. With that constraint, in how many ways can the six people be seated on the toboggan? Draw a diagram to illustrate the number of ways that the people can sit.
14. **Course Schedule** A college student needs to schedule five prescribed courses for next semester. Only three of the five courses are able to be scheduled as the first class of the day. With that constraint, in how many ways can she select her schedule? Draw a diagram to illustrate the number of ways she can select her schedule.
15. **License Plate Numbers** In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. How many distinct license plate numbers can be formed?
16. **License Plate Numbers** In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. To avoid confusion between "O" and "zero" and "I" and "one," the letters "O" and "I" are not used. How many distinct license plate numbers can be formed?
17. **True-False Exam** In how many ways can a 10-question true-false exam be answered? (Assume that no questions are omitted.)
18. **True-False Exam** In how many ways can a 15-question true-false exam be answered? (Assume that no questions are omitted.)
19. **Three-Digit Numbers** How many three-digit numbers can be formed under each condition?
    (a) The leading digit cannot be 0.
    (b) The leading digit cannot be 0 and no repetition of digits is allowed.
    (c) The leading digit cannot be 0 and the number must be a multiple of 5.

20. Four-Digit Numbers How many four-digit numbers can be formed under each condition?

(a) The leading digit cannot be 0.

(b) The leading digit cannot be 0 and no repetition of digits is allowed.

(c) The leading digit cannot be 0 and the number must be a multiple of 5.

21. Combination Lock A combination lock will open when the right choice of three numbers (from 1 to 40, inclusive) is selected. How many different lock combinations are possible?

22. Combination Lock A combination lock will open when the right choice of three numbers (from 1 to 50, inclusive) is selected. How many different lock combinations are possible?

23. Chemistry Lab In a chemistry lab, five oxidation-reduction reactions can be performed in any order. How many different orders are possible for the five reactions?

24. Biology Lab In a plant biology lab, seven different experiments can be performed in any order. How many different orders are possible for the seven experiments?

25. Concert Seats Four couples have reserved seats in a given row for a concert. In how many different ways can they be seated, given the following conditions?

(a) There are no restrictions.

(b) The two members of each couple wish to sit together.

26. Single File In how many orders can five girls and four boys walk through a doorway in single file, given the following conditions?

(a) There are no restrictions.

(b) The boys go before the girls.

(c) The girls go before the boys.

Finding a Permutation In Exercises 27–36, evaluate ${}_nP_r$ using the formula on page 578.

27. ${}_4P_3$ **28.** ${}_5P_4$

29. ${}_4P_4$ **30.** ${}_7P_7$

31. ${}_6P_2$ **32.** ${}_{10}P_5$

33. ${}_8P_0$ **34.** ${}_{11}P_0$

35. ${}_{12}P_4$ **36.** ${}_{48}P_3$



Finding a Permutation Using a Graphing Utility In Exercises 37–40, evaluate ${}_nP_r$ using a graphing utility.

37. ${}_{20}P_6$ **38.** ${}_{10}P_8$

39. ${}_{120}P_4$ **40.** ${}_{100}P_3$

Finding Distinguishable Permutations In Exercises 41–48, find the number of distinguishable permutations of the group of letters. *See Example 7.*

41. B, R, O, O, M **42.** V, I, V, I, D

43. A, S, S, E, T, S **44.** A, C, C, R, U, A, L

45. S, P, E, C, I, E, S **46.** M, A, M, M, A, L

47. A, L, G, E, B, R, A **48.** M, I, S, S, I, S, S, I, P, P, I

Finding a Combination In Exercises 49–58, evaluate ${}_nC_r$ using the formula on page 580. *See Examples 8 and 9.*

49. ${}_4C_3$ **50.** ${}_5C_4$

51. ${}_4C_4$ **52.** ${}_7C_7$

53. ${}_6C_2$ **54.** ${}_{10}C_5$

55. ${}_8C_0$ **56.** ${}_{11}C_0$

57. ${}_{12}C_4$ **58.** ${}_{48}C_3$



Finding a Combination Using a Graphing Utility In Exercises 59–62, evaluate ${}_nC_r$ using a graphing utility.

59. ${}_{20}C_5$ **60.** ${}_{10}C_7$

61. ${}_{42}C_5$ **62.** ${}_{50}C_6$

Simplifying Expressions In Exercises 63–66, simplify the expression.

63. ${}_nP_3$ **64.** ${}_nC_3$

65. ${}_nP_{n-1}$ **66.** ${}_nC_{n-1}$

Choosing Between Permutations and Combinations In Exercises 67–70, decide whether the scenario should be counted using permutations or combinations. Explain your reasoning. *See Example 10.*

67. Number of ways 10 people can line up in a row for concert tickets

68. Number of ways three different roles can be filled by 10 people auditioning for a play

69. Number of different three-topping pizzas that can be made from an assortment of 15 different toppings

70. Number of ways a jury of 12 people can be selected from a group of 50 people

71. Choosing Officers From a pool of 15 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many different ways can the offices be filled if each of the 15 candidates can hold any office?

72. Bike Race There are 10 bicyclists entered in a race. In how many different ways can the top three places be decided?

73. Forming an Experimental Group In order to conduct an experiment, five students are randomly selected from a class of 30. How many different groups of five students are possible?

74. Lab Practical A student may answer any 15 questions from a total of 20 questions on a biology lab practical. In how many ways can the student select the questions?

75. Lottery In California's Fantasy 5 Bonus Bucks game, a player chooses five distinct numbers from 1 to 39. In how many ways can a player select the five numbers? (The order of selection is not important.)

76. Lottery In Massachusetts's Cash WinFall game, a player chooses six distinct numbers from 1 to 46. In how many ways can a player select the six numbers? (The order of selection is not important.)

77. Number of Subsets How many subsets of four elements can be formed from a set of 100 elements?

78. Number of Subsets How many subsets of five elements can be formed from a set of 80 elements?

79. Game Show On a game show, 10 audience members are randomly chosen to be eligible contestants. Six of the 10 eligible contestants are chosen to play a game on stage. You and your friend are two of the 10 eligible contestants. In how many ways can the six players be chosen from the group of eligible contestants given that you and your friend are chosen to play a game?

80. Job Applicants An employer interviews 10 people for four openings at a company. Four of the 10 people are women. All 10 applicants are qualified. In how many ways can the employer fill the four positions, given the following conditions?

(a) There are no restrictions.

(b) Exactly two selections are women.

81. Poker Hand Five cards are selected from a standard deck of 52 playing cards. In how many ways can you get a full house? (A full house consists of three of one kind and two of another.)

82. Poker Hand Five cards are selected from a standard deck of 52 playing cards. In how many ways can you get a straight flush? (A straight flush consists of five cards that are in order and of the same suit.)

83. Committee A six-member research committee at a local college is to be formed. It will consist of one administrator, three faculty members, and two students. There are seven administrators, 12 faculty members, and 20 students in contention for the committee. How many six-member committees are possible?

84. Law Enforcement A police department uses computer imaging to create digital photographs of alleged perpetrators from eyewitness accounts. One software package contains 195 hairlines, 99 sets of eyes and eyebrows, 89 noses, 105 mouths, and 74 chins and cheek structures.

(a) Find the possible number of different faces that the software could create.

(b) An eyewitness can clearly recall the hairline and eyes and eyebrows of a suspect. How many different faces can be produced with this information?

85. Think About It Can your calculator evaluate ${}_{100}P_{80}$? If not, explain why.

86. HOW DO YOU SEE IT? Without calculating, determine whether the value of ${}_nP_r$ is greater than the value of ${}_nC_r$ for the values of n and r given in the table. Complete the table using yes (Y) or no (N). Is the value of ${}_nP_r$ always greater than the value of ${}_nC_r$? Explain.

n \ r	0	1	2	3	4	5	6	7
1								
2								
3								
4								
5								
6								
7								



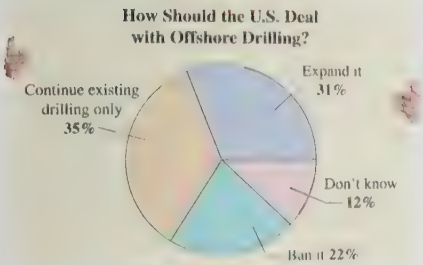
Business Capsule

Advanced Process Combinatorics applies mathematical approaches to a wide range of endeavors, from scheduling of pharmaceutical manufacturing to improving oncology radiation treatments. The company's Dispatch Planning System (DPS) software, created for the beverage industry, downloads customer orders and uses combinations and permutations to find the most efficient way to build pallets for optimized loading and delivery.

87. Research Project Use your campus library, the Internet, or some other reference source to find information about a company that uses combinations or permutations to perform pallet optimization or some other business function. Write a brief report about the company. Explain why the use of combinations or permutations is needed to perform the business function.

7.6 Probability







In Exercise 84 on page 595, you will find probabilities based on the attitudes of Americans about offshore drilling for gas and oil.

- Determine the sample space of an experiment and find the probability of an event.
- Find the probability of mutually exclusive events.
- Find the probability of independent events.

The Probability of an Event

In measuring the uncertainties of everyday life, ambiguous terminology, such as "fairly certain" and "highly unlikely," is often used. Mathematics attempts to assign a number to the likelihood of the occurrence of an event. This measurement is called the **probability** that the event will occur. For instance, when a fair coin is tossed, the probability that it will land heads up is $\frac{1}{2}$.

Any happening whose result is uncertain is called an **experiment.** The possible results of an experiment are **outcomes,** the set of all possible outcomes of an experiment is the **sample space,** and any subcollection of a sample space is an **event.**

Example 1 Finding the Sample Space

Find the sample space for each experiment.

a. One coin is tossed.

b. Two coins are tossed.

SOLUTION

a. Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space is $S = \{H, T\}$.

b. Because either coin can land heads up or tails up, the possible outcomes are

HH = heads up on both coins

HT = heads up on first coin and tails up on second coin

TH = tails up on first coin and heads up on second coin

TT = tails up on both coins.

The sample space is $S = \{HH, HT, TH, TT\}$. This list distinguishes between the two cases HT and TH, even though these two outcomes appear to be similar.

Another way to list the possible outcomes is by using a **tree diagram,** as shown in Figure 7.1.



FIGURE 7.1

Checkpoint 1

Find the sample space when a six-sided die is tossed.

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When an event E has equally likely outcomes and its sample space S has equally likely outcomes, the probability of event E is

$$P(E) = \frac{\text{Number of outcomes in event } E}{\text{Number of outcomes in sample space } S}.$$

More formally, the number of outcomes in event E is denoted by $n(E)$, and the number of outcomes in the sample space S is denoted by $n(S)$.

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is

$$P(E) = \frac{n(E)}{n(S)}.$$

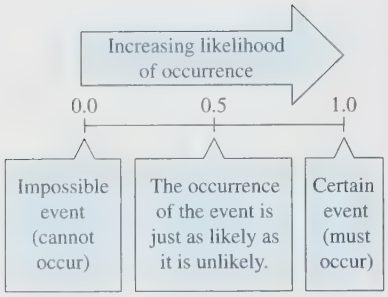


FIGURE 7.2

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number between 0 and 1, as shown in Figure 7.2. So, for any event E, $0 \le P(E) \le 1$.

Properties of the Probability of an Event

Let E be an event that is a subset of a finite sample space S.

1. $0 \le P(E) \le 1$
2. If $P(E) = 0$, then E *cannot occur* and is called an **impossible event.**
3. If $P(E) = 1$, then E *must occur* and is called a **certain event.**



Standard Deck of Playing Cards

FIGURE 7.3

Example 2 Finding the Probability of an Event

a. Two coins are tossed. What is the probability that both land heads up?

b. A card is drawn at random from a standard deck of playing cards (see Figure 7.3). What is the probability that it is an ace?

SOLUTION

a. Following the procedure in Example 1(b), let $E = \{HH\}$ and $S = \{HH, HT, TH, TT\}$. The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

b. Because there are 52 cards in a standard deck of playing cards and there are four aces (one in each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

Checkpoint 2

A card is drawn at random from a standard deck of playing cards. What is the probability that the card is a diamond? ■

A probability can be written as a fraction, a decimal, or a percent. For instance, in Example 2(a), the probability of getting two heads can be written as $\frac{1}{4}$, 0.25, or 25%.

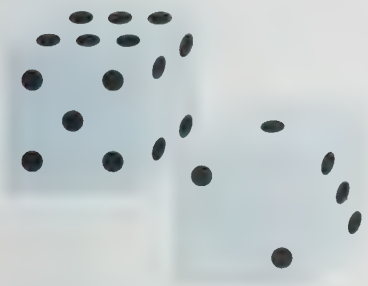


FIGURE 7.4

Example 3 Finding the Probability of an Event

Two six-sided dice are tossed. What is the probability that the total of the two dice is 7? (See Figure 7.4.)

SOLUTION Because there are six possible outcomes on each die, you can use the Fundamental Counting Principle to conclude that there are $6 \cdot 6$ or 36 different outcomes when two dice are tossed. To find the probability of rolling a total of 7, you must first count the number of ways this can occur.

First Die	1	2	3	4	5	6
Second Die	6	5	4	3	2	1

So, a total of 7 can be rolled in six ways, which means that the probability of rolling a total of 7 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

TECH TUTOR

You can use the *random integer* feature of a graphing utility to simulate the tossing of a die or set of dice. For instance, the result of rolling a pair of dice five times and summing the outcomes is shown below.

```
randInt(1,6,5)+r
andInt(1,6,5)
   {11 7 4 10 5}
```

Checkpoint 3

Two six-sided dice are tossed. What is the probability that the total of the two dice is 4?

Example 4 Finding the Probability of an Event

Twelve-sided dice can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 appears twice on each die, as shown in Figure 7.5. Can these dice be used in any game requiring ordinary six-sided dice without changing the probabilities of different outcomes?

SOLUTION For an ordinary six-sided die, each of the numbers

1, 2, 3, 4, 5, and 6

occurs only once, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For a twelve-sided die, each number occurs twice, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

So, the twelve-sided dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of different outcomes.

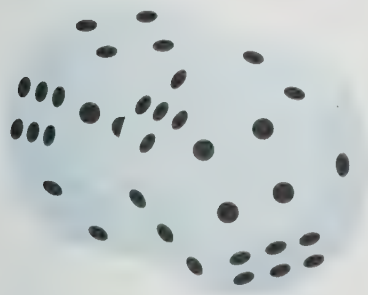


FIGURE 7.5

Checkpoint 4

Two twelve-sided dice (as described in Example 4) are tossed. What is the probability that the total of the two dice is less than 5?

In Examples 2, 3, and 4, you simply counted the outcomes in the desired events. For larger sample spaces, however, you should use the counting principles discussed in Section 7.5.

Example 5 The Probability of Winning a Lottery

In Louisiana's Lotto game, a player chooses six different two-digit numbers from 01 to 40. To win the jackpot, the six numbers must match the six numbers drawn (in any order) by the lottery commission. What is the probability of winning the jackpot?

SOLUTION Because the order of the six numbers does not matter, use the formula for the number of combinations of 40 elements taken six at a time.

$$n(S) = {}_{40}C_6 = \frac{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 3{,}838{,}380$$

For a person buying only one ticket, the probability of winning the jackpot is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{3{,}838{,}380}.$$

Checkpoint 5

In Pennsylvania's Cash 5 game, a player chooses five different numbers from 1 to 43. To win the jackpot, the five numbers must match the five numbers drawn (in any order) by the lottery commission. What is the probability of winning the jackpot?

Example 6 Random Selection

The numbers of colleges and universities in various regions of the United States in 2010 are shown in Figure 7.6. One institution is selected at random. What is the probability that the institution is in one of the three southern regions? *(Source: U.S. National Center for Education Statistics)*

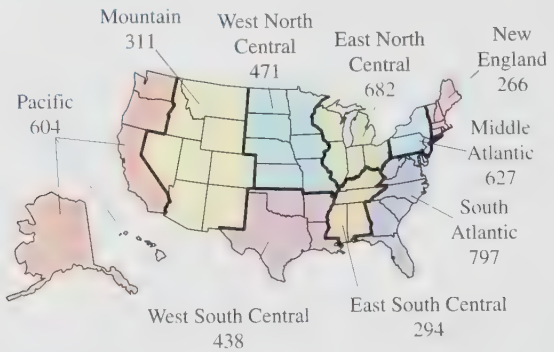


FIGURE 7.6

SOLUTION From the figure, the total number of colleges and universities is 4490. Because there are

$$797 + 294 + 438 = 1529$$

colleges and universities in the three southern regions, the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1529}{4490} \approx 0.341.$$

Checkpoint 6

In Example 6, what is the probability that the randomly selected institution is in the Pacific region?



Roger Jegg / Shutterstock.com

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** when A and B have no outcomes in common. In the terminology of sets, the **intersection of A and B** is the empty set and $P(A \cap B) = 0$. For instance, when two dice are tossed, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, you can *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, then the probability of A or B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example 7 The Probability of a Union

One card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

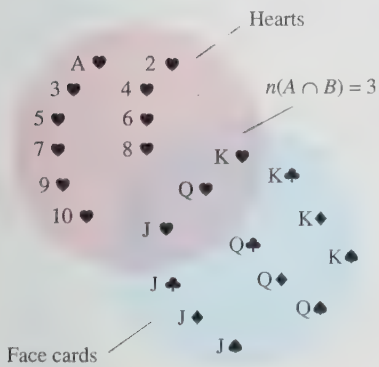


FIGURE 7.7

SOLUTION Because the deck has 13 hearts, the probability of selecting a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, because the deck has 12 face cards, the probability of selecting a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts and face cards (see Figure 7.7), it follows that the events A and B are *not* mutually exclusive, and that

$$P(A \cap B) = \frac{3}{52}.$$

Finally, applying the formula for the probability of the union of two events, you can conclude that the probability of selecting a heart or a face card is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &\approx 0.423. \end{aligned}$$

Checkpoint 7

One card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either a spade or an eight?



Example 8 Probability of Mutually Exclusive Events

The human resources department of a company has compiled data showing the number of years of service for each employee. The results are shown in the table.

Years of service	Number of employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	37
35–39	21
40–44	8

An employee is chosen at random. What is the probability that the employee has had 9 or fewer years of service?

SOLUTION To begin, add the number of employees and find that the total is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. Because there are 157 employees with 0 to 4 years of service, the probability of choosing an employee who has 4 or fewer years of service is

$$P(A) = \frac{157}{529}.$$

Let event B represent choosing an employee with 5 to 9 years of service. Because there are 89 employees with 5 to 9 years of service, the probability of choosing one of these employees is

$$P(B) = \frac{89}{529}.$$

Because A and B have no outcomes in common, you can conclude that these two events are mutually exclusive and that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.465. \end{aligned}$$

So, the probability of choosing an employee who has had 9 or fewer years of service is about 0.465.

Checkpoint 8

In Example 8, what is the probability that the employee chosen at random has had 30 or more years of service?

Independent Events

Two events are **independent** when the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. To find the probability that two independent events will occur, *multiply* the probabilities of the events.

Probability of Independent Events

If A and B are independent events, then the probability that both A and B will occur is

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

This definition can be extended to any number of independent events.

Example 9 Probability of Independent Events

a. The probability that a particular surgery is successful is 0.90. What is the probability that two surgeries are successful? (Assume the surgeries are on different patients.)

b. Two six-sided dice are tossed three times. What is the probability that the total for each toss is 7?

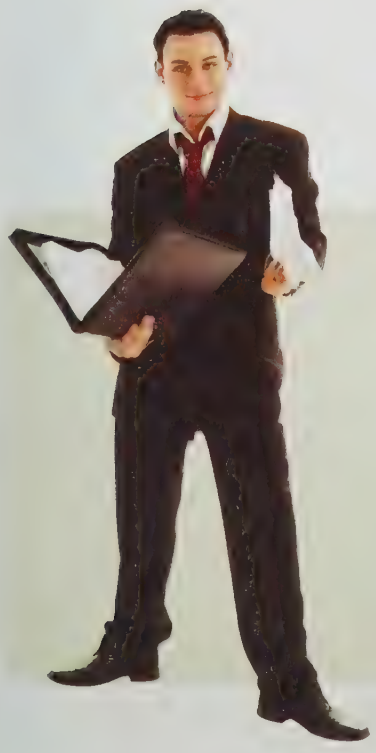
SOLUTION

a. The probability that two surgeries are successful is

$$P(A) \cdot P(A) = (0.90)(0.90) = 0.81.$$

b. From Example 3, you know that the probability of tossing two dice once and getting a total of 7 is $P(A) = \frac{1}{6}$. So, when two dice are tossed three times, the probability that the total is 7 every time is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{216} \approx 0.005.$$



Checkpoint 9

The probability that a particular surgery is successful is 0.65. What is the probability that two surgeries are successful? (Assume the surgeries are on different patients.) ■

SUMMARIZE (Section 7.6)

1. State the definition of sample space *(page 585)*. For an example of finding the sample space for an experiment, see Example 1.
2. State the definition of the probability of an event *(page 586)*. For examples of finding the probability of an event, see Examples 2, 3, 4, 5, and 6.
3. State the definitions of mutually exclusive events and the probability of the union of two events *(page 589)*. For examples of finding the probability of the union of two events, see Examples 7 and 8.
4. State the definitions of independent events and the probability of independent events *(page 591)*. For an example of finding the probability of independent events, see Example 9.

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SKILLS WARM UP 7.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.3, 7.1, and 7.4.

In Exercises 1–8, evaluate the expression.

1. $\frac{1}{4} + \frac{5}{8} - \frac{5}{16}$
2. $\frac{4}{15} + \frac{3}{5} - \frac{1}{3}$
3. $\frac{5 \cdot 4}{5!}$
4. $\frac{5!22!}{27!}$
5. $\frac{4!}{8!12!}$
6. $\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{9!}$
7. $\frac{{}_5C_3}{{}_{10}C_3}$
8. $\frac{{}_{10}C_2 \cdot {}_{10}C_2}{{}_{20}C_4}$

In Exercises 9 and 10, evaluate the expression. (Round your answer to three decimal places.)

9. $\left(\frac{99}{100}\right)^{100}$
10. $1 - \left(\frac{89}{100}\right)^{50}$

Exercises 7.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding the Sample Space In Exercises 1–6, find the sample space for the experiment. *See Example 1.*

1. A coin and a six-sided die are tossed.
2. A six-sided die is tossed twice and the sum of the points is recorded.
3. A taste tester ranks three varieties of yogurt (A, B, and C) according to preference.
4. Two marbles are selected from a sack containing two red marbles, two blue marbles, and one black marble. The color of each marble is recorded.
5. Two county supervisors are selected from five supervisors (A, B, C, D and E) to study a recycling plan.
6. A sales representative makes presentations of a product in three homes. In each home, there may be a sale (denote by S) or there may be no sale (denote by F).

Using a Tree Diagram In Exercises 7–10, make a tree diagram that shows the possible outcomes that make up the sample space for the experiment.

7. Three coins are tossed.
8. A committee rates three programs A, B, and C in order of importance.
9. Two of four vendors A, B, C, and D are given contracts for the upcoming year.
10. Team A plays team B in one semifinal game and team C plays team D in the other semifinal game to determine the two teams in the championship game.

Finding the Probability of an Event In Exercises 11–14, a computer randomly generates an integer from 1 through 50. Find the probability of the event. *See Examples 2, 3, and 4.*

11. An odd number is generated.
12. A number less than 25 is generated.
13. A prime number is generated.
14. A multiple of 5 is generated.

Finding the Probability of an Event In Exercises 15–18, a coin is tossed three times. Find the probability of the event. *See Examples 2, 3, and 4.*

15. Getting exactly one head
16. Getting a tail on the second toss
17. Getting at least one tail
18. Getting at least two heads

Finding the Probability of an Event In Exercises 19–24, two six-sided dice are tossed. Find the probability of the event. *See Examples 2, 3, and 4.*

19. The sum is 5.
20. The sum is less than 10.
21. The sum is at least 7.
22. The sum is 2, 3, or 8.
23. The sum is odd and no more than 7.
24. The sum is odd or a prime number.

Finding the Probability of an Event In Exercises 25–30, a card is selected at random from a standard deck of 52 playing cards. Find the probability of the event. *See Examples 2, 3, and 4.*

25. Getting a red card
26. Getting a face card
27. Not getting a face card
28. Getting a red card that is not a face card
29. Getting a card that is a 6 or less (aces are low)
30. Getting a card that is a 9 or higher (aces are low)

Using Combinations In Exercises 31–36, two marbles are drawn randomly (without replacement) from a bag containing two green, three yellow, and four red marbles. Find the probability of the event.

31. Drawing exactly one red marble

32. Drawing exactly one yellow marble

33. Drawing two green marbles

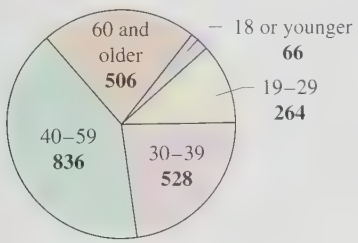
34. Drawing two red marbles

35. Drawing none of the yellow marbles

36. Drawing marbles of different colors

Random Selection In Exercises 37–40, one of a team's 2200 season ticket holders is selected at random to win a prize. The circle graph shows the ages of the season ticket holders. Find the probability of the event. *See Example 6.*

Ages of Season Ticket Holders



37. The winner is younger than 19 years old.

38. The winner is older than 39 years old.

39. The winner is 19 to 39 years old.

40. The winner is younger than 19 years old or older than 59 years old.

The Probability of a Union In Exercises 41–44, one card is selected at random from a standard deck of 52 playing cards. Use a formula to find the probability of the union of the two events. *See Example 7.*

41. The card is a club or a king.

42. The card is a face card or a black card.

43. The card is a face card or a 2.

44. The card is a heart or a spade.

Probability of Mutually Exclusive Events In Exercises 45–48, use the table, which shows the age groups of students in a college sociology class. *See Example 8.*

Age	Number of students
18–19	11
20–21	18
22–30	2
31–40	1

A student from the class is chosen randomly for a project. Find the probability that the student is the given age.

45. 20 or 21 years old

46. 18 to 21 years old

47. Older than 21 years old

48. Younger than 31 years old

Probability of Independent Events In Exercises 49–52, a random number generator selects three numbers from 1 through 10. Find the probability of the event. *See Example 9.*

49. All three numbers are even.

50. All three numbers are less than or equal to 4.

51. Two numbers are less than 5 and the other number is 10.

52. One number is 2, 4, or 6, and the other two numbers are odd.

Finding the Probability of a Complement The *complement of an event A* is the collection of all outcomes in the sample space that are *not* in A. If the probability of A is $P(A)$, then the probability of the complement A' is given by $P(A') = 1 - P(A)$. In Exercises 53–56, you are given the probability that an event *will* happen. Find the probability that the event *will not* happen.

53. $P(E) = 0.75$

54. $P(E) = 0.32$

55. $P(E) = \frac{2}{3}$

56. $P(E) = \frac{7}{8}$

Using the Probability of a Complement In Exercises 57–60, you are given the probability that an event *will not* happen. Find the probability that the event *will* happen.

57. $P(E') = 0.12$

58. $P(E') = 0.84$

59. $P(E') = \frac{13}{20}$

60. $P(E') = \frac{61}{100}$

61. Winning an Election Three people have been nominated for president of a college class. From a small poll, it is estimated that the probability of Jane winning the election is 0.46, and the probability of Larry winning the election is 0.32. What is the probability of the third candidate winning the election?

62. Winning an Election Taylor, Moore, and Jenkins are candidates for a public office. It is estimated that Moore and Jenkins have about the same probability of winning, and Taylor is believed to be twice as likely to win as either of the others. Find each candidate's probability of winning the election.

63. College Bound In a high school graduating class of 198 students, 43 are on the honor roll. Of these, 37 are going on to college, and of the other 155 students, 102 are going on to college. A student is selected at random from the class. What is the probability that the person chosen is (a) going to college, (b) not going to college, and (c) on the honor roll, but not going to college?

64. Alumni Association The alumni office of a college sends a survey to selected members of the class of 2010. The numbers of graduates, and the numbers of graduates who attended graduate school, are shown in the table. An alumnus is selected at random. What is the probability that the person is (a) female, (b) male, and (c) female and did not attend graduate school?

	Number of graduates	Number of graduates who attended graduate school
Women	672	124
Men	582	198

65. Random Number Generator Two integers (from 1 to 30, inclusive) are chosen by a random number generator on a computer. What is the probability that (a) both numbers are even, (b) one number is even and one is odd, (c) both numbers are less than 10, and (d) the same number is chosen twice?

66. Preparing for a Test A biology instructor gives her class a list of eight study problems, from which she will select five to be answered on an exam. A student knows how to solve six of the problems. Find the probability that the student will be able to answer all five questions on the exam.

67. Drawing Cards from a Deck Two cards are selected at random from a standard deck of 52 playing cards. Find the probability that two hearts are selected under each condition.

(a) The cards are drawn in sequence, with the first card being replaced and the deck reshuffled prior to the second drawing.

(b) The two cards are drawn consecutively, without replacement.

68. Game Show On a game show, you are given five different digits to arrange in the proper order to represent the price of a car. If you are correct, then you win the car. Find the probability of winning under each condition.

(a) You must guess the position of each digit.

(b) You know the first digit but must guess the remaining four.

(c) You know the first and last digits but must guess the remaining three.

69. Game Show On a game show, you are given six different digits to arrange in the proper order to represent the price of a house. If you are correct, then you win the house. Find the probability of winning under each condition.

(a) You must guess the position of each digit.

(b) You know the first digit but must guess the remaining five.

(c) You know the first and last digits but must guess the remaining four.

70. Letter Mix-Up Five letters and envelopes are addressed to five different people. The letters are inserted randomly into the envelopes. What is the probability that (a) exactly one is inserted in the correct envelope and (b) at least one is inserted in the correct envelope?

71. Payroll Mix-Up Three paychecks and envelopes are addressed to three different people. The paychecks get mixed up and are inserted randomly into the envelopes.

(a) What is the probability that exactly one is inserted in the correct envelope?

(b) What is the probability that at least one is inserted in the correct envelope?

72. Poker Hand Five cards are drawn randomly from a standard deck of 52 playing cards. What is the probability of getting a full house? (A full house consists of three of one kind and two of another. For example, A-A-A-5-5 and K-K-K-10-10 are full houses.)

73. Poker Hand Five cards are drawn randomly from a standard deck of 52 playing cards. What is the probability of getting a straight flush? (A straight flush consists of five cards that are in order and of the same suit. For example. A♥, 2♥, 3♥, 4♥, 5♥ and 10♠, J♠, Q♠, K♠, A♠ are straight flushes.)

74. Defective Units A shipment of 1000 compact disc players contains four defective units. A retail outlet has ordered 20 units.

(a) What is the probability that all 20 units are good?

(b) What is the probability that at least one unit is defective?

75. Defective Units A shipment of 12 stereos contains three defective units. Four of the units are shipped to a retail store. What is the probability that (a) all four units are good, (b) exactly two units are good, and (c) at least two units are good?

76. Houses for Sale by Region In 2009, about 8.29% of all houses for sale in the United States were in the northeast, about 14.44% were in the midwest, about 54.01% were in the south, and about 23.26% were in the west. A house is selected at random from the houses for sale in the United States. What is the probability that the house is in the midwest or the south? *(Source: U.S. Census Bureau)*

77. Making a Sale A sales representative makes sales on approximately one-third of all calls. On a given day, the representative calls on four offices. What is the probability that sales are made at (a) all four offices, (b) none of the offices, and (c) at least one office?

78. A Boy or a Girl? Assume that the probability of the birth of a child of a particular gender is 50%. In a family with four children, what is the probability that (a) all four children are boys, (b) all four children are of the same gender, and (c) there is at least one boy?

Probability In Exercises 79–81, consider n independent trials of an experiment in which each trial has two possible outcomes, called success and failure. The probability of a success on each trial is p, and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.

79. A fair coin is tossed eight times. To find the probability of obtaining five heads, evaluate the term

$${}_8C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

80. The probability of a baseball player getting a hit during any given at bat is $\frac{1}{5}$. To find the probability that the player will get four hits during the next 10 at bats, evaluate the term

$${}_{10}C_4\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^6$$

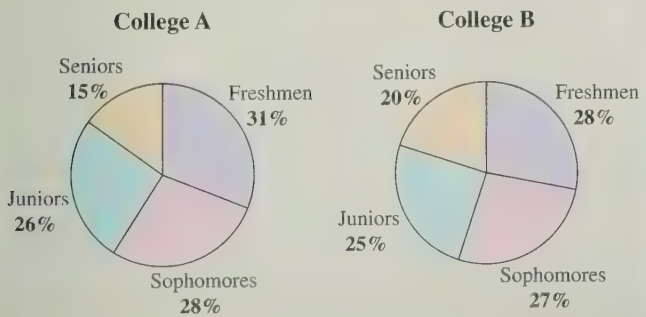
in the expansion of $\left(\frac{1}{5} + \frac{4}{5}\right)^{10}$.

81. The probability of a sales representative making a sale with any one customer is $\frac{1}{4}$. The sales representative makes 10 contacts a day. To find the probability of making four sales, evaluate the term

$${}_{10}C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^6$$

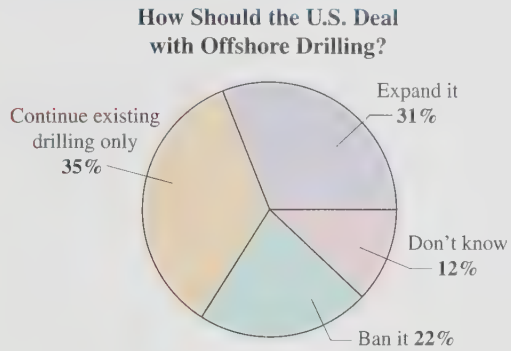
in the expansion of $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

82. HOW DO YOU SEE IT? The circle graphs show the percents of undergraduate students by class level at two colleges. A student is chosen at random from the combined undergraduate population of the two colleges. The probability that the student is a freshman, sophomore, or junior is 81%. Which college has a greater number of undergraduate students? Explain your reasoning.



83. Writing Write a paragraph describing in your own words the difference between mutually exclusive events and independent events.

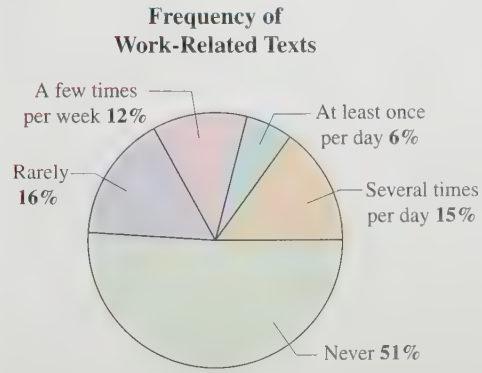
84. Offshore Drilling In a survey, Americans were asked how they felt about offshore drilling for oil and gas. The results are shown in the circle graph. Two people from the survey were chosen at random. *(Source: Pew Research Center)*



(a) What is the probability that neither person felt the U.S. should expand offshore drilling?

(b) What is the probability that both people felt the U.S. should participate in offshore drilling?

85. Texting In a survey, adults with cell phones were asked how often they sent or received text messages related to work. The results are shown in the circle graph. Two people from the survey were chosen at random. *(Source: Pew Research Center's Internet and American Life Project)*



(a) What is the probability that both people sent or received work-related texts *at least* a few times per week?

(b) What is the probability that neither person sent or received work-related texts?

ALGEBRA TUTOR

Evaluating Factorial Expressions

Much of the algebra in this chapter involves evaluating factorial expressions. When evaluating these types of expressions, use order of operations. Factorials follow the same conventions as exponents.

Example 1 Evaluating Simple Factorial Expressions

Evaluate each expression.

a. $\frac{4}{5!}$ **b.** $\frac{12}{3!}$ **c.** $\frac{99!}{100!}$

d. $\frac{20!}{18!}$ **e.** $\frac{2^3}{8!}$

SOLUTION

a. $\frac{4}{5!} = \frac{4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ Definition of factorial

$= \frac{\cancel{4}}{1 \cdot 2 \cdot 3 \cdot \cancel{4} \cdot 5}$ Divide out common factors.

$= \frac{1}{30}$ Simplify.

b. $\frac{12}{3!} = \frac{12}{1 \cdot 2 \cdot 3}$ Definition of factorial

$= \frac{12}{6}$ Multiply.

$= 2$ Divide.

c. $\frac{99!}{100!} = \frac{1 \cdot 2 \cdot 3 \cdots 99}{1 \cdot 2 \cdot 3 \cdots 99 \cdot 100}$ Definition of factorial

$= \frac{\cancel{1 \cdot 2 \cdot 3 \cdots 99}}{\cancel{1 \cdot 2 \cdot 3 \cdots 99} \cdot 100}$ Divide out common factors.

$= \frac{1}{100}$ Simplify.

d. $\frac{20!}{18!} = \frac{1 \cdot 2 \cdot 3 \cdots 18 \cdot 19 \cdot 20}{1 \cdot 2 \cdot 3 \cdots 18}$ Definition of factorial

$= \frac{\cancel{1 \cdot 2 \cdot 3 \cdots 18} \cdot 19 \cdot 20}{\cancel{1 \cdot 2 \cdot 3 \cdots 18}}$ Divide out common factors.

$= 380$ Simplify.

e. $\frac{2^3}{8!} = \frac{8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}$ Evaluate exponent and expand factorial.

$= \frac{\cancel{8}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \cancel{8}}$ Divide out common factors.

$= \frac{1}{5040}$ Simplify.

TECH TUTOR



Try entering the expressions in Example 2 into your calculator. Do you get the same results?

Example 2 Evaluating Other Factorial Expressions

Evaluate each expression.

a. $\frac{8!}{4! + 4!}$ **b.** $\frac{9!}{(9 - 3)!}$ **c.** $\frac{6!}{2! \cdot 3!}$

d. $\frac{1}{3!} + \frac{1}{2!}$ **e.** $4(5 \cdot 3!)$ **f.** $\frac{1}{5}(8 - 3)!$

SOLUTION

a. $\frac{8!}{4! + 4!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{(1 \cdot 2 \cdot 3 \cdot 4) + (1 \cdot 2 \cdot 3 \cdot 4)}$ Definition of factorial

$= \frac{40{,}320}{48}$ Simplify.

$= 840$ Divide.

b. $\frac{9!}{(9 - 3)!} = \frac{9!}{6!}$ Subtract inside parentheses.

$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$ Definition of factorial

$= \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 7 \cdot 8 \cdot 9}{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}}$ Divide out common factors.

$= 504$ Simplify.

c. $\frac{6!}{2! \cdot 3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{(1 \cdot 2) \cdot (1 \cdot 2 \cdot 3)}$ Definition of factorial

$= \frac{720}{12}$ Multiply.

$= 60$ Divide.

d. $\frac{1}{3!} + \frac{1}{2!} = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2}$ Definition of factorial

$= \frac{1}{6} + \frac{1}{2}$ Simplify.

$= \frac{2}{3}$ Add fractions.

e. $4(5 \cdot 3!) = 4(5 \cdot 1 \cdot 2 \cdot 3)$ Definition of factorial

$= 4(30)$ Multiply inside parentheses.

$= 120$ Multiply.

f. $\frac{1}{5}(8 - 3)! = \frac{1}{5} \cdot 5!$ Subtract inside parentheses.

$= \frac{1}{5}(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)$ Definition of factorial

$= \frac{1}{5}(120)$ Multiply inside parentheses.

$= 24$ Multiply.

SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 600. Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 7.1

Review Exercises

- Use sequence notation to write the terms of a sequence. *1–12*
- Evaluate a factorial expression. *13–16*

  $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1) \cdot n$

  As a special case, zero factorial is defined as $0! = 1$.
- Find the sum of a finite sequence and use summation notation to write the sum of a sequence. *17–24*

  $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$
- Use a sequence to solve an application problem. *25, 26*

Section 7.2

- Recognize and find the nth term of an arithmetic sequence. *27–36*
- Find a formula for an arithmetic sequence. *37–42*

  $a_{n+1} = a_n + d \qquad a_n = a_1 + (n-1)d$
- Find the nth partial sum of an arithmetic sequence. *43–46*

  $S_n = \frac{n}{2}(a_1 + a_n)$
- Use an arithmetic sequence to solve an application problem. *47, 48*

Section 7.3

- Recognize and find the nth term of a geometric sequence. *49–60*

  $a_n = a_1 r^{n-1}$
- Find the sum of a finite geometric sequence. *61–64*

  $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$
- Find the sum of an infinite geometric series. *65–68*

  If $|r| < 1$, then $S = \sum_{i=0}^{\infty} a_1 r^i = \frac{a_1}{1-r}$.
- Use a geometric sequence to solve an application problem. *69–72*

* A wide range of valuable study aids are available to help you master the material in this chapter. The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *www.cengagebrain.com* offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

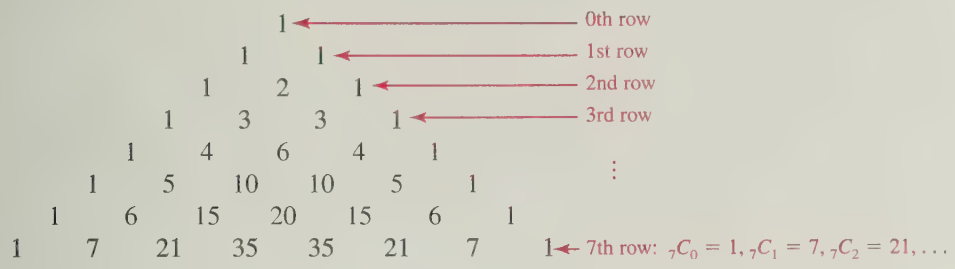
Section 7.4

Review Exercises

- Find a binomial coefficient. *73–76*

  Binomial Theorem: In the expansion of $(x + y)^n$, the coefficient of $x^{n-r}y^r$, the $(r + 1)$th term, is given by ${}_nC_r = \dfrac{n!}{(n-r)!r!}$.

- Use Pascal's Triangle to determine a binomial coefficient. *77, 78*



- Expand a binomial using binomial coefficients and find a specific term in a binomial expansion. *79–88*

Section 7.5

- Use the Fundamental Counting Principle to solve counting problems. *89–92*

  Fundamental Counting Principle: Event E_1 can occur in m_1 ways. After event E_1 occurs, event E_2 can occur in m_2 ways. The number of ways that both events can occur is $m_1 \cdot m_2$.

- Use permutations to solve counting problems. *93–100*

  There are $n!$ different ways that n distinct elements can be ordered. The number of permutations of n distinct elements taken r at a time is

  $${}_nP_r = \frac{n!}{(n-r)!}.$$

- Use combinations to solve counting problems. *101, 102*

  The number of combinations of n elements taken r at a time is ${}_nC_r = \dfrac{n!}{(n-r)!r!}$.

Section 7.6

- Determine the sample space of an experiment and find the probability of an event. *103–106, 117–122*

  $$P(E) = \frac{n(E)}{n(S)}$$

- Find the probability of mutually exclusive events. *107–114, 120–124*

  A and B mutually exclusive: $P(A \cup B) = P(A) + P(B)$

  A and B not mutually exclusive: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- Find the probability of independent events. *115, 116, 123, 124*

  A and B independent: $P(A \text{ and } B) = P(A) \cdot P(B)$

Study Strategies

- **Studying Relationships Between Concepts** To reinforce your understanding of the concepts presented in this chapter, review how they are related. For instance, look back at the similarities and distinctions between concepts such as sequences and series, geometric and arithmetic sequences, combinations and binomial coefficients, combinations and permutations, counting methods and probability, and so on.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Writing Terms of a Sequence In Exercises 1–6, write the first five terms of the sequence. (Assume n begins with 1.)

1. $a_n = 4n + 1$
2. $a_n = 3n - 2$
3. $a_n = \dfrac{n}{n+1}$
4. $a_n = \dfrac{3n-1}{n}$
5. $a_n = (-1)^{n+1}\dfrac{2^n}{n+1}$
6. $a_n = (-1)^n\dfrac{3^n-1}{3^n+1}$

Writing the nth Term of a Sequence In Exercises 7–12, write an expression for the apparent nth term a_n of the sequence. (Assume n begins with 1.)

7. $1, 2, 4, 8, \ldots$
8. $4, -4, 4, -4, \ldots$
9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
10. $-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$
11. $3, -9, 27, -81, 243, \ldots$
12. $-2, 4, -8, 16, -32, \ldots$

Simplifying a Factorial Expression In Exercises 13–16, simplify the factorial expression.

13. $\dfrac{16!}{12!}$
14. $\dfrac{24!}{22!}$
15. $\dfrac{(n+5)!}{(n+4)!}$
16. $\dfrac{(n+4)!}{(n+2)!}$

Using Summation Notation In Exercises 17–20, find the sum.

17. $\sum_{i=1}^{6}(3i-2)$
18. $\sum_{i=1}^{6}(4i+5)$
19. $\sum_{k=1}^{6}(k-1)(k+2)$
20. $\sum_{k=1}^{8}k^2+1$

Writing a Sum in Summation Notation In Exercises 21–24, write the partial sum in summation notation.

21. $\dfrac{3}{1+1} + \dfrac{3}{1+2} + \dfrac{3}{1+3} + \dfrac{3}{1+4} + \cdots + \dfrac{3}{1+12}$
22. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{60}$
23. $2 - 4 + 8 - 16 + 32 - 64 + \cdots + 8192$
24. $3 + 8 + 18 + 32 + \cdots + 98$

25. **Compound Interest** On your next birthday, you plan to deposit \$4000 in an account that earns 6% compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 4000\left(1 + \frac{0.06}{4}\right)^n, \quad n = 1, 2, 3, \ldots$$

(a) Compute the first eight terms of the sequence.

(b) Find the balance in the account after 10 years by computing the 40th term of the sequence.

(c) Find the balance in the account after 30 years by computing the 120th term of the sequence.

26. **Compound Interest** You deposit \$10,000 in an account that earns 8% compounded monthly. The balance in the account after n months is given by

$$A_n = 10{,}000\left(1 + \frac{0.08}{12}\right)^n, \quad n = 1, 2, 3, \ldots$$

(a) Compute the first 10 terms of the sequence.

(b) Find the balance in the account after 10 years by computing the 120th term of the sequence.

Identifying an Arithmetic Sequence In Exercises 27–30, determine whether the sequence is arithmetic. If it is, find the common difference.

27. $3, 8, 13, 18, \ldots$
28. $1, 3, 9, 27, \ldots$
29. $4, \frac{7}{2}, 3, \frac{5}{2}, 2, \ldots$
30. $6.4, 5.6, 4.8, 4.0, 3.2, \ldots$

Writing the Terms of an Arithmetic Sequence In Exercises 31–36, write the first five terms of the arithmetic sequence.

31. $a_1 = 9, d = 5$
32. $a_1 = 8, d = -4$
33. $a_5 = 14, a_8 = 29$
34. $a_9 = 0, a_{15} = -9$
35. $a_7 = 12, a_{10} = 18$
36. $a_{11} = 1, a_{15} = -3$

Finding the nth Term of an Arithmetic Sequence In Exercises 37–42, find a formula for the nth term of the arithmetic sequence.

37. $a_1 = 2, d = 3$
38. $a_1 = 15, d = -3$
39. $2, 7, 12, 17, 22, \ldots$
40. $39, 33, 27, 21, 15, \ldots$
41. $a_1 = 7, a_5 = 27$
42. $a_1 = -5, a_7 = -17$

Finding a Partial Sum of an Arithmetic Sequence In Exercises 43–46, find the indicated nth partial sum of the arithmetic sequence.

43. $3, 9, 15, 21, 27, \ldots, n = 8$

44. $-8, -4, 0, 4, 8, \ldots, n = 12$

45. $a_1 = 9, a_{15} = 93, n = 15$

46. $a_1 = 27, a_{40} = 300, n = 20$

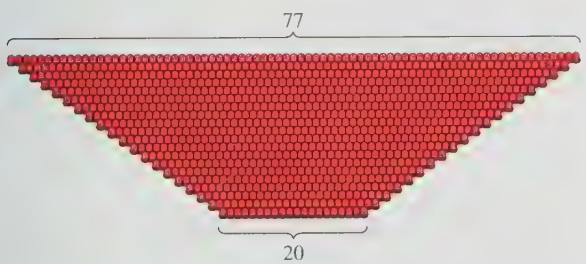
47. Job Offers You have job offers from two companies. The position with Company A has a salary of \$32,500 for the first year with a guaranteed annual raise of \$1600 per year. The position with Company B has a salary of \$28,000 for the first year with a guaranteed annual raise of \$2500 per year.

(a) What will your salary be during the sixth year of employment at Company A? at Company B?

(b) How much will Company A have paid you at the end of 6 years?

(c) How much will Company B have paid you at the end of 6 years?

48. Seating You are setting up 20 rows of chairs for a concert. There will be 20 chairs in the first row, 23 chairs in the second row, 26 chairs in the third row, and so on. How many chairs do you need?



Identifying a Geometric Sequence In Exercises 49–52, determine whether the sequence is geometric. If it is, find the common ratio and a formula for the nth term.

49. $1, -3, 9, -27, \ldots$

50. $3, 9, 15, 21, \ldots$

51. $16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

52. $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}, \ldots$

Writing the Terms of a Geometric Sequence In Exercises 53–56, write the first five terms of the geometric sequence.

53. $a_1 = 3, r = 2$

54. $a_1 = 2, r = e$

55. $a_1 = 10, r = -\frac{1}{5}$

56. $a_1 = 1, r = \frac{3}{4}$

Finding the nth Term of a Geometric Sequence In Exercises 57–60, find the indicated term of the geometric sequence.

57. $a_1 = 8, r = \frac{1}{2}$, 40th term

58. $a_1 = 100, r = -\frac{1}{10}$, 5th term

59. $a_2 = -100, a_5 = 12.5$, 10th term

60. $a_1 = 10, a_3 = 11.025$, 19th term

Finding the Sum of a Finite Geometric Sequence In Exercises 61–64, find the sum of the finite geometric sequence.

61. $\sum_{n=1}^{10} 4(2^n)$

62. $\sum_{n=0}^{20} 2\left(\frac{2}{3}\right)^n$

63. $\sum_{i=1}^{15} 20(0.1)^{i-1}$

64. $\sum_{i=1}^{30} 100(1.06)^{i-1}$

Finding the Sum of an Infinite Geometric Series In Exercises 65–68, find the sum of the infinite geometric series, if it exists.

65. $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

66. $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

67. $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$

68. $\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$

69. Compound Interest You deposit \$100 at the beginning of each month for 10 years in an account that pays 6% compounded monthly. The balance A at the end of 10 years is as follows.

$$A = 100\left(1 + \frac{0.06}{12}\right)^1 + \cdots + 100\left(1 + \frac{0.06}{12}\right)^{120}$$

(a) Find the balance after 10 years.

(b) Is the balance after 20 years twice what it is after 10 years? Explain your reasoning.

70. Compound Interest You deposit \$200 in an account at the beginning of each month for 10 years. The account pays 6% compounded monthly. What will your balance be at the end of 10 years? Suppose instead that the interest were compounded continuously. What would your balance be at the end of the 10 years?

71. Profit The annual profits a_n (in millions of dollars) for a pharmaceuticals company from 2000 through 2012 can be approximated by the model

$$a_n = 165.3e^{0.15n}, \quad n = 0, 1, 2, 3, \ldots, 12$$

where n represents the year, with $n = 0$ corresponding to 2000.

(a) Construct a bar graph that represents the company's profits during the 13-year period.

(b) Use the formula for the sum of a finite geometric sequence to approximate the total profit earned during the 13-year period.

72. Sales The annual sales a_n (in millions of dollars) for a furniture company from 2000 through 2012 can be approximated by the model

$$a_n = 225e^{0.135n}, \quad n = 0, 1, 2, 3, \ldots, 12$$

where n represents the year, with $n = 0$ corresponding to 2000. Use the formula for the sum of a finite geometric sequence to approximate the total sales during the 13-year period.

Finding Binomial Coefficients In Exercises 73–76, find the binomial coefficient.

73. ${}_7C_4$

74. ${}_{12}C_8$

75. $\binom{20}{20}$

76. $\binom{20}{10}$

Using Pascal's Triangle In Exercises 77 and 78, evaluate using Pascal's Triangle.

77. ${}_{12}C_5$

78. $\binom{7}{5}$

Expanding a Binomial In Exercises 79–84, use the Binomial Theorem to expand the expression. Simplify your answer.

79. $(x + 1)^6$

80. $(x - 1)^7$

81. $(3x + y)^5$

82. $(2x - y)^6$

83. $(x - 2y)^8$

84. $(x^2 + 4)^5$

Finding a Term in a Binomial Expansion In Exercises 85 and 86, find the specified nth term in the expansion of the binomial.

85. $(a - 3b)^5, \ n = 3$

86. $(2x + y)^{10}, \ n = 8$

Finding a Coefficient in a Binomial Expansion In Exercises 87 and 88, find the coefficient a of the term in the expansion of the binomial.

Binomial	*Term*
87. $(x + 2)^{12}$	ax^7
88. $(2x - 5y)^9$	ax^4y^5

89. Computer Systems A customer in a computer store can choose one of six monitors, one of five keyboards, and one of seven computers. All of the choices are compatible. How many different systems can be chosen?

90. Chemistry Experiment To set up an experiment, a chemist can use one of two beakers, one of two graduated cylinders, one of six hot plates, and one of three stirring rods. How many different equipment setups are possible for the experiment?

91. Four-Digit Numbers How many four-digit numbers can be formed under each condition?

(a) The leading digit cannot be 0, 1, or 2.

(b) The leading digit cannot be 0, 1, or 2, and no repetition of digits is allowed.

(c) The leading digit cannot be 0 and the number must be divisible by 2.

92. Five-Digit Numbers How many five-digit numbers can be formed under each condition?

(a) The leading digit cannot be 0.

(b) The leading digit cannot be 0 and no repetition of digits is allowed.

(c) The leading digit cannot be 0 and the number must be odd.

93. Genetics A geneticist is using gel electrophoresis to analyze DNA. The DNA is divided into five samples. Each sample is treated with a different restriction enzyme and then injected into one of five wells that have been formed in a bed of gel. In how many orders can the five samples be injected into the wells?

94. Aircraft Boarding Ten people are boarding an aircraft. Two have tickets for first class and board before the other eight people. In how many orders can the ten people board the aircraft?

Finding a Permutation In Exercises 95–98, evaluate ${}_nP_r$ using the formula on page 578.

95. ${}_9P_3$

96. ${}_{30}P_1$

97. ${}_8P_8$

98. ${}_{15}P_2$

Finding Distinguishable Permutations In Exercises 99 and 100, find the number of distinguishable permutations of the group of letters.

99. E, N, T, R, E, P, R, E, N, E, U, R

100. P, H, O, T, O, S, Y, N, T, H, E, S, I, S

101. Zebrafish A researcher has 24 zebrafish embryos and must choose a sample of them for study. How many different samples of each size are possible?

(a) 5 embryos

(b) 10 embryos

(c) 18 embryos

102. Job Applicants An employer interviews 10 people for five identical job openings in the marketing department of a company. All 10 applicants are qualified and seven of them are women. For each situation below, how many different groups of five people are possible?

(a) The people are chosen at random.

(b) Exactly three of those hired are women.

(c) All five of those hired are women.

Finding the Sample Space In Exercises 103 and 104, find the sample space for the experiment.

103. Guessing the last digit in a telephone number

104. Tossing four coins

Finding the Probability of an Event In Exercises 105 and 106, a card is selected at random from a standard deck of 52 playing cards. Find the probability of the event.

105. Getting a 10, jack, or queen

106. Getting a black card that is not a face card

Probability of Mutually Exclusive Events In Exercises 107–110, use the table, which shows the weights of the players on a college football team.

Weight (in pounds)	Number of players
160–199	12
200–239	31
240–279	34
280–319	8

A player from the team is chosen randomly to compete in a tug of war for charity. Find the probability that the player's weight is in the given range.

107. 280 to 319 pounds

108. 240 to 319 pounds

109. Greater than 199 pounds

110. Less than 280 pounds

The Probability of a Union In Exercises 111 and 112, one card is selected at random from a deck of 52 playing cards. Use a formula to find the probability of the union of the two events.

111. The card is red or an ace.

112. The card is a face card or a heart.

Finding the Probability of a Complement In Exercises 113 and 114, you are given the probability p that an event *will* happen. Find the probability that the event *will not* happen.

113. $p = 0.57$ **114.** $p = 0.23$

Probability of Independent Events In Exercises 115 and 116, a random number generator selects three numbers from 1 through 20. Find the probability of the event.

115. All three numbers are odd.

116. Two numbers are even and the third number is 11.

117. Education In a high school graduating class of 72 students, 28 are on the honor roll. Of these, 18 are going on to college, and of the other 44 students, 12 are going on to college. A student is selected at random from the class. What is the probability that the person chosen is (a) going to college, (b) not going to college, and (c) on the honor roll, but not going to college?

118. Game Show On a game show, you are given four different digits to arrange in the proper order to represent the price of a trip. If you are correct, then you win the trip. Find the probability of winning under each condition.

(a) You must guess the position of each digit.

(b) You know the first digit, but must guess the remaining three.

119. Poker Hand Five cards are drawn randomly from a standard deck of 52 playing cards. What is the probability of getting two pair? (Two pair consists of two of one kind, two of another kind, and one of a third kind. For example, K-K-9-9-5 is two pair.)

120. Preparing for a Test An instructor gives a class a list of 12 study problems, from which eight will be selected to construct an exam. A student knows how to solve 10 of the problems. Find the probability that the student will be able to answer (a) all eight questions on the exam, (b) exactly seven questions on the exam, and (c) at least seven questions on the exam.

121. Payroll Mix-Up Four paychecks and envelopes are addressed to four different people. The paychecks get mixed up and are inserted randomly into the envelopes.

(a) What is the probability that exactly one will be inserted in the correct envelope?

(b) What is the probability that at least one will be inserted in the correct envelope?

122. Defective Units A shipment of 1000 radar detectors contains five defective units. A retail outlet has ordered 50 units.

(a) What is the probability that all 50 units are good?

(b) What is the probability that at least one unit is defective?

123. A Boy or a Girl? Assume that the probability of the birth of a child of a particular gender is 50%. In a family with five children, what is the probability that (a) all the children are boys, (b) all the children are of the same gender, and (c) there is at least one boy?

124. A Boy or a Girl? Assume that the probability of the birth of a child of a particular gender is 50%. In a family with seven children, what is the probability that (a) all the children are girls, (b) all the children are of the same gender, and (c) there is at least one girl?

TEST YOURSELF

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, write the first five terms of the sequence. (Assume n begins with 1.)

1. $a_n = 2n + 1$ **2.** $a_n = (-1)^n n^2$ **3.** $a_n = (n + 1)!$ **4.** $a_n = \left(\frac{1}{2}\right)^n$

In Exercises 5–8, determine whether the sequence is arithmetic, geometric, or neither. If possible, find the common difference or common ratio.

5. $a_n = 5n - 2$ **6.** $a_n = 2^n + 2$ **7.** $a_n = 5(2^n)$ **8.** $a_n = 3n^2 - 2$

9. Find a formula for the nth term of an arithmetic sequence 5, 13, 21, ... Then find the 20th term of the sequence.

10. The 11th term of an arithmetic sequence is 360 and the common difference is 35. What is the first term of the sequence?

11. Find a formula for the nth term of the geometric sequence 4, 28, 196, ... Then find the eighth term of the sequence.

12. Find the 18th term of the geometric sequence whose first term is 4 and whose common ratio is 1.1.

In Exercises 13–15, find the sum.

13. $\sum_{n=1}^{50} (2n + 1)$ **14.** $\sum_{n=1}^{20} 3\left(\frac{3}{2}\right)^n$ **15.** $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

16. A deposit of \$8000 is made in an account that pays 5% compounded monthly. The balance in the account after n months is given by

$$A = 8000\left(1 + \frac{0.05}{12}\right)^n, \quad n = 1, 2, 3, \ldots.$$

Find the balance in the account after 10 years.

17. Use the Binomial Theorem to expand $(2x - 3)^5$.

18. Determine the coefficient of the x^7 term in the expansion of $(x - 3)^{12}$.

In Exercises 19 and 20, evaluate each expression.

19. (a) ${}_9P_2$ (b) ${}_{70}P_3$ **20.** (a) ${}_{11}C_4$ (b) ${}_{66}C_5$

In Exercises 21 and 22, find the number of distinguishable permutations of the group of letters.

21. P, I, Z, Z, A **22.** I, N, T, E, R, N, E, T

23. A customer in an electronics store can choose one of eight digital cameras, one of four memory cards, one of two compact tripods, and one of ten cases. How many different camera packages are possible?

24. An ecology quiz consists of 12 multiple choice questions with four possible answers for each question. In how many ways can the quiz be answered? (Assume that no question is omitted.)

25. How many different five-digit numbers can be formed if the leading digit cannot be 0 and the number must be odd?

26. In how many ways can eight people sit in an eight-passenger van?

27. In how many ways can three toppings for a pizza be selected from a list of 12 toppings?

28. You roll three six-sided dice. What is the probability that you roll a five on each die?

A An Introduction to Graphing Utilities

Graphing utilities such as graphing calculators and computers with graphing software are very valuable tools for visualizing mathematical principles, verifying solutions of equations, exploring mathematical ideas, and developing mathematical models. Although graphing utilities are extremely helpful in learning mathematics, their use does not mean that learning algebra is any less important. In fact, the combination of knowledge of mathematics and the use of graphing utilities enables you to explore mathematics more easily and to a greater depth. If you are using a graphing utility in this course, then it is up to you to learn its capabilities and to practice using this tool to enhance your mathematical learning.

In this text, there are many opportunities to use a graphing utility, some of which are described below.

Uses of a Graphing Utility

A graphing utility can be used to

- check or validate answers to problems obtained using algebraic methods.
- discover and explore algebraic properties, rules, and concepts.
- graph functions and approximate solutions of equations involving functions.
- efficiently perform complicated mathematical procedures such as those found in many real-life applications.
- find mathematical models for sets of data.

In this appendix, the features of graphing utilities are discussed from a generic perspective. To learn how to use the features of a specific graphing utility, consult your user's manual or the website for this text, found at *www.cengagebrain.com*, where you will find specific keystrokes for most graphing utilities. Your college library may also have a video on how to use your graphing utility.

The Equation Editor

Many graphing utilities are designed to act as "function graphers." In this course, functions and their graphs are studied in detail. A function can be thought of as a rule that describes the relationship between two variables. These rules are frequently written in terms of x and y. For example, the equation $y = 3x + 5$ represents y as a function of x.

Many graphing utilities have an *equation editor* that requires an equation to be written in "$y =$" form in order to be entered, as shown in Figure A.1. (You should note that your *equation editor* screen may not look like the screen shown in Figure A.1.)

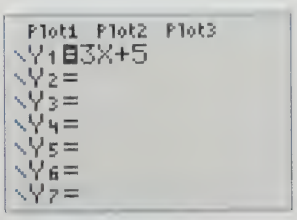


FIGURE A.1

Table Feature

Most graphing utilities are capable of displaying a table of values with x-values and one or more corresponding y-values. These tables can be used to check solutions of an equation and to generate ordered pairs to assist in graphing an equation by hand.

To use the *table* feature, enter an equation in the *equation editor* in "$y =$" form. The table may have a setup screen, which allows you to select the starting x-value and the table step or x-increment. You may then have the option of automatically generating values for x and y or building your own table using *ask* mode (see Figure A.2). In *ask* mode, you enter a value for x and the graphing utility displays the y-value.

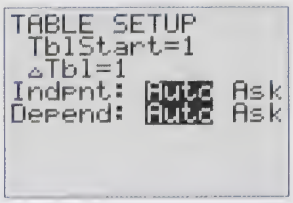


FIGURE A.2

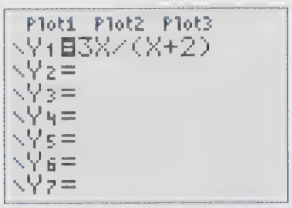


FIGURE A.3

For example, enter the equation

$$y = \frac{3x}{x + 2}$$

in the *equation editor,* as shown in Figure A.3. In the table setup screen, set the table to start at $x = -4$ and set the table step to 1. When you view the table, notice that the first x-value is -4 and each value after it increases by 1. Also notice that the Y1 column gives the resulting y-value for each x-value, as shown in Figure A.4. The table shows that the y-value when $x = -2$ is ERROR. This means that the equation is undefined when $x = -2$.

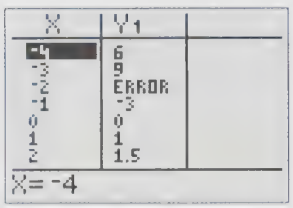


FIGURE A.4

With the same equation in the *equation editor*, set the independent variable in the table setup screen to *ask* mode. In this mode, you do not need to set the starting x-value or the table step, because you are entering any value you choose for x. You may enter any real value for x—integers, fractions, decimals, irrational numbers, and so forth. When you enter $x = 1 + \sqrt{3}$, the graphing utility may rewrite the number as a decimal approximation, as shown in Figure A.5. You can continue to build your own table by entering additional x-values in order to generate y-values.

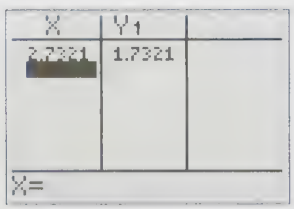


FIGURE A.5

When you have several equations in the *equation editor*, the table may generate y-values for each equation.

Creating a Viewing Window

A *viewing window* for a graph is a rectangular portion of the coordinate plane. A viewing window is determined by the following six values.

Xmin = the minimum value of x

Xmax = the maximum value of x

Xscl = the number of units per tick mark on the x-axis

Ymin = the minimum value of y

Ymax = the maximum value of y

Yscl = the number of units per tick mark on the y-axis

When you enter these six values into a graphing utility, you are setting the viewing window. On some graphing utilities, there is a seventh value on the viewing window labeled Xres. This sets the pixel resolution. For instance, when Xres = 1, functions are evaluated and graphed at each pixel on the x-axis. Some graphing utilities have a standard viewing window, as shown in Figure A.6.

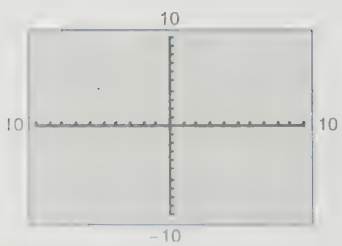


FIGURE A.6

By choosing different viewing windows for a graph, it is possible to obtain very different impressions of the graph's shape. For instance, Figure A.7 shows four different viewing windows for the graph of

$$y = 0.1x^4 - x^3 + 2x^2.$$

Of these, the view shown in part (a) is the most complete.

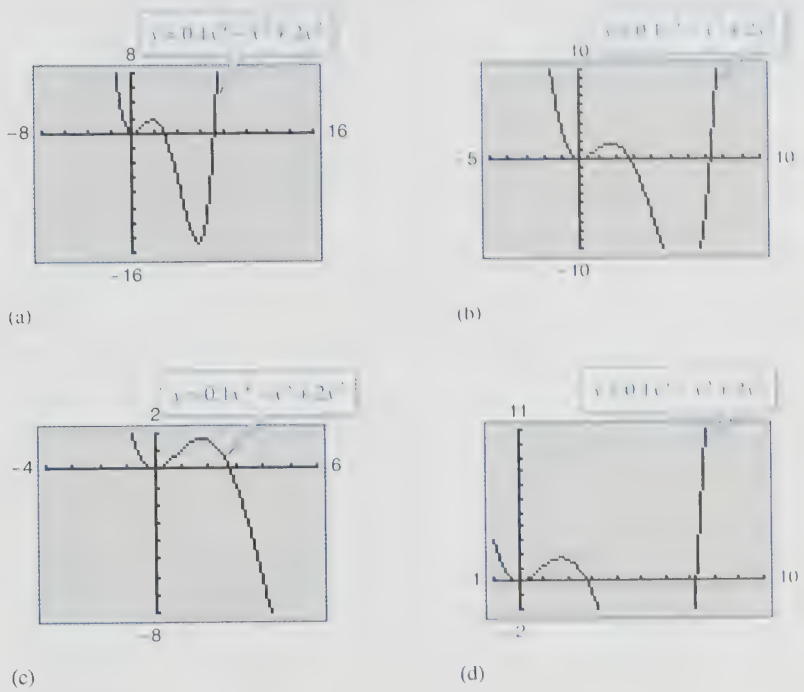


FIGURE A.7

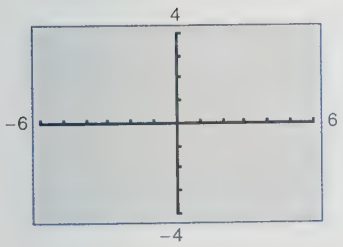


FIGURE A.8

On most graphing utilities, the display screen is two-thirds as high as it is wide. On such screens, you can obtain a graph with a true geometric perspective by using a *square setting*—one in which

$$\frac{\text{Ymax} - \text{Ymin}}{\text{Xmax} - \text{Xmin}} = \frac{2}{3}.$$

One such setting is shown in Figure A.8. Notice that the x and y tick marks are equally spaced on a square setting, but not on a standard setting.

To see how the viewing window affects the geometric perspective, graph the semicircles

$$y_1 = \sqrt{9 - x^2} \quad \text{and} \quad y_2 = -\sqrt{9 - x^2}$$

using a standard viewing window. Then graph y_1 and y_2 using a square viewing window. Note the difference in the shapes of the circles.

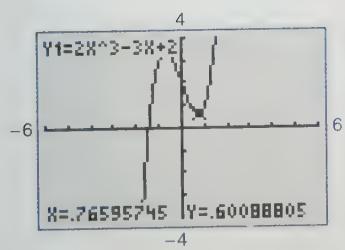


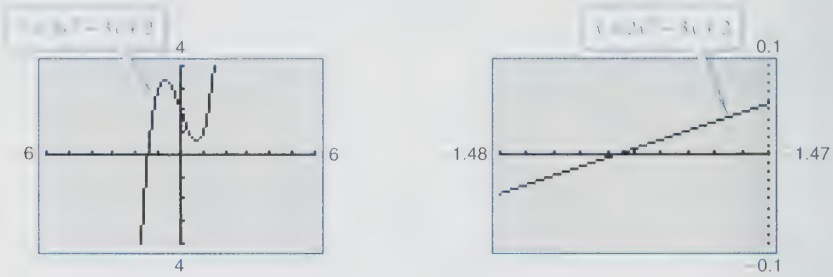
FIGURE A.9

Zoom and Trace Features

When you graph an equation, you can move from point to point along its graph using the *trace* feature. As you trace the graph, the coordinates of each point are displayed, as shown in Figure A.9. The *trace* feature combined with the *zoom* feature enables you to obtain better and better approximations of desired points on a graph.

For instance, you can use the *zoom* feature of a graphing utility to approximate the x-intercept(s) of a graph. Suppose you want to approximate the x-intercept(s) of the graph of $y = 2x^3 - 3x + 2$. Begin by graphing the equation, as shown in Figure A.10(a).

From the viewing window shown, the graph appears to have only one x-intercept. This intercept lies between -2 and -1. By zooming in on the intercept, you can improve the approximation, as shown in Figure A.10(b). To three decimal places, the x-intercept is $x \approx -1.476$.



(a) (b)

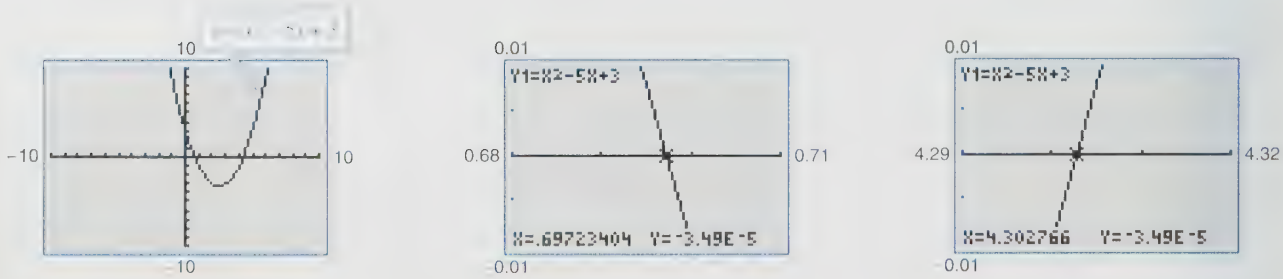
FIGURE A.10

Here are some suggestions for using the *zoom* feature.

1. With each successive zoom-in, adjust the x-scale so that the viewing window shows at least one tick mark on each side of the x-intercept.
2. The error in your approximation will be less than the distance between two tick marks.
3. The *trace* feature can usually be used to add one more decimal place of accuracy without changing the viewing window.

Figure A.11(a) shows the graph of $y = x^2 - 5x + 3$. Figures A.11(b) and A.11(c) show "zoom-in views" of the two x-intercepts. Using these views and the *trace* feature, you can approximate the x-intercepts to be

$$x \approx 0.697 \quad \text{and} \quad x \approx 4.303.$$



(a) (b) (c)

FIGURE A.11

Zero or Root Feature

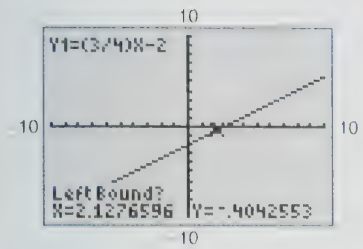
Using the *zero* or *root* feature, you can find the real zeros of functions of the various types studied in this text—polynomial, exponential, and logarithmic. To find the zeros of a function such as

$$f(x) = \tfrac{3}{4}x - 2$$

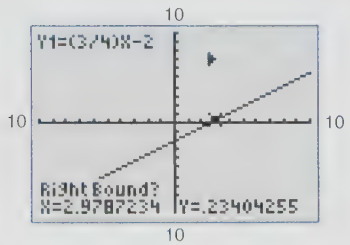
first enter the function as

$$y_1 = \tfrac{3}{4}x - 2$$

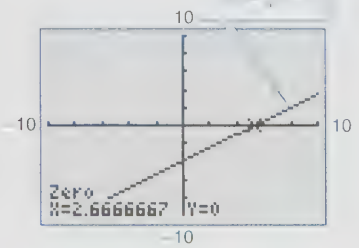
in the *equation editor*. Then use the *zero* or *root* feature, which may require entering lower and upper bound estimates of the zero, as shown in Figures A.12(a) and A.12(b).



(a)



(b)



(c)

FIGURE A.12

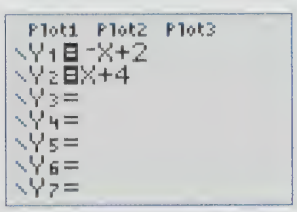
In Figure A.12(c), you can see that the zero is $x \approx 2.6666667 \approx 2\frac{2}{3}$.

Intersect Feature

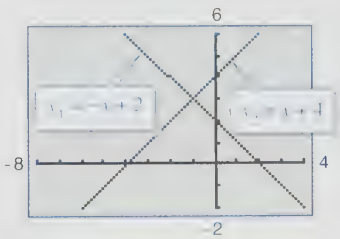
To find the point(s) of intersection of two graphs, you can use the *intersect* feature. For instance, to find the point(s) of intersection of the graphs of

$$y_1 = -x + 2 \quad \text{and} \quad y_2 = x + 4$$

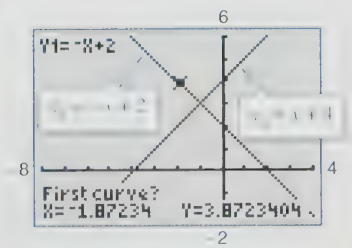
enter these two functions in the *equation editor* and use the *intersect* feature, as shown in Figure A.13.



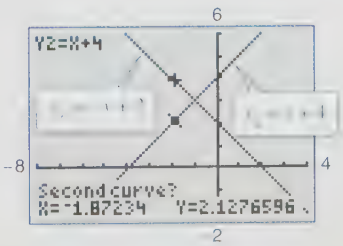
(a)



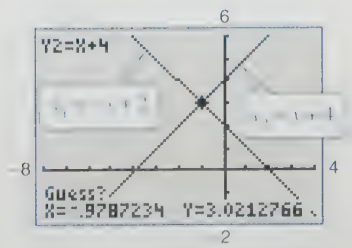
(b)



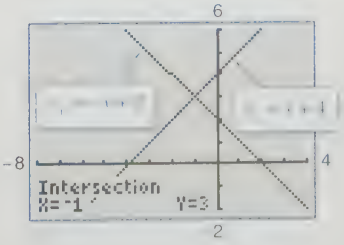
(c)



(d)



(e)



(f)

FIGURE A.13

From Figure A.13(f), you can see that the point of intersection is $(-1, 3)$.

Regression Feature

Throughout the text, you are asked to use the *regression* feature of a graphing utility to find models for sets of data. Most graphing utilities have built-in regression programs for the following.

Regression	*Form of Model*
Linear	$y = ax + b$ or $y = a + bx$
Quadratic	$y = ax^2 + bx + c$
Cubic	$y = ax^3 + bx^2 + cx + d$
Quartic	$y = ax^4 + bx^3 + cx^2 + dx + e$
Logarithmic	$y = a + b \ln x$
Exponential	$y = ab^x$
Power	$y = ax^b$
Logistic	$y = \frac{c}{1 + ae^{-bx}}$ or $y = \frac{a}{1 + be^{cx}} + d$
Sine	$y = a \sin(bx + c) + d$

For example, you can find a linear model for the data in the table.

x	9	10	11	12	13	14	15
y	142.8	155.4	166.7	180.1	192.2	198.6	208.4

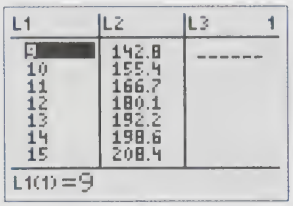


FIGURE A.14

Enter the data in the *list editor*, as shown in Figure A.14. Note that the list in the first column contains the x-values and the list in the second column contains the y-values. Now use your graphing utility's *linear regression* feature to obtain the coefficients a and b for the model

$$y = ax + b$$

as shown in Figure A.15. So, a linear model for the data is

$$y = 11.03x + 45.4.$$

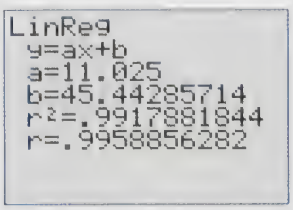


FIGURE A.15

When you use the *regression* feature of a graphing calculator or computer program to find a linear model for data, you will notice that the program may also output an "r-value." The r-value, or **correlation coefficient,** measures how well the linear model fits the data. The correlation coefficient r varies between -1 and 1. The closer $|r|$ is to 1, the better the data can be described by a line. For the data in the table above, you can see from Figure A.15 that $r \approx 0.996$, which implies that the linear model

$$y = 11.03x + 45.4$$

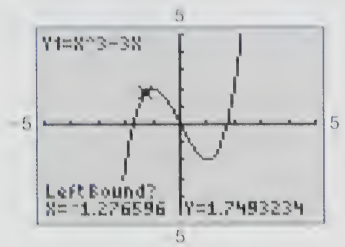
is a good fit for the data.

Maximum and Minimum Features

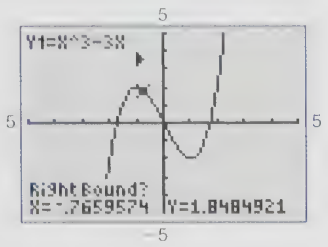
The *maximum* and *minimum* features of a graphing utility find the relative extrema of a function. To find the relative maximum of a function such as

$$f(x) = x^3 - 3x$$

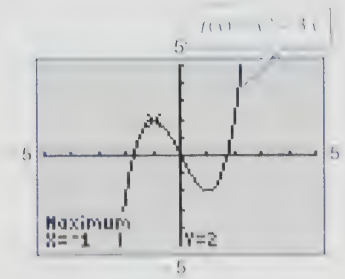
first enter the function as $y_1 = x^3 - 3x$ in the *equation editor*. Then use the *maximum* feature, which may require entering lower and upper bound estimates for the maximum, as shown in Figures A.16(a) and A.16(b).



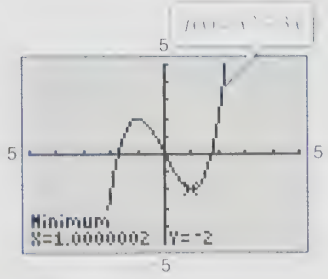
(a)



(b)



(c)



(d)

FIGURE A.16

In Figure A.16(c), you can see that the relative maximum occurs at $(-1, 2)$. The relative minimum of the function can be found in a similar manner. From Figure A.16(d), you can see that the relative minimum occurs at $(1, -2)$.

B Conic Sections

B.1 Conic Sections

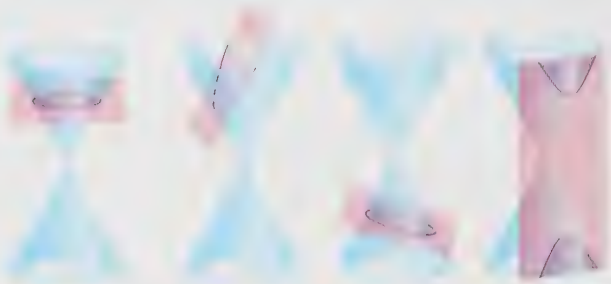
- Recognize the four basic conics: circles, parabolas, ellipses, and hyperbolas.
- Recognize, graph, and write equations of parabolas (vertex at origin).
- Recognize, graph, and write equations of ellipses (center at origin).
- Recognize, graph, and write equations of hyperbolas (center at origin).

Introduction to Conic Sections

Conic sections were discovered during the classical Greek period, which lasted from 600 to 300 B.C. By the beginning of the Alexandrian period, enough was known of conics for Apollonius (262–190 B.C.) to produce an eight-volume work on the subject.

This early Greek study was largely concerned with the geometric properties of conics. It was not until the early seventeenth century that the broad applicability of conics became apparent.

A **conic section** (or simply **conic**) can be described as the intersection of a plane and a double-napped cone. Notice from Figure B.1 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure B.2.



Conic Sections
FIGURE B.1



Degenerate Conics
FIGURE B.2

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach in which each of the conics is defined as a **locus,** or collection, of points satisfying a certain geometric property. For example, in Section 2.1 you saw how the definition of a circle as *the collection of all points* (x, y) *that are equidistant from a fixed point* (h, k) led easily to the standard equation of a circle,

$$(x - h)^2 + (y - k)^2 = r^2.$$

You will restrict your study of conics in Appendix B.1 to parabolas with vertices at the origin, and ellipses and hyperbolas with centers at the origin. In Appendix B.2, you will look at the general cases.

Parabolas

STUDY TIP

Note that the term *parabola* is a technical term used in mathematics and does not simply refer to *any* U-shaped curve.

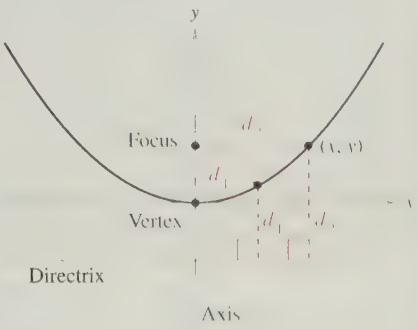
In Section 3.1, you determined that the graph of the quadratic function given by

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward or downward. The definition of a parabola given below is more general in the sense that it is independent of the orientation of the parabola.

Definition of a Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus** (not on the line). The midpoint between the focus and the directrix is called the **vertex,** and the line passing through the focus and the vertex is called the **axis** of the parabola.



Using this definition, you can derive the following standard form of the equation of a parabola.

Standard Equation of a Parabola (Vertex at Origin)

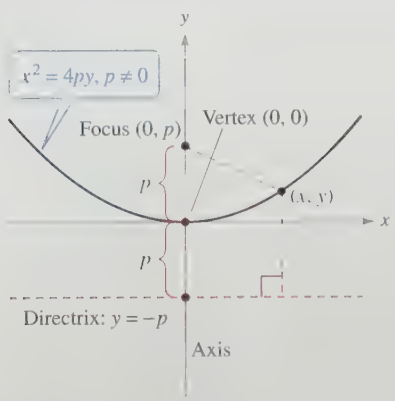
The **standard form of the equation of a parabola** with vertex at $(0, 0)$ and directrix $y = -p$ is given by

$$x^2 = 4py, \quad p \neq 0.$$ Vertical axis

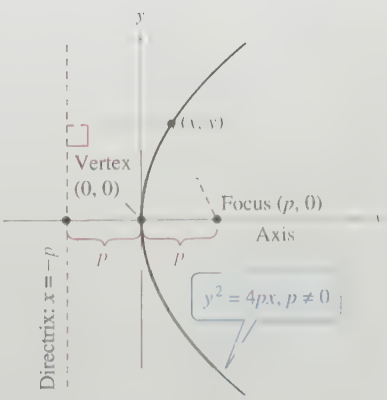
For directrix $x = -p$, the equation is given by

$$y^2 = 4px, \quad p \neq 0.$$ Horizontal axis

The focus is on the axis p units (directed distance) from the vertex. See Figure B.3.



Parabola with Vertical Axis



Parabola with Horizontal Axis

FIGURE B.3

Example 1 Finding the Focus of a Parabola

Find the focus of the parabola whose equation is $y = -2x^2$.

SOLUTION Because the squared term in the equation involves x, you know that the axis is vertical, and the equation is of the form

$x^2 = 4py.$ Standard form, vertical axis

You can write the original equation in this form, as shown.

$-2x^2 = y$ Write original equation.

$x^2 = -\frac{1}{2}y$ Divide each side by -2.

$x^2 = 4\left(-\frac{1}{8}\right)y.$ Write in standard form.

So, $p = -\frac{1}{8}$. Because p is negative, the parabola opens downward and the focus of the parabola is $(0, p) = \left(0, -\frac{1}{8}\right)$, as shown in Figure B.4.

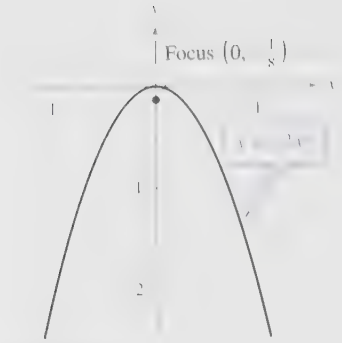


FIGURE B.4

Checkpoint 1

Find the focus of the parabola whose equation is $y^2 = 2x$.

Example 2 Finding the Standard Equation of a Parabola

Write the standard form of the equation of the parabola with vertex at the origin and focus at $(2, 0)$.

SOLUTION The axis of the parabola is horizontal, passing through $(0, 0)$ and $(2, 0)$, as shown in Figure B.5. So, the standard form is

$y^2 = 4px.$ Standard form, horizontal axis

Because the focus is $p = 2$ units from the vertex, the equation is

$y^2 = 4(2)x$ Standard form

$y^2 = 8x.$

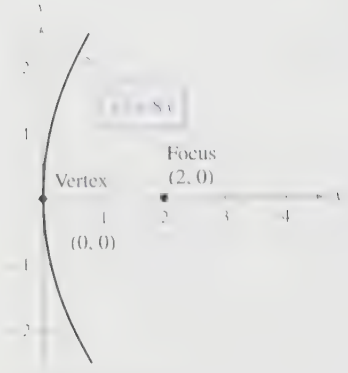


FIGURE B.5

Checkpoint 2

Write the standard form of the equation of the parabola with vertex at the origin and focus at $(0, 2)$.

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of the parabolic reflector used in a flashlight are all reflected parallel to one another, as shown in Figure B.6.

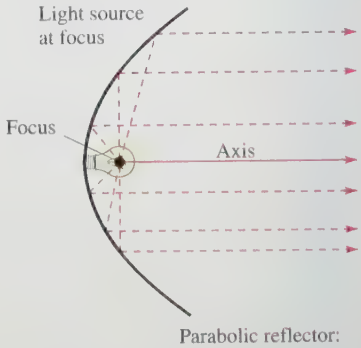


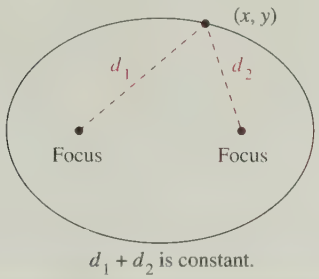
FIGURE B.6

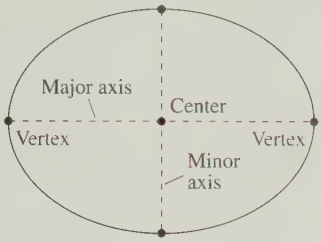
Ellipses

Another basic type of conic is called an **ellipse.**

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane the sum of whose distances from two distinct fixed points, called **foci,** is constant.





The line through the foci intersects the ellipse at two points, called the **vertices.** The chord joining the vertices is called the **major axis,** and its midpoint is called the **center** of the ellipse. The chord perpendicular to the major axis at the center is called the **minor axis** of the ellipse. The endpoints of the minor axis of an ellipse are commonly referred to as the **co-vertices.**

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure B.7. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

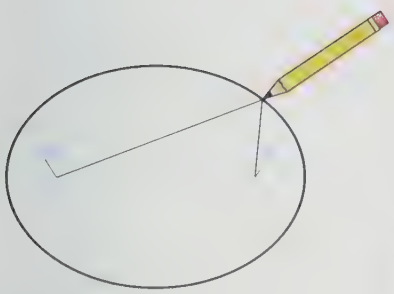


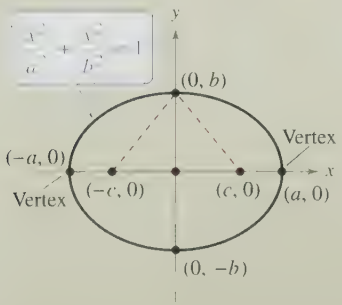
FIGURE B.7

The standard form of the equation of an ellipse takes one of two forms, depending on whether the major axis is horizontal or vertical.

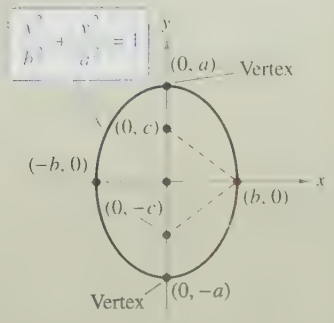
Standard Equation of an Ellipse (Center at Origin)

The **standard form of the equation of an ellipse** with the center at the origin and major and minor axes of lengths $2a$ and $2b$, respectively (where $0 < b < a$), is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$



Major axis is horizontal.
Minor axis is vertical.



Major axis is vertical.
Minor axis is horizontal.

The vertices and foci lie on the major axis, a and c units, respectively, from the center. Moreover, a, b, and c are related by the equation $c^2 = a^2 - b^2$.

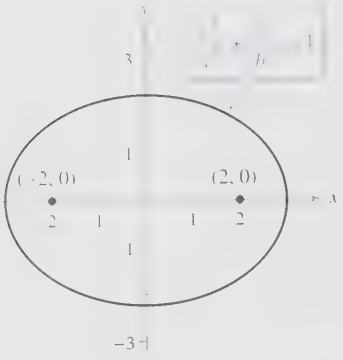


FIGURE B.8

Example 3 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse that has a major axis of length 6 and foci at $(-2, 0)$ and $(2, 0)$, as shown in Figure B.8.

SOLUTION Because the foci occur at $(-2, 0)$ and $(2, 0)$, the center of the ellipse is $(0, 0)$, and the major axis is horizontal. So, the ellipse has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{Standard form, horizontal major axis}$$

Because the length of the major axis is 6, you have $2a = 6$, which implies that $a = 3$. Moreover, the distance from the center to either focus is $c = 2$. Finally, you have

$$b^2 = a^2 - c^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Substituting $a^2 = 3^2$ and $b^2 = \left(\sqrt{5}\right)^2$ yields the following equation in standard form.

$$\frac{x^2}{3^2} + \frac{y^2}{\left(\sqrt{5}\right)^2} = 1 \qquad \text{Standard form}$$

This equation simplifies to

$$\frac{x^2}{9} + \frac{y^2}{5} = 1.$$

Checkpoint 3

Find the standard form of the equation of the ellipse that has a major axis of length 8 and foci at $(0, -3)$ and $(0, 3)$.

Example 4 Sketching an Ellipse

Sketch the ellipse given by

$$4x^2 + y^2 = 36$$

and identify the vertices.

SOLUTION Begin by writing the equation in standard form.

$$4x^2 + y^2 = 36 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \qquad \text{Divide each side by 36.}$$

$$\frac{x^2}{9} + \frac{y^2}{36} = 1 \qquad \text{Simplify.}$$

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1 \qquad \text{Write in standard form.}$$

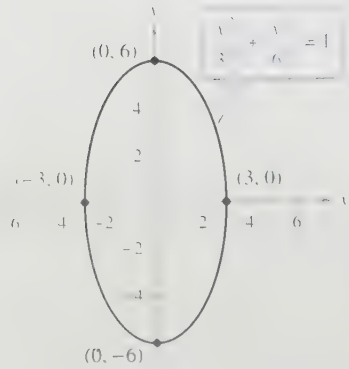


FIGURE B.9

Because the denominator of the y^2-term is greater than the denominator of the x^2-term, you can conclude that the major axis is vertical. Also, because $a^2 = 6^2$, the endpoints of the major axis lie six units *up and down* from the center $(0, 0)$. So, the vertices of the ellipse are $(0, 6)$ and $(0, -6)$. Similarly, because the denominator of the x^2-term is $b^2 = 3^2$, the endpoints of the minor axis (or co-vertices) lie three units to the *right and left* of the center at $(3, 0)$ and $(-3, 0)$. The ellipse is shown in Figure B.9.

Checkpoint 4

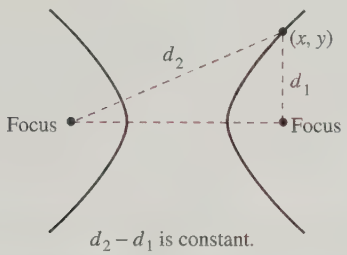
Sketch the ellipse given by $x^2 + 4y^2 = 64$, and identify the vertices.

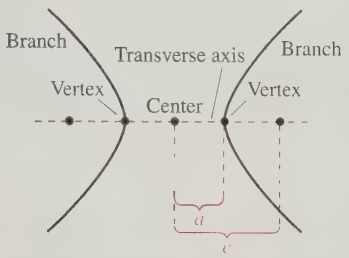
Hyperbolas

The definition of a **hyperbola** is similar to that of an ellipse. The distinction is that, for an ellipse, the *sum* of the distances between the foci and a point on the ellipse is constant, whereas for a hyperbola, the *difference* of the distances between the foci and a point on the hyperbola is constant.

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane the difference of whose distances from two distinct fixed points, called **foci,** is constant.



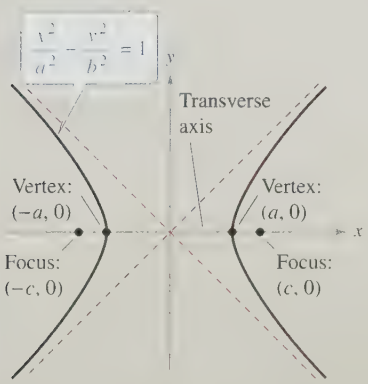


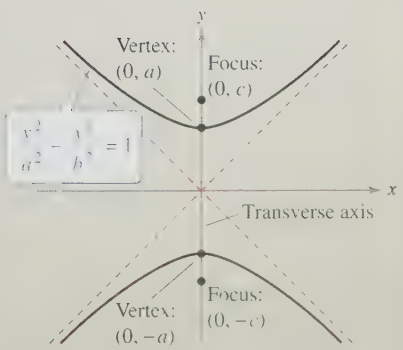
The graph of a hyperbola has two disconnected parts, called **branches.** The line through the two foci intersects the hyperbola at two points, called **vertices.** The line segment connecting the vertices is called the **transverse axis,** and the midpoint of the transverse axis is called the **center** of the hyperbola.

Standard Equation of a Hyperbola (Center at Origin)

The **standard form of the equation of a hyperbola** with the center at the origin (where $a \neq 0$ and $b \neq 0$) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$





The vertices and foci are, respectively, a and c units from the center. Moreover, a, b, and c are related by the equation

$$b^2 = c^2 - a^2.$$

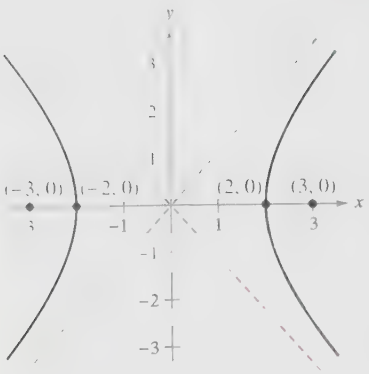


FIGURE B.10

Example 5 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci at $(-3, 0)$ and $(3, 0)$ and vertices at $(-2, 0)$ and $(2, 0)$, as shown in Figure B.10.

SOLUTION From the graph, $c = 3$ because the foci are three units from the center. Also, $a = 2$ because the vertices are two units from the center. So,

$$b^2 = c^2 - a^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Because the transverse axis is horizontal, the standard form of the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$ Standard form, horizontal transverse axis

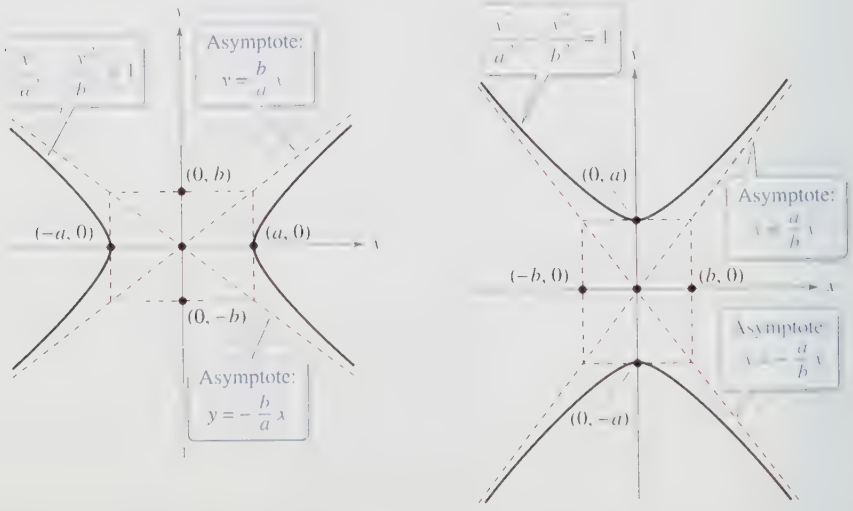
Finally, substituting $a^2 = 2^2$ and $b^2 = \left(\sqrt{5}\right)^2$, you have

$$\frac{x^2}{2^2} - \frac{y^2}{\left(\sqrt{5}\right)^2} = 1.$$ Write in standard form.

Checkpoint 5

Find the standard form of the equation of the hyperbola with foci at $(0, -4)$ and $(0, 4)$ and vertices at $(0, -3)$ and $(0, 3)$.

An important aid in sketching the graph of a hyperbola is the determination of its *asymptotes*, as shown in Figure B.11. Each hyperbola has two asymptotes that intersect at the center of the hyperbola. Furthermore, the asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$. The line segment of length $2b$, joining $(0, b)$ and $(0, -b)$ [or $(-b, 0)$ and $(b, 0)$], is referred to as the **conjugate axis** of the hyperbola.



Transverse axis is horizontal. Transverse axis is vertical.

FIGURE B.11

Asymptotes of a Hyperbola (Center at Origin)

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$ Transverse axis is horizontal.

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x$$ Transverse axis is vertical.

Example 6 Sketching a Hyperbola

Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

SOLUTION

$$4x^2 - y^2 = 16$$ Write original equation.

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16}$$ Divide each side by 16.

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$$ Write in standard form.

Because the x^2-term is positive, you can conclude that the transverse axis is horizontal and the vertices occur at $(-2, 0)$ and $(2, 0)$. Moreover, the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$, and you can sketch the rectangle shown in Figure B.12. Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch shown in Figure B.13.

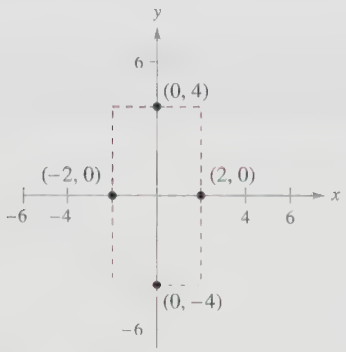


FIGURE B.12

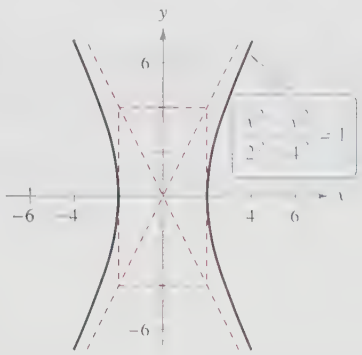


FIGURE B.13

Checkpoint 6

Sketch the hyperbola whose equation is $9y^2 - x^2 = 9$.

Example 7 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola that has vertices at $(0, -3)$ and $(0, 3)$ and asymptotes $y = -2x$ and $y = 2x$, as shown in Figure B.14.

SOLUTION Because the transverse axis is vertical, the asymptotes are of the form

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.$$ Transverse axis is vertical.

Using the fact that $y = 2x$ and $y = -2x$, you can determine that $a/b = 2$. Because $a = 3$, you can determine that $b = \frac{3}{2}$. Finally, you can conclude that the hyperbola has the equation

$$\frac{y^2}{3^2} - \frac{x^2}{(3/2)^2} = 1.$$ Write in standard form.

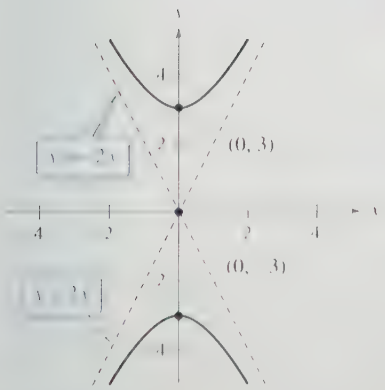


FIGURE B.14

Checkpoint 7

Find the standard form of the equation of the hyperbola that has vertices at $(-5, 0)$ and $(5, 0)$ and asymptotes $y = -x$ and $y = x$.

SKILLS WARM UP B.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.7, 1.3, and 2.1.

In Exercises 1–4, rewrite the equation so that it has no fractions.

1. $\frac{x^2}{16} + \frac{y^2}{9} = 1$
2. $\frac{x^2}{32} + \frac{4y^2}{32} = \frac{32}{32}$
3. $\frac{x^2}{1/4} - \frac{y^2}{4} = 1$
4. $\frac{3x^2}{1/9} + \frac{4y^2}{9} = 1$

In Exercises 5–8, solve for c. (Assume $c > 0$.)

5. $c^2 = 3^2 - 1^2$
6. $c^2 = 2^2 + 3^2$
7. $c^2 + 2^2 = 4^2$
8. $c^2 - 1^2 = 2^2$

In Exercises 9 and 10, find the distance between the point and the origin.

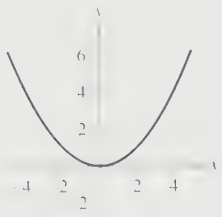
9. $(0, -4)$
10. $(-2, 0)$

Exercises B.1

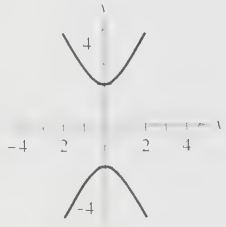
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

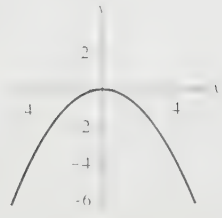
(a)



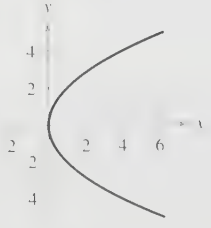
(b)



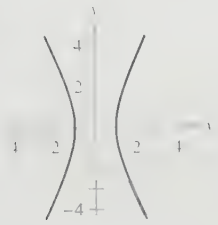
(c)



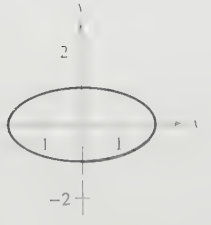
(d)



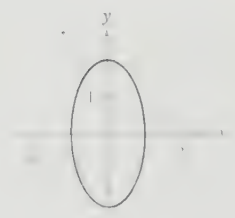
(e)



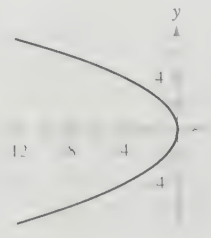
(f)



(g)



(h)



1. $x^2 = 4y$
2. $x^2 = -4y$
3. $y^2 = 4x$
4. $y^2 = -4x$
5. $\frac{x^2}{1} + \frac{y^2}{4} = 1$
6. $\frac{x^2}{4} + \frac{y^2}{1} = 1$
7. $\frac{x^2}{1} - \frac{y^2}{4} = 1$
8. $\frac{y^2}{4} - \frac{x^2}{1} = 1$

Finding the Vertex and Focus of a Parabola In Exercises 9–16, find the vertex and focus of the parabola and sketch its graph. *See Example 1.*

9. $y = 4x^2$
10. $y = \frac{1}{2}x^2$
11. $y^2 = -6x$
12. $y^2 = 3x$
13. $x^2 + 8y = 0$
14. $x^2 + 12y = 0$
15. $x + y^2 = 0$
16. $y^2 - 8x = 0$

Finding the Standard Equation of a Parabola In Exercises 17–26, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin. *See Example 2.*

17. Focus: $(0, -\frac{3}{2})$
18. Focus: $(0, -2)$

19. Focus: $(-2, 0)$

20. Focus: $\left(\frac{5}{2}, 0\right)$

21. Directrix: $y = -1$

22. Directrix: $y = 2$

23. Directrix: $x = 3$

24. Directrix: $x = -2$

25. Passes through the point $(4, 6)$; horizontal axis

26. Passes through the point $(-2, -2)$; vertical axis

Finding the Standard Equation of an Ellipse In Exercises 27–34, find the standard form of the equation of the ellipse with the given characteristics and center at the origin. *See Example 3.*

27. Vertices: $(0, \pm 2)$; minor axis of length 2

28. Vertices: $(\pm 2, 0)$; minor axis of length 3

29. Vertices: $(\pm 5, 0)$; foci: $(\pm 2, 0)$

30. Vertices: $(0, \pm 10)$; foci: $(0, \pm 4)$

31. Foci: $(\pm 5, 0)$; major axis of length 12

32. Foci: $(\pm 2, 0)$; major axis of length 8

33. Vertices: $(0, \pm 5)$; passes through the point $(4, 2)$

34. Major axis vertical; passes through the points $(0, 4)$ and $(2, 0)$

Sketching an Ellipse In Exercises 35–42, find the center and vertices of the ellipse and sketch its graph. *See Example 4.*

35. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

36. $\frac{x^2}{144} + \frac{y^2}{169} = 1$

37. $\frac{x^2}{\frac{25}{9}} + \frac{y^2}{\frac{16}{9}} = 1$

38. $\frac{x^2}{4} + \frac{y^2}{\frac{1}{4}} = 1$

39. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

40. $\frac{x^2}{28} + \frac{y^2}{64} = 1$

41. $5x^2 + 3y^2 = 15$

42. $x^2 + 4y^2 = 4$

Finding the Standard Equation of a Hyperbola In Exercises 43–46, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin. *See Example 5.*

43. Vertices: $(0, \pm 2)$; foci: $(0, \pm 4)$

44. Vertices: $(0, \pm 5)$; foci: $(0, \pm 8)$

45. Vertices: $(\pm 6, 0)$; foci: $(\pm 9, 0)$

46. Vertices: $(\pm 3, 0)$; foci: $(\pm 5, 0)$

Sketching a Hyperbola In Exercises 47–54, find the center and vertices of the hyperbola and sketch its graph. *See Example 6.*

47. $x^2 - y^2 = 1$

48. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

49. $\frac{y^2}{1} - \frac{x^2}{4} = 1$

50. $\frac{y^2}{9} - \frac{x^2}{1} = 1$

51. $\frac{y^2}{25} - \frac{x^2}{144} = 1$

52. $\frac{x^2}{36} - \frac{y^2}{4} = 1$

53. $2x^2 - 3y^2 = 6$

54. $3y^2 - 5x^2 = 15$

Finding the Standard Equation of a Hyperbola In Exercises 55–60, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin. *See Example 7.*

55. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 3x$

56. Vertices: $(0, \pm 3)$; asymptotes: $y = \pm 3x$

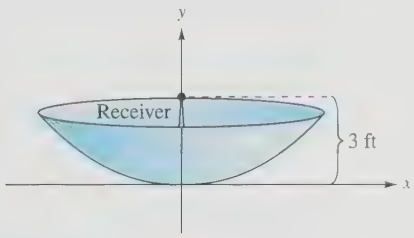
57. Foci: $(0, \pm 4)$; asymptotes: $y = \pm \frac{1}{2}x$

58. Foci: $(\pm 10, 0)$; asymptotes: $y = \pm \frac{3}{4}x$

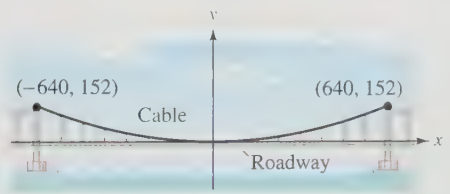
59. Vertices: $(0, \pm 3)$; passes through the point $(-2, 5)$

60. Vertices: $(\pm 2, 0)$; passes through the point $\left(3, \sqrt{3}\right)$

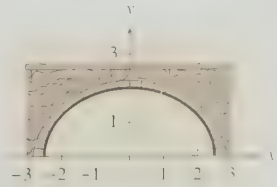
61. Satellite Antenna The receiver in a parabolic television dish antenna is 3 feet from the vertex and is located at the focus (see figure). Write an equation for a cross section of the reflector. (Assume that the dish is directed upward and the vertex is at the origin.)



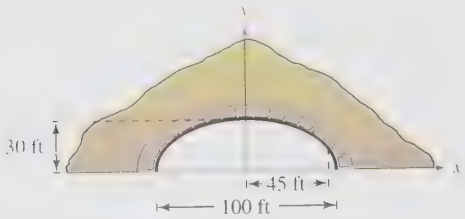
62. Suspension Bridge Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway (see figure). The cables touch the roadway at the midpoint between the towers. Write an equation for the parabolic shape of each cable.



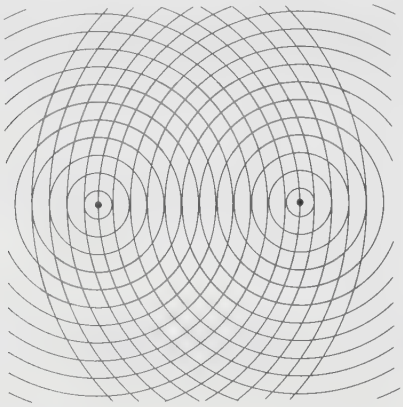
63. Architecture A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 5 feet along the base (see figure). The contractor draws the outline of the ellipse by the method shown in Figure B.7. Where should the tacks be placed and what should be the length of the piece of string?



64. Mountain Tunnel A semielliptical arch over a tunnel for a road through a mountain has a major axis of 100 feet, and its height at the center is 30 feet (see figure). Determine the height of the arch 5 feet from the edge of the tunnel.

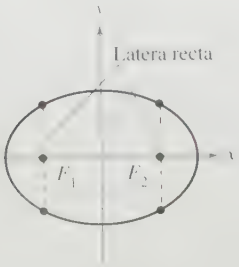


65. Sketching an Ellipse Sketch a graph of the ellipse that consists of all points (x, y) such that the sum of the distances between (x, y) and two fixed points is 15 units and the foci are located at the centers of the two sets of concentric circles, as shown in the figure.



66. Think About It A line segment through a focus of an ellipse with endpoints on the ellipse and perpendicular to its major axis is called a **latus rectum** of the ellipse. An ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because this information yields other points on the curve (see figure). Show that the length of each latus rectum is

$$\frac{2b^2}{a}.$$



Using Latera Recta In Exercises 67–70, sketch the ellipse using the latera recta (see Exercise 66).

67. $\frac{x^2}{4} + \frac{y^2}{1} = 1$

68. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

69. $9x^2 + 4y^2 = 36$

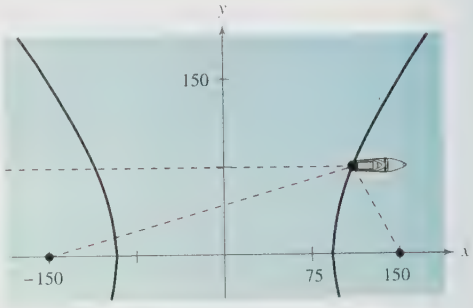
70. $5x^2 + 3y^2 = 15$

71. Think About It Consider the ellipse

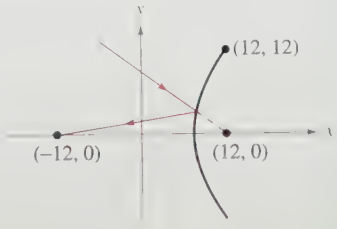
$$\frac{x^2}{328} + \frac{y^2}{327} = 1.$$

Is this ellipse better described as *elongated* or *nearly circular*? Explain your reasoning.

72. Navigation Long-range navigation for aircraft and ships is accomplished by synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations 300 miles apart are positioned on a rectangular coordinate system at points with coordinates $(-150, 0)$ and $(150, 0)$ and that a ship is traveling on a path with coordinates $(x, 75)$ (see figure). Find the x-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).



73. Hyperbolic Mirror A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at one focus will be reflected to the other focus (see figure). The focus of the hyperbolic mirror has coordinates $(12, 0)$. Find the vertex of the mirror if its mount at the top edge of the mirror has coordinates $(12, 12)$.



74. Writing Explain how the central rectangle of a hyperbola can be used to sketch its asymptotes.

B.2 Conic Sections and Translations

- Recognize equations of conics that have been shifted vertically or horizontally in the plane.
- Write and graph equations of conics that have been shifted vertically or horizontally in the plane.

Vertical and Horizontal Shifts of Conics

In Appendix B.1, you studied conic sections whose graphs were in **standard position.** In this section, you will study the equations of conic sections that have been shifted vertically or horizontally in the plane. The following summary lists the standard forms of the equations of the four basic conics.

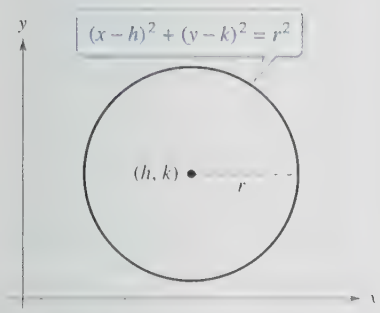
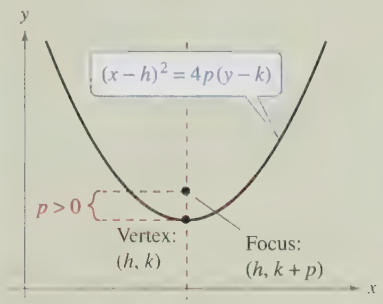


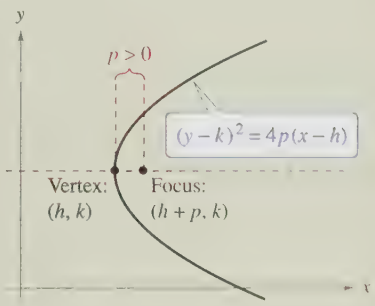
FIGURE B.15

Standard Forms of Equations of Conics

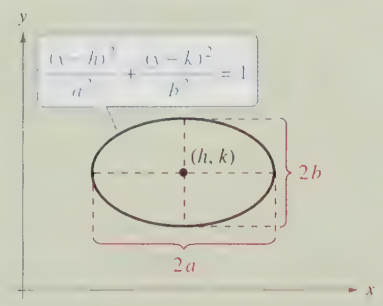
Circle: Center $= (h, k)$; Radius $= r$; See Figure B.15.

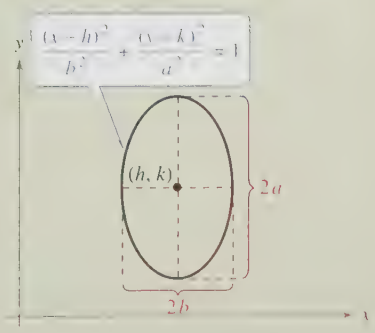
Parabola: Vertex $= (h, k)$; Directed distance from vertex to focus $= p$



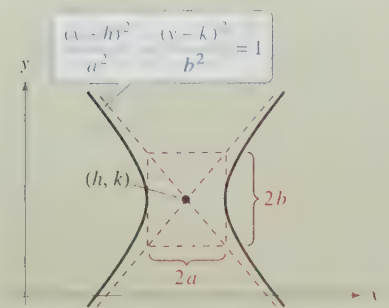


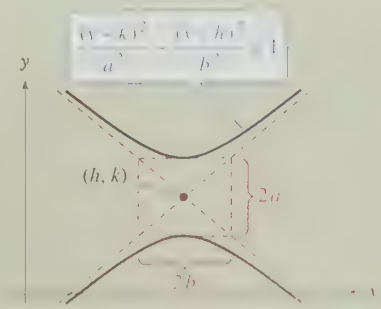
Ellipse: Center $= (h, k)$; Major axis length $= 2a$; Minor axis length $= 2b$





Hyperbola: Center $= (h, k)$; Transverse axis length $= 2a$;
Conjugate axis length $= 2b$





Example 1 Equations of Conic Sections

Describe the translation of the graph of each conic.

a. $(x - 1)^2 + (y + 2)^2 = 3^2$ **b.** $(x - 2)^2 = 4(-1)(y - 3)$

c. $\dfrac{(x - 3)^2}{1^2} - \dfrac{(y - 2)^2}{3^2} = 1$ **d.** $\dfrac{(x - 2)^2}{3^2} + \dfrac{(y - 1)^2}{2^2} = 1$

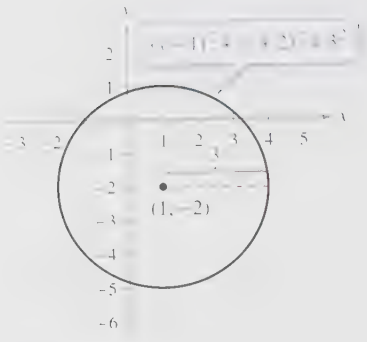


FIGURE B.16

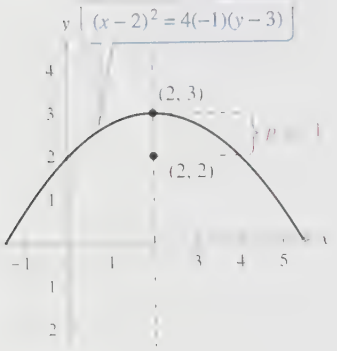


FIGURE B.17

SOLUTION

a. The graph of $(x - 1)^2 + (y + 2)^2 = 3^2$ is a circle whose center is the point $(1, -2)$ and whose radius is 3, as shown in Figure B.16. The graph of the circle has been shifted one unit to the right and two units downward from standard position.

b. The graph of $(x - 2)^2 = 4(-1)(y - 3)$ is a parabola whose vertex is the point $(2, 3)$. The axis of the parabola is vertical. Moreover, because $p = -1$, it follows that the focus lies *below* the vertex, as shown in Figure B.17. The graph of the parabola has been reflected in the x-axis, and shifted two units to the right and three units upward from standard position.

c. The graph of

$$\frac{(x - 3)^2}{1^2} - \frac{(y - 2)^2}{3^2} = 1$$

is a hyperbola whose center is the point $(3, 2)$. The transverse axis is horizontal with a length of $2(1) = 2$. The conjugate axis is vertical with a length of $2(3) = 6$, as shown in Figure B.18. The graph of the hyperbola has been shifted three units to the right and two units upward from standard position.

d. The graph of

$$\frac{(x - 2)^2}{3^2} + \frac{(y - 1)^2}{2^2} = 1$$

is an ellipse whose center is the point $(2, 1)$. The major axis of the ellipse is horizontal with a length of $2(3) = 6$. The minor axis of the ellipse is vertical with a length of $2(2) = 4$, as shown in Figure B.19. The graph of the ellipse has been shifted two units to the right and one unit upward from standard position.

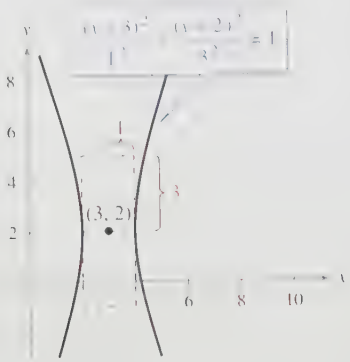


FIGURE B.18

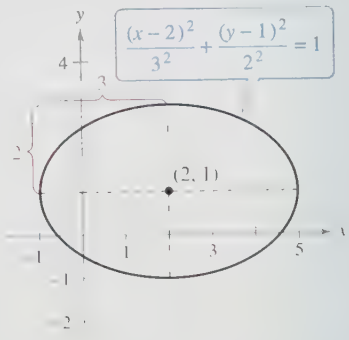


FIGURE B.19

Checkpoint 1

Describe the translation of the graph of

$$(x + 4)^2 + (y - 3)^2 = 5^2.$$

Writing Equations of Conics in Standard Form

Example 2 Finding the Standard Form of a Parabola

Find the vertex and focus of the parabola given by $x^2 - 2x + 4y - 3 = 0$.

SOLUTION Complete the square to write the equation in standard form.

$$x^2 - 2x + 4y - 3 = 0$$ Write original equation.

$$x^2 - 2x = -4y + 3$$ Group terms.

$$x^2 - 2x + 1 = -4y + 3 + 1$$ Complete the square.

$$(x - 1)^2 = -4y + 4$$ Write in completed square form.

$$(x - 1)^2 = 4(-1)(y - 1)$$ Standard form, $(x - h)^2 = 4p(y - k)$

From this standard form, it follows that $h = 1$, $k = 1$, and $p = -1$. Because the axis is vertical and p is negative, the parabola opens downward. The vertex is $(h, k) = (1, 1)$ and the focus is $(h, k + p) = (1, 0)$. (See Figure B.20.)

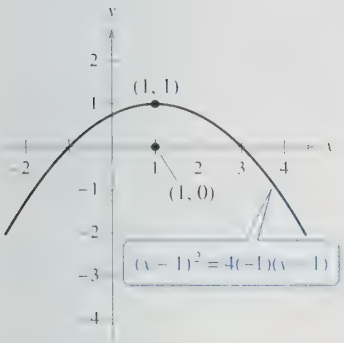


FIGURE B.20

Checkpoint 2

Find the vertex and focus of the parabola given by $x^2 + 2x - 4y + 5 = 0$.

In Examples 1(b) and 2, p is the *directed distance* from the vertex to the focus. Because the axis of the parabola is vertical and $p = -1$, the focus is one unit *below* the vertex, and the parabola opens downward.

Example 3 Sketching an Ellipse

Sketch the ellipse given by $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

SOLUTION Complete the square to write the equation in standard form.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0$$ Write original equation.

$$(x^2 + 6x + \quad) + (4y^2 - 8y + \quad) = -9$$ Group terms.

$$(x^2 + 6x + \quad) + 4(y^2 - 2y + \quad) = -9$$ Factor 4 out of y-terms.

$$(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -9 + 9 + 4(1)$$ Complete the squares.

$$(x + 3)^2 + 4(y - 1)^2 = 4$$ Write in completed square form.

$$\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{1} = 1$$ Divide each side by 4.

$$\frac{(x + 3)^2}{2^2} + \frac{(y - 1)^2}{1^2} = 1$$ $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

From this standard form, it follows that the center is $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, the endpoints of the minor axis lie one unit up and down from the center. The ellipse is shown in Figure B.21.

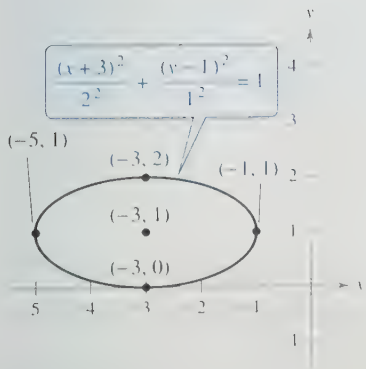


FIGURE B.21

Checkpoint 3

Sketch the ellipse given by $x^2 + 9y^2 - 4x + 18y + 4 = 0$.

Example 4 **Sketching a Hyperbola**

Sketch the hyperbola given by

$$y^2 - 4x^2 + 4y + 24x - 41 = 0.$$

SOLUTION Complete the square to write the equation in standard form.

$$y^2 - 4x^2 + 4y + 24x - 41 = 0$$ Write original equation.

$$(y^2 + 4y + \quad) - (4x^2 - 24x + \quad) = 41$$ Group terms.

$$(y^2 + 4y + \quad) - 4(x^2 - 6x + \quad) = 41$$ Factor 4 out of x-terms.

$$(y^2 + 4y + 4) - 4(x^2 - 6x + 9) = 41 + 4 - 4(9)$$ Complete the squares.

$$(y + 2)^2 - 4(x - 3)^2 = 9$$ Write in completed square form.

$$\frac{(y + 2)^2}{9} - \frac{4(x - 3)^2}{9} = 1$$ Divide each side by 9.

$$\frac{(y + 2)^2}{9} - \frac{(x - 3)^2}{9/4} = 1$$ Change 4 to $\frac{1}{1/4}$.

$$\frac{(y + 2)^2}{3^2} - \frac{(x - 3)^2}{(3/2)^2} = 1$$ $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$

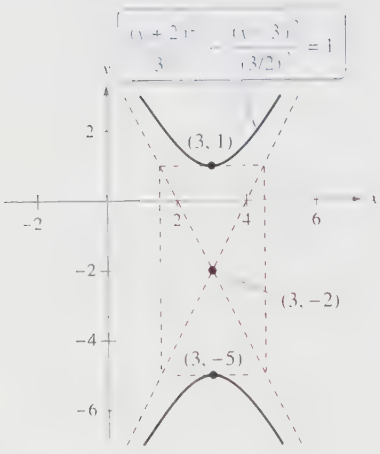


FIGURE B.22

From the standard form, it follows that the transverse axis is vertical and the center lies at $(h, k) = (3, -2)$. Because the denominator of the y-term is $a^2 = 3^2$, you know that the vertices lie three units above and below the center.

Vertices: $(3, 1)$ and $(3, -5)$

To sketch the hyperbola, draw a rectangle whose top and bottom pass through the vertices. Because the denominator of the x-term is $b^2 = \left(\frac{3}{2}\right)^2$, locate the sides of the rectangle $\frac{3}{2}$ units to the right and left of the center, as shown in Figure B.22. Finally, sketch the asymptotes by drawing lines through the opposite corners of the rectangle. Using these asymptotes, you can complete the graph of the hyperbola, as shown in Figure B.22.

Checkpoint 4

Sketch the hyperbola given by

$$4x^2 - y^2 + 16x - 2y - 1 = 0.$$

To find the foci in Example 4, first find c. Recall from Appendix B.1 that $b^2 = c^2 - a^2$. So,

$$c^2 = a^2 + b^2$$

$$c^2 = 9 + \frac{9}{4}$$

$$c^2 = \frac{45}{4}$$

$$c = \frac{3\sqrt{5}}{2}.$$

Because the transverse axis is vertical, the foci lie c units above and below the center.

Foci: $\left(3, -2 + \frac{3\sqrt{5}}{2}\right)$ and $\left(3, -2 - \frac{3\sqrt{5}}{2}\right)$

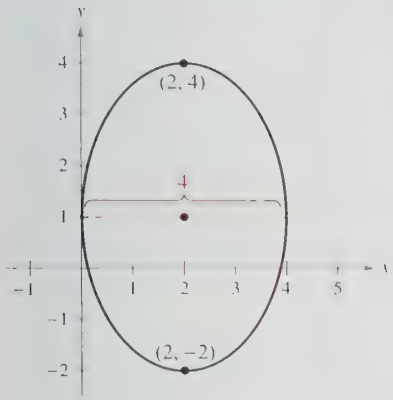


FIGURE B.23

Example 5 Writing the Equation of an Ellipse

Write the standard form of the equation of the ellipse whose vertices are $(2, -2)$ and $(2, 4)$. The length of the minor axis of the ellipse is 4, as shown in Figure B.23.

SOLUTION The center of the ellipse lies at the midpoint of its vertices. So, the center is

$$(h, k) = \left(\frac{2 + 2}{2}, \frac{4 + (-2)}{2}\right)$$

$$= (2, 1). \qquad \text{Center}$$

Because the vertices lie on a vertical line and are six units apart, it follows that the major axis is vertical and has a length of $2a = 6$. So, $a = 3$. Moreover, because the minor axis has a length of 4, it follows that $2b = 4$, which implies that $b = 2$. So, you can conclude that the standard form of the equation of the ellipse is

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \qquad \text{Major axis is vertical.}$$

$$\frac{(x - 2)^2}{2^2} + \frac{(y - 1)^2}{3^2} = 1. \qquad \text{Write in standard form.}$$

Checkpoint 5

Write the standard form of the equation of the ellipse whose vertices are $(-1, 4)$ and $(4, 4)$. The length of the minor axis of the ellipse is 3.

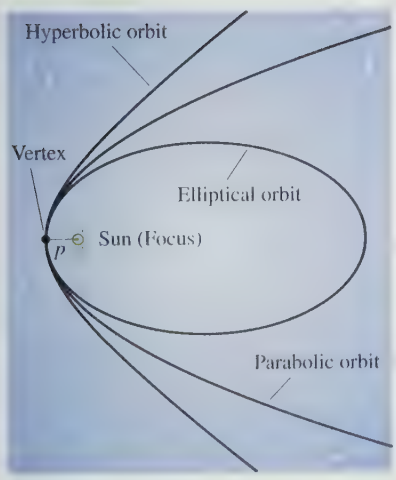


FIGURE B.24

An interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. For example, Halley's comet has an elliptical orbit, and reappearance of this comet can be predicted every 76 years. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure B.24.

If p is the distance between the vertex and the focus (in meters), and v is the speed of the comet at the vertex (in meters per second), then the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{\dfrac{2GM}{p}}$
2. Parabola: $v = \sqrt{\dfrac{2GM}{p}}$
3. Hyperbola: $v > \sqrt{\dfrac{2GM}{p}}$

In these expressions, $M = 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

SKILLS WARM UP B.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section B.1.

In Exercises 1–10, identify the conic represented by the equation.

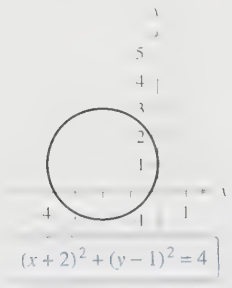
1. $\frac{x^2}{4} - \frac{y^2}{4} = 1$
2. $\frac{x^2}{9} + \frac{y^2}{1} = 1$
3. $2x + y^2 = 0$
4. $\frac{x^2}{4} + \frac{y^2}{16} = 1$
5. $4x^2 + 4y^2 = 25$
6. $\frac{y^2}{4} - \frac{x^2}{2} = 1$
7. $x^2 - 6y = 0$
8. $3x - y^2 = 0$
9. $\frac{9x^2}{16} + \frac{y^2}{4} = 1$
10. $3y^2 - 3x^2 = 48$

Exercises B.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

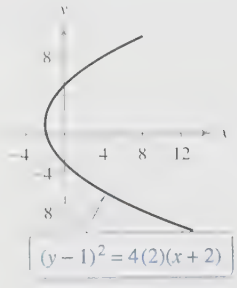
Equations of Conic Sections In Exercises 1–6, describe the translation of the graph of the conic from the standard position. *See Example 1.*

1.



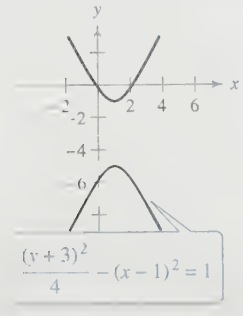
$(x+2)^2 + (y-1)^2 = 4$

2.



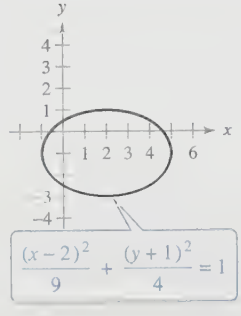
$(y-1)^2 = 4(2)(x+2)$

3.



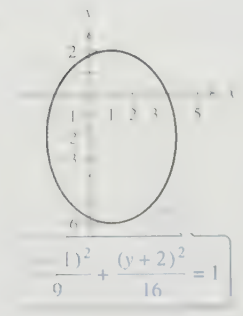
$\frac{(y+3)^2}{4} - (x-1)^2 = 1$

4.



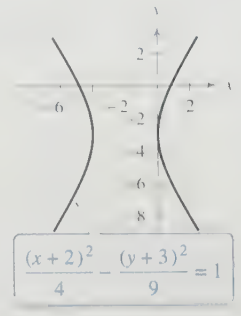
$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$

5.



$\frac{1)^2}{9} + \frac{(y+2)^2}{16} = 1$

6.



$\frac{(x+2)^2}{4} - \frac{(y+3)^2}{9} = 1$

Finding the Standard Form of a Parabola In Exercises 7–16, find the vertex, focus, and directrix of the parabola. Then sketch its graph. *See Example 2.*

7. $(x-1)^2 + 8(y+2) = 0$
8. $(x+3) + (y-2)^2 = 0$
9. $\left(y + \frac{1}{2}\right)^2 = 2(x-5)$
10. $\left(x + \frac{1}{2}\right)^2 = 4(y-3)$
11. $y = \frac{1}{4}(x^2 - 2x + 5)$
12. $y = -\frac{1}{6}(x^2 + 4x - 2)$
13. $4x - y^2 - 2y - 33 = 0$
14. $y^2 + x + y = 0$
15. $y^2 + 6y + 8x + 25 = 0$
16. $x^2 - 2x + 8y + 9 = 0$

Finding the Standard Form of a Parabola In Exercises 17–24, find the standard form of the equation of the parabola with the given characteristics.

17. Vertex: $(3, 2)$; focus: $(1, 2)$
18. Vertex: $(-1, 2)$; focus: $(-1, 0)$
19. Vertex: $(0, 4)$; directrix: $y = 2$
20. Vertex: $(-2, 1)$; directrix: $x = 1$
21. Focus: $(2, 2)$; directrix: $x = -2$
22. Focus: $(0, 0)$; directrix: $y = 4$
23. Vertex: $(0, 4)$; passes through $(-2, 0)$ and $(2, 0)$
24. Vertex: $(2, 4)$; passes through $(0, 0)$ and $(4, 0)$

Sketching an Ellipse In Exercises 25–32, find the center, foci, and vertices of the ellipse. Then sketch its graph. *See Example 3.*

25. $\dfrac{(x-1)^2}{9}+\dfrac{(y-5)^2}{25}=1$

26. $(x+2)^2+\dfrac{(y+4)^2}{1/4}=1$

27. $9x^2+4y^2+36x-24y+36=0$

28. $9x^2+4y^2-36x+8y+31=0$

29. $16x^2+25y^2-32x+50y+16=0$

30. $9x^2+25y^2-36x-50y+61=0$

31. $12x^2+20y^2-12x+40y-37=0$

32. $36x^2+9y^2+48x-36y+43=0$

Sketching a Hyperbola In Exercises 33–42, find the center, vertices, and foci of the hyperbola. Then sketch its graph, using asymptotes as sketching aids. *See Example 4.*

33. $\dfrac{(x-1)^2}{4}-\dfrac{(y+2)^2}{1}=1$

34. $\dfrac{(x+1)^2}{144}-\dfrac{(y-4)^2}{25}=1$

35. $(y+6)^2-(x-2)^2=1$

36. $\dfrac{(y-1)^2}{1/4}-\dfrac{(x+3)^2}{1/9}=1$

37. $9x^2-y^2-36x-6y+18=0$

38. $x^2-9y^2+36y-72=0$

39. $9y^2-x^2+2x+54y+62=0$

40. $16y^2-x^2+2x+64y+63=0$

41. $x^2-9y^2+2x-54y-107=0$

42. $9x^2-y^2+54x+10y+55=0$

Finding the Standard Equation of an Ellipse In Exercises 43–52, find the standard form of the equation of the ellipse with the given characteristics. *See Example 5.*

43. Vertices: $(0, 2), (4, 2)$; minor axis of length 2

44. Vertices: $(3, 1), (3, 9)$; minor axis of length 6

45. Foci: $(0, 0), (4, 0)$; major axis of length 8

46. Foci: $(0, 0), (0, 8)$; major axis of length 16

47. Center: $(2, -1)$; vertex: $\left(2, \frac{1}{2}\right)$; minor axis of length 2

48. Center: $(4, 0)$; vertex: $(4, -4)$; minor axis of length 6

49. Center: $(3, 2)$; $a = 3c$; foci: $(1, 2), (5, 2)$

50. Center: $(0, 4)$; $a = 2c$; vertices $(-4, 4), (4, 4)$

51. Vertices: $(5, 0), (5, 12)$; endpoints of minor axis: $(0, 6), (10, 6)$

52. Vertices: $(-12, 0), (0, 0)$; endpoints of minor axis: $(-6, 3), (-6, -3)$

Finding the Standard Equation of a Hyperbola In Exercises 53–60, find the standard form of the equation of the hyperbola with the given characteristics.

53. Vertices: $(2, 0), (6, 0)$; foci: $(0, 0), (8, 0)$

54. Vertices: $(2, 3), (2, -3)$; foci: $(2, 5), (2, -5)$

55. Vertices: $(4, 1), (4, 9)$; foci: $(4, 0), (4, 10)$

56. Vertices: $(-2, 1), (2, 1)$; foci: $(-3, 1), (3, 1)$

57. Vertices: $(2, 3), (2, -3)$; passes through $(0, 5)$

58. Vertices: $(-2, 1), (2, 1)$; passes through $(4, 3)$

59. Vertices: $(0, 2), (6, 2)$; asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

60. Vertices: $(3, 0), (3, 4)$; asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

Identifying Conics In Exercises 61–68, identify the conic by writing the equation in standard form. Then sketch its graph.

61. $x^2+y^2-6x+4y+9=0$

62. $x^2+4y^2-6x+16y+21=0$

63. $4x^2-y^2-4x-3=0$

64. $y^2-4y-4x=0$

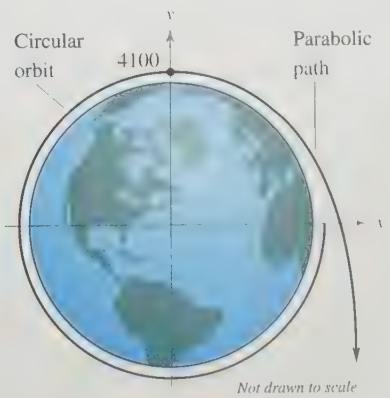
65. $4x^2+3y^2+8x-24y+51=0$

66. $4y^2-2x^2-4y+8x-15=0$

67. $25x^2-10x-200y-119=0$

68. $4x^2+4y^2-16y+15=0$

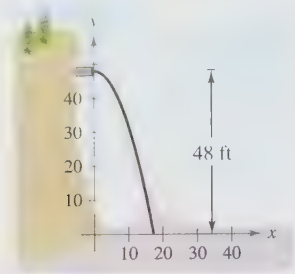
69. Satellite Orbit A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour. When this velocity is multiplied by $\sqrt{2}$, the satellite has the minimum velocity necessary to escape Earth's gravity, and it follows a parabolic path with the center of Earth as the focus (see figure).



(a) Find the escape velocity of the satellite.

(b) Find an equation of its path (assume the radius of Earth is 4000 miles).

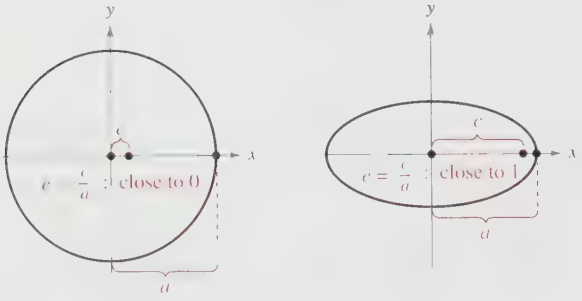
70. Fluid Flow Water is flowing from a horizontal pipe 48 feet above the ground. The falling stream of water has the shape of a parabola whose vertex (0, 48) is at the end of the pipe (see figure). The stream of water strikes the ground at the point $(10\sqrt{3}, 0)$. Find the equation of the path taken by the water.



Eccentricity of an Ellipse In Exercises 71–78, the flatness of an ellipse is measured by its **eccentricity** e, defined by

$$e = \frac{c}{a}, \text{ where } 0 < e < 1.$$

When an ellipse is nearly circular, e is close to 0. When an ellipse is elongated, e is close to 1 (see figures).



71. Find an equation of the ellipse with vertices $(\pm 5, 0)$ and eccentricity $e = \frac{3}{5}$.

72. Find an equation of the ellipse with vertices $(0, \pm 8)$ and eccentricity $e = \frac{1}{2}$.

73. Planetary Motion Earth moves in an elliptical orbit with the sun at one of the foci (see figure). The length of half of the major axis is $a = 9.2956 \times 10^7$ miles and the eccentricity is 0.017. Find the shortest distance (perihelion) and the greatest distance (aphelion) between Earth and the sun.

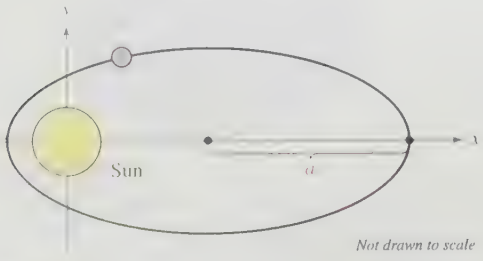
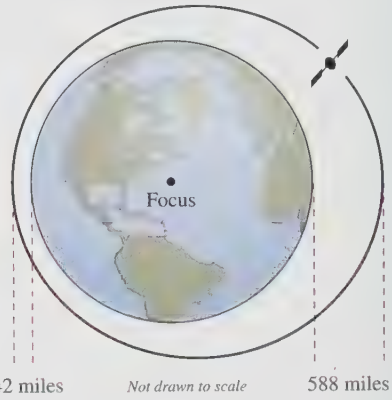


Figure for 73–75

74. Planetary Motion The dwarf planet Pluto moves in an elliptical orbit with the sun at one of the foci (see figure). The length of half of the major axis is 3.670×10^9 miles and the eccentricity is 0.249. Find the shortest distance and the greatest distance between Pluto and the sun.

75. Planetary Motion Saturn moves in an elliptical orbit with the sun at one of the foci (see figure). The shortest distance and the greatest distance between Saturn and the sun are 1.3495×10^9 kilometers and 1.5040×10^9 kilometers, respectively. Find the eccentricity of the orbit.

76. Satellite Orbit The first artificial satellite to orbit Earth was Sputnik I (launched by the former Soviet Union in 1957). Its highest point above Earth's surface was 588 miles, and its lowest point was 142 miles (see figure). Assume that the center of Earth is one of the foci of the elliptical orbit and that the radius of Earth is 4000 miles. Find the eccentricity of the orbit.



77. Alternate Form of Equation of an Ellipse Show that the equation of an ellipse can be written as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

78. Comet Orbit Halley's comet has an elliptical orbit with the sun at one focus. The eccentricity of the orbit is approximately 0.97. The length of the major axis of the orbit is approximately 35.88 astronomical units. (An astronomical unit is about 93 million miles.) Find the standard form of the equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

79. Australian Football In Australia, football by *Australian Rules* (or rugby) is played on elliptical fields. The fields can be a maximum of 170 yards wide and a maximum of 200 yards long. Let the center of a field of maximum size be represented by the point (0, 85). Find the standard form of the equation of the ellipse that represents this field.

Answers to Selected Exercises

Chapter 0

Section 0.1 *(page 8)*

1. (a) Natural: $\{5\}$
(b) Integer: $\{-9, 5\}$
(c) Rational: $\{-9, -\frac{7}{2}, 5, \frac{2}{3}, 0.1\}$
(d) Irrational: $\{\sqrt{2}\}$

3. (a) Natural: $\{12, 1, \sqrt{4}\}$ (*Note*: $\sqrt{4} = 2$)
(b) Integer: $\{12, -13, 1, \sqrt{4}\}$
(c) Rational: $\{12, -13, 1, \sqrt{4}, \frac{3}{2}\}$
(d) Irrational: $\{\sqrt{6}\}$

5. (a) Natural: $\{\frac{8}{2}, 9\}$ (*Note*: $\frac{8}{2} = 4$)
(b) Integer: $\{\frac{8}{2}, -4, 9\}$
(c) Rational: $\{\frac{8}{2}, -\frac{8}{3}, -4, 9, 14.2\}$
(d) Irrational: $\{\sqrt{10}\}$

7. $0.\overline{6}$ **9.** $0.\overline{126}$ **11.** $-1 < 2.5$

13. $-6 < 7$



15. $1 > -3.5$



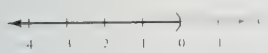
17. $\frac{5}{6} > \frac{2}{3}$



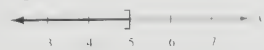
19. $\frac{204}{60}, \frac{31}{9}, 3.45, 2\sqrt{3}, \frac{7}{2}$

21. $\frac{127}{90}, \frac{584}{413}, \frac{7071}{5000}, \sqrt{2}, \frac{47}{33}$ **23.** $x \leq 4$ **25.** $0 < x \leq 3$

27. $x < 0$ denotes all negative real numbers.



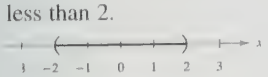
29. $x \leq 5$ denotes all real numbers less than or equal to 5.



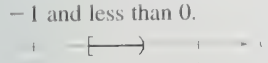
31. $x > 3.5$ denotes all real numbers greater than 3.5.



33. $-2 < x < 2$ denotes all real numbers greater than -2 and less than 2.



35. $-1 \leq x < 0$ denotes all real numbers greater than or equal to -1 and less than 0.



37. $x > 0$ **39.** $5 < y \leq 12$ **41.** $A \geq 21$
43. $0.5\% \leq r \leq 3.8\%$ **45.** 10 **47.** -3 **49.** -1.6
51. -1 **53.** $\pi - 3$ **55.** $|-7| = |7|$
57. $|-3| > -|-3|$ **59.** $-|-2| = -|2|$ **61.** 4
63. 1 **65.** 7 **67.** 51 **69.** 3.67 **71.** $|z - 6| > 1$
73. $|x + 10| \geq 6$ **75.** $|y - 0| \geq 6 \Rightarrow |y| \geq 6$
77. $|x - m| > 5$ **79.** 81 miles **81.** 89°F

		Passes Budget Variance Test
$\lvert a - b\rvert$	*0.05b*	
83. \$876.55	\$1500	No
85. \$264.32	\$600	Yes

		Passes Quality Control Test
$\lvert a - b\rvert$	*0.0012b*	
87. 0.002	0.0168	Yes
89. 0.027	0.0192	No
91. 0.022	0.0156	No

93. When u and v have the same sign, $|u + v| = |u| + |v|$. When u and v have different signs, $|u + v| < |u| + |v|$.

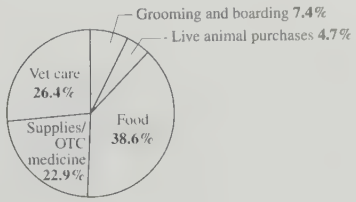
95. $a \leq 0$; If the original value of a is negative, then $|a|$ results in a positive number. Because a is negative, the expression $|a| = a$ states that $|a|$ is equal to a negative number, which can never happen. So, if a is originally negative, $|a|$ must equal $-a$, which is a positive value.

Section 0.2 *(page 18)*

Skills Warm Up *(page 18)*

1. $-4 < -2$ **2.** $0 > -3$ **3.** $-\frac{11}{9} > -\frac{13}{9}$
4. $-\frac{12}{33} < \frac{11}{33}$ **5.** $|6 - 4| = 2$ **6.** $|2 - (-2)| = 4$
7. $|0 - (-5)| = 5$ **8.** $|3 - (-1)| = 4$

1. $7x, 4$ **3.** $x^2, -4x, 8$ **5.** $2x^2, -9x, 13$ **7.** -2
9. 6 **11.** (a) -10 (b) -6 **13.** (a) 14 (b) 2
15. (a) $\frac{1}{2}$ (b) Undefined. You cannot divide by zero.
17. 1 **19.** 35 **21.** $\frac{1}{4}$ **23.** Commutative (addition)
25. Inverse (addition) **27.** Distributive Property
29. Inverse (multiplication) **31.** Identity (addition)
33. Identity (multiplication) **35.** Associative (addition)
37. $x(3y) = (x \cdot 3)y$ Associative (multiplication)
$= (3x)y$ Commutative (multiplication)
39. $2^4 \cdot 3$ **41.** $2^4 \cdot 3 \cdot 5$ **43.** $3x$ **45.** $\frac{17x}{20}$ **47.** $\frac{12}{x}$
49. 7 **51.** $\frac{1}{24}$ **53.** $\frac{7}{20}$ **55.** $\frac{5}{6}$ **57.** -0.13
59. 1.56 **61.** -4 **63.** 5 **65.** 23.8 **67.** 46.25
69. (a) 20.7%
(b) National Defense: about \$617 billion
Other: about \$495 billion
Income Security: about \$433 billion
Education and Veterans' Benefits: about \$149 billion
Health and Medicare: about \$671 billion
Social Security: about \$617 billion
71. $\approx$1695 patients
73. Food: 38.6%
Vet care: 26.4%
Supplies/OTC medicine: 22.9%
Live animal purchases: 4.7%
Grooming and boarding: 7.4%



75. Answers will vary.

Section 0.3 *(page 27)*

Skills Warm Up *(page 27)*

1. 1 **2.** 5 **3.** -4 **4.** $-\frac{3}{2}$ **5.** $-\frac{1}{4}$ **6.** 1
7. $\frac{3}{2}$ **8.** 0 **9.** $\frac{3}{8}$ **10.** 1

1. 64 **3.** 8 **5.** 729 **7.** -81 **9.** 5184 **11.** $-\frac{3}{5}$
13. 1 **15.** $-125z^3$ **17.** $16x^7$ **19.** $10x^4$ **21.** $-3z^7$
23. $\frac{5y^4}{3}$ **25.** $\frac{5184}{y^7}$ **27.** $\frac{7}{x}$ **29.** $\frac{1}{x}$ **31.** 27^n
33. $\frac{5x}{3} + 5$ **35.** $1, x \neq 0$ **37.** $\frac{1}{2}$ **39.** 8 **41.** $-\frac{3}{10}$
43. $\frac{4}{x^2}$ **45.** $\frac{3a^3}{b^3}$ **47.** $\frac{10}{x}$ **49.** $\frac{125x^9}{y^{12}}$ **51.** 18
53. $\frac{7}{16}$ **55.** $10^3 = 1000$
$10^2 = 100$ As the exponents decrease
$10^1 = 10$ by 1, the answers decrease
$10^0 = 1$ by a factor of 10.
$10^{-1} = 0.1$
$10^{-2} = 0.01$
57. 5.73×10^7 square miles **59.** 9.46×10^{12} kilometers
61. 1×10^{-7} meter **63.** 350,000,000 air sacs
65. 0.00000000000000000001602 coulomb
67. (a) 6.0×10^4 (b) 2.0×10^{11}
69. (a) $(4.8 \times 10^{10})(2.5 \times 10^8) = 1.2 \times 10^{19}$
(b) $\frac{1.2 \times 10^{-8}}{6.4 \times 10^{-6}} = 1.875 \times 10^{-3}$
71. 161.100 **73.** 0.325 **75.** 1.007843137
77. (a) 3.071×10^6 (b) 3.077×10^{10}
79. (a) 4.907×10^{17} (b) 1.479
81. (a) \$15,682.76 (b) \$15,680.58
(c) \$15,672.12 (d) \$15,650.28
As the number of compoundings per year decreases, the balance in the account also decreases.
83. About 4.32% **85.** About 4 hours

Section 0.4 *(page 36)*

Skills Warm Up *(page 36)*

1. $\frac{4}{25}$ **2.** 48 **3.** $-8x^3$ **4.** $6x^7$ **5.** $28x^6$
6. $\frac{1}{5}x^2$ **7.** $3z^4$ **8.** $\frac{25}{4x^2}$ **9.** 1 **10.** $(x + 2)^{10}$

1. $\sqrt{4} = \pm 2$ **3.** $\sqrt[3]{-8} = -2$ **5.** $\sqrt[3]{27} = 3$
7. $\sqrt[4]{-1}$ is not a real number. **9.** $\sqrt[6]{64} = \pm 2$ **11.** 3
13. 3 **15.** 2 **17.** $\sqrt[4]{-16}$ is not a real number.
19. $-\frac{4}{5}$ **21.** $-3\sqrt[3]{2}$ **23.** -125 **25.** $2x\sqrt[3]{2x^2}$
27. $4x^3$ **29.** $\frac{z}{5}$ **31.** 90 **33.** 45 **35.** 5 **37.** $\frac{\sqrt{5}}{5}$
39. $4\sqrt[3]{4}$ **41.** $\frac{x(5 + \sqrt{3})}{11}$ **43.** $3(\sqrt{6} - \sqrt{5})$
45. $9^{1/2} = 3$ **47.** $\sqrt[5]{32} = 2$ **49.** $(-216)^{1/3} = -6$
51. $81^{3/4} = 27$ **53.** $\sqrt[3]{125^2} = 25$ **55.** 4 **57.** $\frac{27}{8}$
59. -4 **61.** 25 **63.** $2^{1/2}$ **65.** 1 **67.** x^3
69. $8x^6y^3$ **71.** $\sqrt{y}$ **73.** $2\sqrt[4]{2}$ **75.** $\sqrt[4]{(x + 1)^3}$
77. $2\sqrt{x}$ **79.** $31\sqrt{2}$ **81.** $-2\sqrt{y}$ **83.** 3.557
85. 2.140 **87.** 1.787 **89.** 0.382
91. (4 − (√) 7)) ÷ 3 **93.** $\sqrt{5} + \sqrt{3} > \sqrt{5 + 3}$
95. $\sqrt{3} \cdot \sqrt[4]{3} > \sqrt[8]{3}$ **97.** $10\sqrt{11}$ feet $\times$ $10\sqrt{11}$ feet
99. 25.01% **101.** $\approx$ 2.221 seconds **103.** No
105. $\approx$ 40.2 miles per hour **107.** $\approx$ 17.4°F
109. $\approx$ 494 vibrations per second
111. a; Higher notes have higher frequencies.
113. $(2/\sqrt{5})^2 = \frac{4}{5}$. This is not the same as rationalizing the denominator, which gives the result $2/\sqrt{5} \cdot \sqrt{5}/\sqrt{5} = (2\sqrt{5})/5$.
115. No; $\sqrt{4x^2} \geq 0$, whereas $2x$ can be less than zero.

Quiz Yourself *(page 39)*

1. $-|-7| < |-7|$ **2.** $-(-3) = |-3|$
3. Natural: $\{\sqrt{1}\}$
Integer: $\{-6, \sqrt{1}\}$
Rational: $\{-6, \sqrt{1}, \frac{3}{4}, -\frac{1}{2}\}$
Irrational: $\{\sqrt{11}\}$
4. $x \geq 0$ **5.** $|5 - (-6)| = 11$ **6.** $3x^2, -7x, 2$
7. -4 **8.** -13 **9.** $\frac{5}{14}$ **10.** $\frac{11}{9}$ **11.** $-\frac{2}{x}$
12. $\frac{y^{10}}{3}$ **13.** 1.4×10^{-5} **14.** \$5579.71
15. -1 **16.** -64 **17.** 9 **18.** $-\sqrt[3]{3}$ **19.** $2\sqrt{3}$
20. 22 cm $\times$ 22 cm $\times$ 22 cm

Section 0.5 *(page 46)*

Skills Warm Up *(page 46)*

1. $42x^3$ **2.** $-45x^5$ **3.** $-20z^2$ **4.** $-15x^{-1}$
5. $-27x^6$ **6.** $-3x^6$ **7.** $\frac{9}{4}z^3, z \neq 0$
8. $\frac{8}{5}x^3, x \neq 0$ **9.** $\frac{9}{4x^2}$ **10.** xy^2

1. $4x^2 + 3x + 2$
Degree: 2; leading coefficient: 4
3. $x^7 - 8$
Degree: 7; leading coefficient: 1
5. $-2x^5 + 6x^4 - x + 1$
Degree: 5; leading coeficient: -2
7. Polynomial, $-3x^3 + 2x + 8$, degree 3
9. Not a polynomial
11. Polynomial, $-w^4 + 2w^3 + w^2$, degree 4
13. (a) -3 (b) 1 (c) 5 (d) 17
15. (a) -10 (b) -1 (c) 4 (d) 5
17. $-2x - 10$ **19.** $2x^3 - 4x - 5$ **21.** $8x^3 + 29x^2 + 11$
23. $3x^3 - 6x^2 + 3x$ **25.** $9x^3 - 21x^2$ **27.** $x^2 + 7x + 12$
29. $6x^2 - 7x - 5$ **31.** $x^4 + 3x^3 - x^2 - 6x$
33. $-3x^4 - x^3 - 12x^2 - 19x - 5$ **35.** $x^2 - 25$
37. $x^2 + 12x + 36$ **39.** $4x^2 - 20xy + 25y^2$
41. $x^2 + 2xy + y^2 - 6x - 6y + 9$

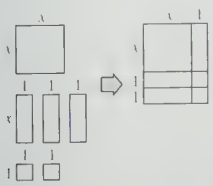
43. $x^3 + 3x^2 + 3x + 1$ **45.** $8x^3 - 12x^2y + 6xy^2 - y^3$
47. $9y^4 - 1$ **49.** $m^2 - n^2 - 6m + 9$ **51.** $x^4 + x^2 + 1$
53. $24x^2 - 53x - 7$ **55.** $4x^3 + 6x^2 - 8x - 5$
57. The middle term was omitted when squaring the binomial.
$(x - 3)^2 = x^2 - 2(x)(3) + 3^2 = x^2 - 6x + 9$
59. $1000r^3 + 3000r^2 + 3000r + 1000$ **61.** Yes; yes; no
63. 77,918.76; 82,562.74; In the years 2006 and 2007, the total amounts of federal student aid disbursed were approximately \$77,918,760,000 and \$82,562,740,000.
65. $x = 4$ inches: $V = 6076$ cubic inches
$x = 6$ inches: $V = 7290$ cubic inches
$x = 10$ inches: $V = 7030$ cubic inches
$x = 6$ inches produces the greatest volume.
67. Answers will vary.

Section 0.6 *(page 53)*

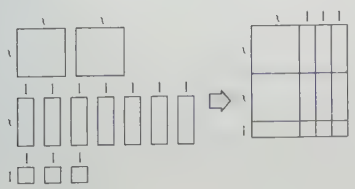
Skills Warm Up *(page 53)*

1. $15x^2 - 6x$ **2.** $-2y^2 - 2y$ **3.** $4x^2 + 12x + 9$
4. $9x^2 - 48x + 64$ **5.** $2x^2 + 13x - 24$
6. $-5z^2 - z + 4$ **7.** $4y^2 - 1$ **8.** $x^2 - a^2$
9. $x^3 + 12x^2 + 48x + 64$
10. $8x^3 - 36x^2 + 54x - 27$

1. $3(x + 2)$ **3.** $3x(x^2 - 2)$ **5.** $(x - 5)(3x + 8)$
7. $(x + 6)(x - 6)$ **9.** $(4x + 3y)(4x - 3y)$
11. $(x + 1)(x - 3)$ **13.** $(9x^2 + 1)(3x + 1)(3x - 1)$
15. $3(x + 1)(x - 1)$ **17.** $2(2x + 5y)(2x - 5y)$
19. $(5 - x)(5 + x)(x - 1)$ **21.** $(x - 2)^2$ **23.** $(2y + 3)^2$
25. $\left(y - \frac{1}{3}\right)^2$ **27.** $(x - 2)(x^2 + 2x + 4)$
29. $(y + 5)(y^2 - 5y + 25)$ **31.** $\left(x - \frac{2}{3}\right)\left(x^2 + \frac{2}{3}x + \frac{4}{9}\right)$
33. $(x + 2)(x - 1)$ **35.** $(w - 2)(w - 3)$
37. $(y + 5)(y - 4)$ **39.** $(x - 20)(x - 10)$
41. $(3x - 2)(x - 1)$ **43.** $(3x + 1)(3x - 2)$
45. $(6x + 1)(x + 6)$ **47.** $(x - 1)(x^2 + 2)$
49. $(2x - 1)(x^2 - 3)$ **51.** $(2 - y^3)(3 + y)$
53. $(2x - 1)(x + 2)$ **55.** $(x - 2)(6x + 5)$ **57.** $4x(x - 2)$
59. $y\left(y - \frac{1}{3}\right)\left(y + \frac{1}{3}\right)$ **61.** $3(x + 4)(x - 4)$ **63.** $(x - 1)^2$
65. $(1 - 2x)^2$ **67.** $y(2y + 3)(y - 5)$ **69.** Prime
71. $(3x + 1)(x^2 + 5)$ **73.** $x(x - 4)(x^2 + 1)$
75. $-x(x + 10)$ **77.** $(x + 1)^2(x - 1)^2$
79. $2(t - 2)(t^2 + 2t + 4)$
81.



83.



85. $(2x - 3)$ feet

87. (a) $(x + 8)(x + 3)$; factoring by grouping
(b) $(3x - 5)(x + 4)$; factoring by trial and error
89. $c = \{7, 12, 15, 16\}$; Answers will vary.
91. Answers will vary. Sample answer:
(1) Find a combination of factors of 2 and -15 such that the outer and inner products add up to the middle term $-7x$.
$2x^2 - 7x - 15 = (2x + 3)(x - 5)$
(2) Rewrite -7 as the sum of two factors of the product $2(-15)$. Then factor by grouping.
$2x^2 - 7x - 15 = 2x^2 - 10x + 3x - 15$
$= 2x(x - 5) + 3(x - 5)$
$= (x - 5)(2x + 3)$
93. Box 1: $V = (a - b)a^2$
Box 2: $V = (a - b)ab$
Box 3: $V = (a - b)b^2$
Multiplying $(a - b)$ by each term of $(a^2 + ab + b^2)$ produces the volumes of the three boxes.

Section 0.7 *(page 60)*

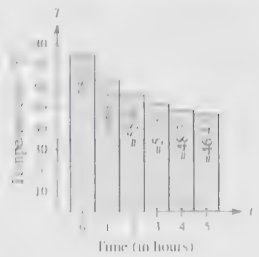
Skills Warm Up *(page 60)*

1. $5x^2(1 - 3x)$ **2.** $(4x + 3)(4x - 3)$
3. $(3x - 1)^2$ **4.** $(2y + 3)^2$ **5.** $(z + 3)(z + 1)$
6. $2(x - 5)(x - 10)$ **7.** $(3 - x)(1 + 3x)$
8. $(3x - 1)(x - 15)$ **9.** $(s + 1)(s + 2)(s - 2)$
10. $2(x - 4)(x^2 + 4x + 16)$

1. (a) No (b) Yes **3.** (a) Yes (b) Yes
5. All real numbers **7.** All real numbers x except $x = 3$
9. All real numbers except $x = 0$ and $x = 4$
11. All real numbers greater than or equal to -1
13. $3x$, $x \neq 0$ **15.** $x - 2$, $x \neq 2$, $x \neq 0$
17. $x + 2$, $x \neq -2$, $x \neq 3$ **19.** $\frac{3x}{2}$, $x \neq 0$
21. $\frac{x}{2(x + 1)}$ **23.** $-\frac{1}{2}$, $x \neq 5$
25. $-(x + 5)$, $x \neq 5$ **27.** $\frac{x(x + 3)}{x - 2}$, $x \neq -2$
29. $\frac{y - 4}{y + 6}$, $y \neq 3$ **31.** $-1 - x^2$, $x \neq 2$ **33.** $z - 3$
35. $\frac{1}{5(x - 2)}$, $x \neq 1$ **37.** $-\frac{x(x + 7)}{x + 1}$, $x \neq 9$
39. $\frac{r + 1}{r}$, $r \neq 1$ **41.** $\frac{t - 3}{(t + 3)(t - 2)}$, $t \neq -2$
43. $\frac{x - 1}{x(x + 1)^2}$, $x \neq -2$ **45.** $\frac{3}{2}$, $x \neq -y$
47. $x(x + 1)$, $x \neq -1, 0$ **49.** $(x - 6)(x + 4)$
51. $x^2(x - 1)$ **53.** $\frac{5x}{x - 2}$ **55.** $\frac{4x}{x - 4}$ **57.** $\frac{4x - 17}{x - 5}$
59. $\frac{x - 4}{(x + 2)(x - 2)(x - 1)}$ **61.** $\frac{2 - x}{x^2 + 1}$, $x \neq 0$
63. $\frac{1}{2}$, $x \neq 2$ **65.** $\frac{1}{x}$, $x \neq -1$ **67.** $\frac{2x - 1}{2x}$, $x > 0$
69. (a) 4.88% (b) $\frac{288(NM - P)}{N(12P + NM)}$; 4.88%

CHAPTER 0

71.



No

73. False. The domain of the left-hand side is $x^n \neq 1$.

75. Answers will vary.

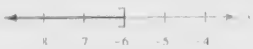
Review Exercises *(page 66)*

1. (a) Natural: $\{9\}$
   (b) Integer: $\{9, -7\}$
   (c) Rational: $\left\{9, -7, -\frac{8}{9}, \frac{5}{2}, 0.4\right\}$
   (d) Irrational: $\{\sqrt{6}\}$

3. $-4 < -3$



5. $x \leq -6$ denotes all real numbers less than or equal to -6.



7. $x \geq 0$ 9. $2 < x \leq 5$ 11. -14
13. 4 15. 10 17. $|x - 7| \geq 4$
19. $5x^4, -2x, 7$ 21. (a) 2 (b) 0
23. Distributive Property 25. Commutative (addition)
27. 2 29. $\frac{z}{3}$ 31. 0.10 33. $8x, x \neq 0$ 35. $\frac{5}{x^4}$
37. 3.087×10^8 39. 7,280,000 41. 11,414.125
43.

Year	5	10	15
Balance	\$10,014.37	\$12,535.94	\$15,692.44

Year	20	25
Balance	\$19,643.73	\$24,589.94

45. $16^{1/2} = 4$ 47. 13 49. $2x^2$ 51. $2\sqrt{3}$
53. $2 + \sqrt{3}$ 55. $-3\sqrt{3}$ 57. $\frac{1}{16}$ 59. $\sqrt{5}$
61. 11.269 63. $-x + 5$ 65. $x^2 - 8x - 14$
67. $x^2 - x - 2$ 69. $x^3 + 64$ 71. $x^2 + 8x + 16$
73. 254,691; In 2010, the number of mobile telephone sites in the United States was 254,691.
75. $4(x + 3)(x - 3)$ 77. $(2x - 5)(4x^2 + 10x + 25)$
79. $(x + 1)(x - 5)$ 81. $(x^2 - 2)(x - 4)$
83. All real numbers except $x = 4$
85. All real numbers where $x \geq -1$ 87. $3x, x \neq 0$
89. $\frac{x - 2}{2}, x \neq -2$ 91. $\frac{x + 3}{x - 1}, x \neq 0, 3$
93. $\frac{x - 1}{x - 3}, x \neq -1, \frac{1}{2}$ 95. $\frac{3x^2 - 4x}{(x - 1)(x - 2)}$
97. $\frac{6}{\quad}, x \neq -1$ 99. $\frac{x + 1}{\quad}, x \neq 0$

Test Yourself *(page 68)*

1. -12
2.

Year	5	10	15
Balance	\$4763.77	\$5673.38	\$6756.67

Year	20	25
Balance	\$8046.81	\$9583.29

The longer you leave the \$4000 in the account, the more money you earn.

3. $-64x^6$ 4. $-4\sqrt{x}$ 5. 25 6. $4(\sqrt{3} - \sqrt{5})$
7. $2x\sqrt{3x}$ 8. $\frac{5 + \sqrt{7}}{9}$ 9. $9x^2 + 42x + 49$
10. $-5x^2 + 29x$ 11. $5(x + 4)(x - 4)$ 12. $(2x + 3)^2$
13. $(x^2 - 3)(x - 6)$ 14. $(x + 2)^2(x - 2)$
15. $\frac{1}{3}(x - 4), x \neq -4$ 16. $\frac{x + 4}{3x + 5}, x \neq -3, \frac{5}{3}$
17. $\frac{4x^2 - 13x}{(x - 3)(x - 4)}$ 18. $-\frac{x + 26}{(x + 5)(x - 2)}$
19. All real numbers greater than or equal to 10
20. All real numbers except $x = -9$
21. $\frac{2x^2 - 5x - 18}{5 + 5x - x^2}, x \neq 1, -2$
22. $x = 0$: 5.35, $x = 9$: 7.36
In 2000 and 2009, the average prices of a movie ticket in the United States were \$5.35 and \$7.36, respectively.

Chapter 1

Section 1.1 *(page 77)*

Skills Warm Up *(page 77)*

1. $-3x - 10$ 2. $5x - 12$ 3. x 4. $x + 26$
5. $\frac{9x}{14}$ 6. $\frac{3x}{4}$ 7. $-\frac{1}{x(x + 1)}$ 8. $\frac{8}{x}$
9. $\frac{7x - 8}{x(x - 2)}$ 10. $\frac{2}{x^2 - 1}$

1. Identity 3. Conditional equation
5. Conditional equation
7. (a) No (b) No (c) Yes (d) No
9. (a) Yes (b) Yes (c) No (d) No
11. (a) Yes (b) No (c) No (d) No
13. (a) No (b) No (c) No (d) Yes
15. (a) Yes (b) No (c) No (d) No
17. 5 19. -4 21. 3 23. No solution
25. Infinitely many solutions 27. 9 29. -26
31. No solution 33. -4 35. $z = -\frac{6}{5}$ 37. 10
39. 4 41. 3 43. 5 45. No solution 47. $x = 1$
49. 9 51. 0 53. All real numbers 55. No solution
57. Because substituting 2 for x in the equation produces division by zero, $x = 2$ cannot be a solution to the equation.

59. Extraneous solutions may arise when a fractional expression is multiplied by factors involving the variable.

61. Equivalent equations have the same solutions.
Example: $2x - 6 = 0$ and $x - 3 = 0$ both have the solution $x = 3$.

63. $x \approx 138.889$ **65.** $x \approx 62.372$ **67.** $x \approx 19.993$

69. Use the *table* feature in ASK mode, or use the scientific calculator part of the graphing utility.

71. (a) 6.46 (b) 6.41
(c) There is a roundoff error because decimals in the original problem have 5 decimal places.

73. (a) 56.09 (b) 56.13
(c) There is a roundoff error because the fractions in the numerator and denominator have infinitely many decimal places.

75. 2009 **77.** 65.8 in. **79.** (a) 2007 (b) 2010

Section 1.2 *(page 88)*

Skills Warm Up *(page 88)*

1. 14 **2.** 4 **3.** -3 **4.** 4 **5.** -2
6. 1 **7.** $\frac{2}{5}$ **8.** $\frac{10}{3}$ **9.** 6 **10.** $-\frac{11}{5}$

1. (a) One natural number + Next consecutive natural number
(b) One natural number $= x$
Next consecutive natural number $= x + 1$
(c) $x + (x + 1) = 2x + 1$

3. (a) Distance $= t$ hours $\times$ Car speed
(b) Hours $= t$
Car speed $= 50$ mi/h
(c) $50t$

5. (a) Perimeter $= 2$(Width) $+ 2$(Length)
(b) Width $= x$
Length $= 2x$
(c) $2x + 2(2x) = 6x$

7. $5 + x = 8$ **9.** $\frac{r}{2} = 9$ **11.** $60{,}000 + 0.08(60{,}000)$

13. (a) Sum $=$ One number + Next consecutive number
(b) One number $= x$
Next consecutive number $= x + 1$
Sum $= 525$
(c) $525 = x + (x + 1)$ (d) 262, 263

15. (a) 148 $=$ One positive number $-$ Another positive number
(b) One positive number $= x$
Another positive number $= 5x$
(c) $148 = 5x - x$ (d) 37, 185

17. (a) Raise $=$ Percent $\times$ Old wage
(b) Old wage $= \$440$
Raise $= \$66$
Percent $= r$
(c) $66 = 440r$ (d) 15%

19. (a) Change in price $=$ Percent $\times$ Original price
(b) Change in price $= \$30$
Percent $= r$
Original price $= \$149$
(c) $30 = 149r$ (d) $\approx 20.13\%$

21. (a) Total paycheck $=$ Your weekly paycheck + Your coworker's paycheck
(b) Total paycheck $= \$848$
Your weekly paycheck $= 1.12x$
Your coworker's paycheck $= x$
(c) $848 = 2.12x$
(d) \$400, \$448

23. $\approx 17.5\%$ decrease **25.** $\approx 16.2\%$ increase

27. $\approx 3.4\%$ increase **29.** $\approx 128.57\%$

31. $\approx 54.29\%$ **33.** $\approx 56.41\%$ decrease

35. (a) \$37,800 (b) \$40,748.40 (c) \$44,578.75

37. (a) 46.6%
(b) Cars: 5,459,515
Light trucks: 4,940,066
Other trucks: 201,419
(c) $\approx 96.1\%$

39. ≈ 0.57 ft $\times$ 0.93 ft **41.** ≈ 8.9 years

43. 97 or greater **45.** \$1411.76 **47.** \$18

49. (a) \$361.25
(b) Percent increase needed: $\approx 17.65\%$
A higher percent increase is needed because you are taking a percentage of a smaller number.

51. $2\frac{1}{3}$ hours **53.** 62.5 feet **55.** 1.28 seconds

57. \$10,500 at 6.5% and \$4500 at 7.5%

59. Stock A: \$2200, Stock B: \$2800

61. 8571 units per month **63.** $h = \dfrac{2A}{b}$ **65.** $l = \dfrac{P - 2w}{2}$

67. $L = \dfrac{S}{1 - R}$ **69.** $P = \dfrac{A}{\left(1 + \dfrac{r}{n}\right)^{nt}}$

71. ≈ 48 ft **73.** ≈ 32.1 gallons

75. Williams: $\approx \$24{,}933$
Gonzalez: $\approx \$27{,}833$
Walters: $\approx \$25{,}633$
Gilbert: $\approx \$26{,}100$
Hart: $\approx \$24{,}433$
Team average: April: \$25,600, May: \$25,920, June: \$25,840

77. "takes 30 minutes"; "from a depth of 150 feet"

Section 1.3 *(page 100)*

Skills Warm Up *(page 100)*

1. $\dfrac{\sqrt{14}}{10}$ **2.** $4\sqrt{2}$ **3.** 14 **4.** $\dfrac{\sqrt{10}}{4}$
5. $x(3x + 5)$ **6.** $(2x - 5)(2x + 5)$
7. $-(x - 7)(x - 15)$ **8.** $(x - 2)(x + 9)$
9. $(5x - 1)(2x + 3)$ **10.** $(6x - 1)(x - 12)$

1. $2x^2 + 5x - 3 = 0$ **3.** $x^2 - 19x = 0$
5. $x^2 - 6x + 4 = 0$ **7.** $2x^2 - 2x + 1 = 0$
9. $3x^2 - 60x - 10 = 0$ **11.** $4, -2$ **13.** $0, -\frac{1}{2}$
15. -5 **17.** $3, -\frac{1}{2}$ **19.** $2, -6$ **21.** $-2, -5$
23. ± 4 **25.** $\pm\sqrt{7} \approx \pm 2.65$ **27.** $\pm 2\sqrt{3} \approx \pm 3.46$
29. 0, 24

31. $-2 + 2\sqrt{3} \approx 1.46$
$-2 - 2\sqrt{3} \approx -5.46$

33. ± 5 **35.** $\pm\sqrt{38} \approx \pm 6.16$ **37.** $\pm\frac{\sqrt{115}}{5} \approx \pm 2.14$

39. $\pm\frac{\sqrt{78}}{3} \approx \pm 2.94$ **41.** ± 8 **43.** 1 **45.** $\pm\frac{3}{4}$

47. $\frac{3}{2}$ **49.** $3, -11$ **51.** $\frac{3}{2}, -\frac{1}{2}$ **53.** $5, -\frac{10}{3}$

55. $9, 3$ **57.** $\frac{1}{5}, 1$ **59.** $-1, -5$ **61.** $-\frac{1}{2}$

63. Algebra argument:

$(x + 2)^2 = (x + 2)(x + 2)$ Definition of exponent
$= x^2 + 2x + 2x + 4$ FOIL
$= x^2 + 4x + 4$ Combine like terms.

So, $(x + 2)^2 \neq x^2 + 4$.

Graphing utility argument:

(1) Let $y_1 = (x + 2)^2$ and $y_2 = x^2 + 4$. Use the table feature with an arbitrary value of x (but not $x = 0$). The table will show that the values of y_1 are not the same as the values of y_2.

(2) Use the scientific calculator portion of the graphing utility to show that if $x = 5$, $(5 + 2)^2 = 49$ and $5^2 + 4 = 29$. So, $(x + 2)^2$ is not equal to $x^2 + 4$.

65. 34 feet × 48 feet

67. Base: $2\sqrt{2}$ feet
Height: $2\sqrt{2}$ feet

69. 5 feet **71.** ≈ 3.54 seconds **73.** ≈ 1.43 seconds

75. 42 seconds faster **77.** ≈ 4.24 centimeters

79. 976 miles **81.** ≈ 494.97 meters **83.** 60,000 units

85. 2019 ($t \approx 8.85$)

87. (a) 1987 ($t \approx 18.74$)

(b) Yes; the model is a good representation through 1890.

(c) 2010: 312,493.19 thousand
They are close.

(d) Yes

89. 2019 ($t \approx 9.13$)

91. 1 P.M. ($t \approx 12.96$)
No; For 7 P.M. ($t = 19$), the model yields a value of $T = 144.81°$F, which is extremely high and unreasonable.

Section 1.4 *(page 110)*

Skills Warm Up *(page 110)*

1. 13 **2.** 5 **3.** $4\sqrt{6}$ **4.** $3\sqrt{73}$
5. 2, 1 **6.** $\frac{3}{2}, -3$ **7.** $5, -1$ **8.** $\frac{1}{2}, 7$
9. 5, 2 **10.** $4, -1$

1. One real solution **3.** Two real solutions
5. No real solutions **7.** Two real solutions
9. $\frac{1}{2}, -1$ **11.** $\frac{1}{4}, -\frac{3}{4}$ **13.** $-3 \pm \sqrt{13}$ **15.** $5 \pm \sqrt{3}$
17. $4 \pm 2\sqrt{5}$ **19.** $\frac{2}{3} \pm \frac{\sqrt{7}}{3}$ **21.** $-\frac{1}{3} \pm \frac{\sqrt{11}}{6}$
23. $\frac{1}{2} \pm \sqrt{2}$ **25.** $\frac{2}{7}$ **27.** $2 \pm \frac{\sqrt{6}}{2}$ **29.** $6 \pm \sqrt{11}$
31. $x \approx 0.976, -0.643$ **33.** $-0.290, -2.200$
35. No real solution **37.** -11 **39.** $\pm\sqrt{10}$
41. $1 \pm \sqrt{2}$ **43.** $-2, 4$
45. $\frac{-1 \pm \sqrt{37}}{6}$ **47.** 50, 50; Real-life problems will vary.
49. 7, 8, or $-8, -7$; Real-life problems will vary.
51. 200 units **53.** 653 units
55. 6 rows, 18 chairs per row
57. 14 in. × 14 in. **59.** ≈ 7.2 seconds
61. Moon: ≈ 14.9 seconds **63.** Moon
Earth: ≈ 2.6 seconds
65. Oklahoma City to Austin: 354.45 mi
Austin to New Orleans: 433.55 mi
67. 4:00 P.M. ($t \approx 3.9$)
69. Southbound: ≈ 550 miles per hour
Eastbound: ≈ 600 miles per hour
71. (a) 2011 ($t \approx 11$) (b) 2008 ($t \approx 8$)
(c) 2016 ($t \approx 16.6$); yes
73. (a) $\approx 16.8°$C (b) ≈ 2.5
75. 5279 units or 94,721 units
77. No; the equation that has a discriminant of 5 has two distinct real solutions, while the equation that has a discriminant of -10 has no real solutions.
79. Answers will vary.

Quiz Yourself *(page 114)*

1. $x = -6$ **2.** $x = 6$ **3.** $x = -2$ **4.** No solution
5. Use the *table* feature in ASK mode or the scientific calculator portion of the graphing utility.
6. 328.954 **7.** 431.398
8. $8.50x + 30{,}000 = 200{,}000$; 20,000 units
9. $300{,}000 = x(75 - 0.0002x)$; 4044 units or 370,956 units
10. $x = \frac{2}{3}, -5$ **11.** $x = \pm\sqrt{5}$; $x \approx \pm 2.24$
12. $x = -3 \pm \sqrt{17}$; $x \approx -7.12, 1.12$
13. $x = -1 \pm \sqrt{7}$; $x \approx -3.65, 1.65$
14. $x = \frac{-7 \pm \sqrt{73}}{6}$; $x \approx -2.59, 0.26$
15. $x \approx 1.568, -0.068$ **16.** No real solutions
17. One real solution
18. Answers will vary. Sample answer: Use the FOIL method $[(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9]$, use the *table* feature of your graphing utility, or use the scientific calculator portion of your graphing utility to evaluate the solution.
19. 4.5 seconds **20.** 9 in. × 9 in. × 6 in.

Section 1.5 *(page 123)*

Skills Warm Up *(page 123)*

1. 11 **2.** $20, -3$ **3.** $5, -45$ **4.** $0, -\frac{1}{5}$
5. $\frac{2}{3}, 2$ **6.** $\frac{11}{6}, -\frac{5}{2}$ **7.** $1, -5$ **8.** $\frac{3}{2}, -\frac{5}{2}$
9. $\frac{3 \pm \sqrt{5}}{2}$ **10.** $2 \pm \sqrt{2}$

1. $3, -1, 0$ **3.** $0, \pm 3$ **5.** ± 3 **7.** $-3, 0$ **9.** $\pm 2, 7$
11. ± 1 **13.** $\pm\sqrt{11}, \pm 1$ **15.** ± 2 **17.** $\pm\frac{1}{5}, \pm 4$
19. $1, -2$ **21.** 50 **23.** 26 **25.** -16 **27.** $\frac{1}{4}$
29. $6, 5$ **31.** $2, -5$ **33.** 0 **35.** $-59, 69$ **37.** 1
39. $\pm\sqrt{69}$ **41.** -1 **43.** $-12, 2$ **45.** $1, -3$

47. $6, -1$ **49.** $1, -3$ **51.** $3, -2$
53. $\sqrt{3}, -3$ **55.** $10, -1$
57. The quadratic equation was not written in general form before the values of a, b, and c were substituted in the Quadratic Formula. The general form for this equation is $3x^2 - 7x - 4 = 0$ ($a = 3, b = -7$, and $c = -4$), and the only solution is

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-4)}}{2(3)}.$$

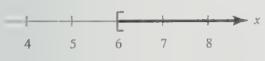
59. $x \approx \pm 1.038$ **61.** $x \approx 16.756$ **63.** 34 **65.** 2.2%
67. $\approx 12.98\%$ **69.** 45,000 passengers **71.** 57 years old
73. 67,760 units; It does not make sense for demand x or price p to be less than zero.
75. Least acceptable weight: 15.6 ounces
Greatest acceptable weight: 16.4 ounces
77. 13 minutes, 20 seconds
79. A polynomial of degree n has at most n real solutions.

Section 1.6 *(page 134)*

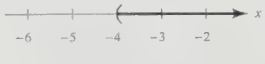
Skills Warm Up *(page 134)*

1. $-\frac{1}{2}$ **2.** $-\frac{1}{6}$ **3.** -3 **4.** -6 **5.** $x \geq 0$
6. $-3 < z < 10$ **7.** $P \leq 2$ **8.** $W \geq 200$
9. 2, 7 **10.** 0, 1

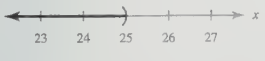
1. $-1 \leq x \leq 5$; Bounded **3.** $x > 11$; Unbounded
5. $x < -2$; Unbounded
7. c **8.** h **9.** f **10.** e
11. g **12.** a **13.** b **14.** d
15. (a) Yes (b) No (c) Yes (d) No
17. (a) Yes (b) No (c) No (d) Yes
19. (a) No (b) Yes (c) Yes (d) Yes
21. (a) No (b) Yes (c) No (d) Yes
23. If $2x > 6$, then $x > 3$. **25.** If $2x \leq -8$, then $x \leq -4$.
27. If $2 - 4x > -10$, then $x < 3$.
29. If $-\frac{2}{3}x \geq -6$, then $x \leq 9$.
31. $x \geq 6$



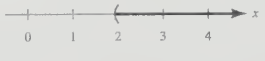
33. $x > -4$



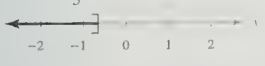
35. $x < 25$



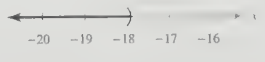
37. $x > 2$



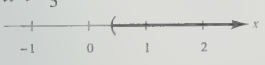
39. $x \leq -\frac{2}{3}$



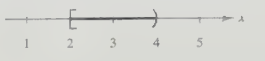
41. $x < -18$



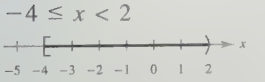
43. $x > \frac{2}{5}$



45. $2 \leq x < 4$



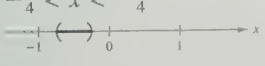
47. $-4 \leq x < 2$



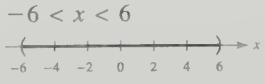
49. $-9 < x < 15$



51. $-\frac{3}{4} < x < -\frac{1}{4}$



53. $-6 < x < 6$



55. $x < -6$ or $x > 6$



57. $-8 < x < 2$



59. No solution

61. $x < -\frac{1}{2}$ or $x > \frac{11}{2}$



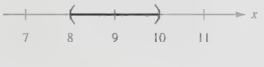
63. $x \leq -\frac{29}{2}$ or $x \geq -\frac{11}{2}$



65. $x \leq -7$ or $x \geq 13$



67. $8 < x < 10$



69. $x < 0$ or $x > 5$



71. $|x| \leq 2$ **73.** $|x - 9| \geq 3$ **75.** $|x - 12| \leq 10$
77. $|x + 3| > 5$ **79.** Less than 18,269.23 miles
81. 2008 ($t > 7.9$) **83.** More than 250 miles
85. (a) 1998 to 2005 ($t < 15.23$); yes; According to the bar graph, the average salary was less than \$1.5 million from 1998 to 2005.
(b) 2011; Yes; The average salary has been increasing.
87. $33\frac{1}{3}$ weeks **89.** ≤ 15 inches
91. $\geq 10{,}417$ units **93.** Greater than 4%
95. [$\approx 106.864, \approx 109.464$]
97. [65.8, 71.2]
99. Minimum = 20%; Maximum = 80%
101. $\approx$ \$0.28

Section 1.7 *(page 145)*

Skills Warm Up *(page 145)*

1. $y < -6$ **2.** $z > -\frac{9}{2}$ **3.** $-3 < x < 1$
4. $x \leq -5$ **5.** $-3 < x$ **6.** $5 < x < 7$
7. $-\frac{7}{2} \leq x \leq \frac{7}{2}$ **8.** $x < 2, x > 4$
9. $x < -6, x > -2$ **10.** $-2 \leq x \leq 6$

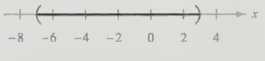
1. $(-\infty, -5), (-5, 5), (5, \infty)$
3. $(-\infty, -4), \left(-4, \frac{1}{2}\right), \left(\frac{1}{2}, \infty\right)$
5. $(-\infty, -1), (-1, 1), (1, \infty)$
7. $[-3, 3]$



9. $(-\infty, -2) \cup (2, \infty)$



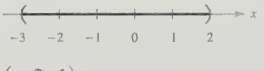
11. $(-7, 3)$



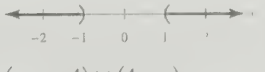
13. $(-\infty, -5] \cup [1, \infty)$



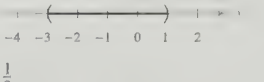
15. $(-3, 2)$



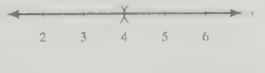
17. $(-\infty, -1) \cup (1, \infty)$



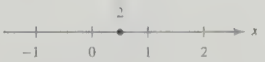
19. $(-3, 1)$



21. $(-\infty, 4) \cup (4, \infty)$



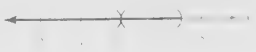
23. $\frac{1}{2}$



25. All real numbers **27.** No solution

CHAPTER 1

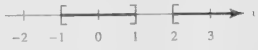
29. $(-\infty, 0) \cup (0, \frac{3}{2})$



31. $[-2, 0] \cup [2, \infty)$



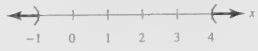
33. $[-1, 1] \cup [2, \infty)$



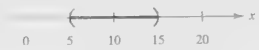
35. $(-\infty, -1) \cup (0, 1)$



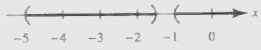
37. $(-\infty, -1) \cup (4, \infty)$



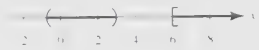
39. $(5, 15)$



41. $(-5, -\frac{3}{2}) \cup (-1, \infty)$



43. $(-\frac{3}{4}, 3) \cup [6, \infty)$



45. $(-\infty, -3] \cup [3, \infty)$ **47.** All real numbers

49. $[-\frac{9}{2}, \frac{9}{2}]$ **51.** $(-\infty, 2) \cup (5, \infty)$ **53.** All real numbers

55. $(\frac{5}{3}, \infty)$ **57.** $(-\infty, -3] \cup [0, 3]$ **59.** $[-2, \infty)$

61. $(-3.51, 3.51)$ **63.** $(-0.13, 25.13)$ **65.** $(2.26, 2.39)$

67. Between about 13.8 meters and about 36.2 meters

69. (a) $90{,}000 \le x \le 100{,}000$ (b) $\$30 \le p \le \$32$

(c) About 185,967 units; The company should produce a maximum of about 185,967 units.

71. 4.8%

73. (a) 6393.56 million; It matches the value shown in the bar graph.

(b) 2012; yes

75. 2014/2015 **77.** $R_1 \ge 2$ ohms

Review Exercises *(page 152)*

1. Conditional equation **3.** Identity

5. (a) No (b) No (c) Yes (d) No

7. (a) No (b) No (c) No (d) No

9. 13 **11.** $-\frac{1}{2}$ **13.** No solution

15. All real numbers **17.** $-\frac{5}{3}$ **19.** $-\frac{2}{3}$

21. 377.778 **23.** 0.033 **25.** 2009

27. (a) Sum = First even integer + Second even integer + Third even integer

(b) Sum = 42

First even integer = x

Second even integer = $x + 2$

Third even integer = $x + 4$

(c) $x + (x + 2) + (x + 4) = 42$

(d) 12

29. (a) New salary = 7%(Annual salary) + Annual salary

(b) New salary = S (dollars)

Annual salary = 28,900 (dollars)

Percent = 0.07 (percent in decimal form)

(c) $S = 0.07(28{,}900) + 28{,}900$

(d) \$30,923

31. \$163.53 **33.** 2 hours **35.** $\approx 26.58\%$

37. $2\frac{2}{9}$ quarts **39.** 29.5 feet × 59 feet **41.** \$20

43. $-\frac{1}{2}, \frac{4}{3}$ **45.** 3, 8 **47.** $\pm\sqrt{11}, \approx \pm 3.32$

49. $-4 + 3\sqrt{2} \approx 0.24$

$-4 - 3\sqrt{2} \approx -8.24$

51. 15 feet × 27 feet

53. 200,000 units or 400,000 units

55. Two real solutions **57.** No real solutions

59. $6 \pm \sqrt{6}$ **61.** $\dfrac{-19 \pm \sqrt{165}}{2}$ **63.** $-3 \pm 2\sqrt{3}$

65. 1.866, −0.283 **67.** 8.544, 0.162

69. Moon: ≈ 8.61 seconds

Earth: ≈ 3.54 seconds

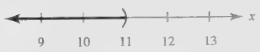
71. $0, -1, 4$ **73.** $\pm 2, \pm 1$ **75.** $\frac{25}{4}$ **77.** No solution

79. $\pm 4\sqrt{2}$ **81.** $-3, \frac{7}{5}$ **83.** $2 \pm \sqrt{19}$ **85.** \$900

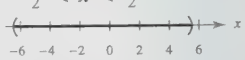
87. $\approx 313{,}251$ units **89.** $-4 \le x \le 7$; bounded

91. $x \le 1000$; unbounded

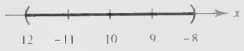
93. $x < 11$



95. $-\frac{13}{2} < x < \frac{11}{2}$



97. $-12 < x < -8$

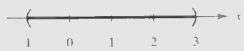


99. $x > 45$ units

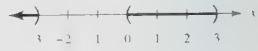
101. $(-\infty, -1), (-1, 1), (1, \infty)$

103. $(-\infty, 1), (1, 5), (5, \infty)$

105. $(-1, 3)$



107. $(-\infty, -3) \cup (0, 3)$



109. $(-\infty, \frac{6}{5}) \cup (4, \infty)$



111. $(-1.69, 1.69)$ **113.** $(1.65, 1.74)$ **115.** $[10, \infty)$

117. $(-\infty, 6] \cup [9, \infty)$

119. Between 2.5 and 3.75 seconds

121. Between about 15.5 meters and about 24.5 meters

123. Greater than 4.25%

125. Between about \$41.34 and about \$58.66

Test Yourself *(page 156)*

1. $\frac{17}{23}$ **2.** (a) All real numbers (b) $-3 \le x \le 3$

3. April: \$325,786.00 **4.** $-\frac{5}{3}, \frac{1}{2}$ **5.** $4, -\frac{3}{2}$

May: \$299,723.12

6. $\pm\sqrt{15}$ **7.** $\dfrac{-13 \pm \sqrt{69}}{2}$ **8.** $\dfrac{11 \pm \sqrt{145}}{6}$

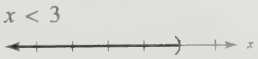
9. 1.038, −0.446 **10.** $-\frac{7}{2}, \frac{13}{2}$

11. 4 (7 is extraneous.)

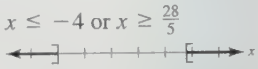
12. $-1, 1, -3, 3$ **13.** $-6, 6$

14. Selling either 341,421 units or 58,579 units will produce a revenue of \$2,000,000.

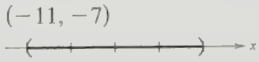
15. $x < 3$



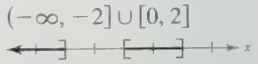
16. $x \le -4$ or $x \ge \frac{28}{5}$



17. $(-11, -7)$



18. $(-\infty, -2] \cup [0, 2]$



19. Between 19,189 units and 143,311 units **20.** 2012

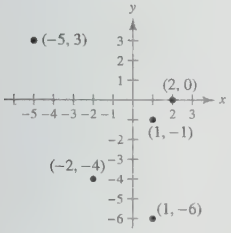
Chapter 2

Section 2.1 *(page 168)*

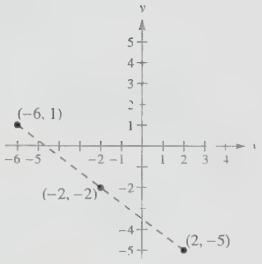
Skills Warm Up *(page 168)*

1. 5 **2.** $3\sqrt{2}$ **3.** 1 **4.** -2
5. $3(\sqrt{2}+\sqrt{5})$ **6.** $2(\sqrt{3}+\sqrt{11})$ **7.** $-3, 11$
8. 9, 1 **9.** $0, \pm 3$ **10.** ± 2

1.

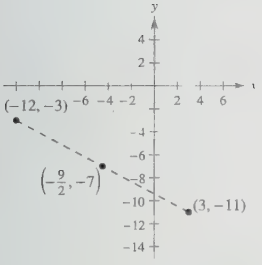


3. (a)



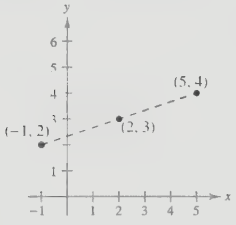
(b) 10 (c) $(-2, -2)$

5. (a)



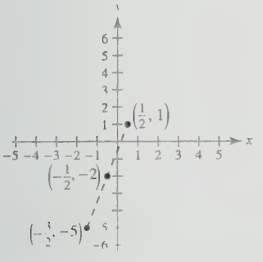
(b) 17 (c) $\left(-\frac{9}{2}, -7\right)$

7. (a)



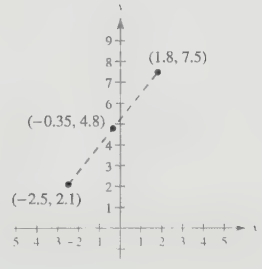
(b) $2\sqrt{10}$ (c) (2, 3)

9. (a)



(b) $2\sqrt{10}$
(c) $\left(-\frac{1}{2}, -2\right)$

11. (a)



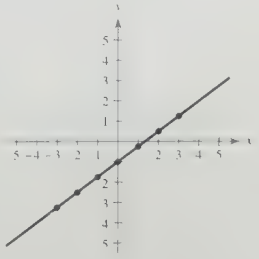
(b) $\sqrt{47.65}$
(c) $(-0.35, 4.8)$

13. 5 **15.** $\sqrt{109}$ **17.** $x = 15, -9$ **19.** $y = 9, -23$
21. (a) Yes (b) No **23.** (a) Yes (b) Yes

25.

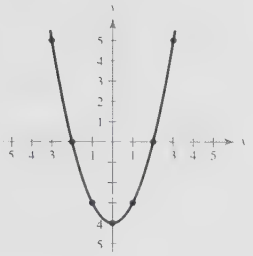
x	-3	-2	-1
y	-3.25	-2.5	-1.75

x	0	1	2	3
y	-1	-0.25	0.5	1.25



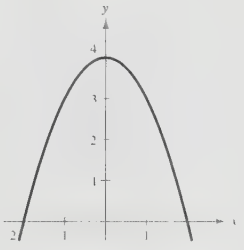
27.

x	-3	-2	-1	0	1	2	3
y	5	0	-3	-4	-3	0	5

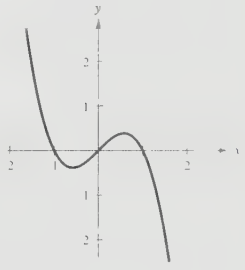


29. $\left(\frac{1}{2}, 0\right), (0, -1)$ **31.** $(-2, 0), (1, 0), (0, -2)$
33. $(-2, 0), (2, 0), (0, 2)$ **35.** $\left(\frac{4}{3}, 0\right), (0, 2)$
37. Every ordered pair on the x-axis has a y-coordinate of zero $[(x, 0)]$, so to find an x-intercept we let $y = 0$. Similarly, every ordered pair on the y-axis has an x-coordinate of zero $[(0, y)]$, so to find a y-intercept we let $x = 0$.
39. y-axis symmetry **41.** x-axis symmetry
43. y-axis symmetry **45.** Origin symmetry
47. Origin symmetry **49.** x-axis, y-axis, and origin symmetry

51.

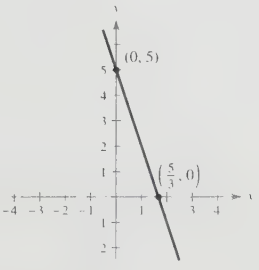


53.



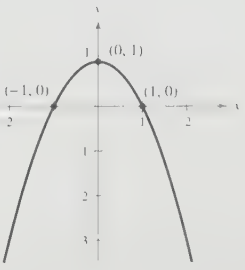
55. c **56.** d **57.** f **58.** a **59.** e **60.** b

61.



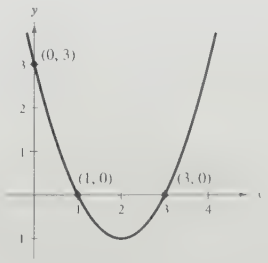
Intercepts:
$\left(\frac{5}{3}, 0\right), (0, 5)$
Symmetry: none

63.



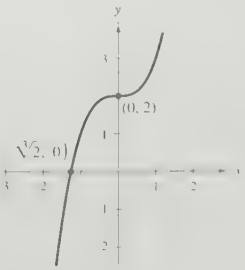
Intercepts:
$(-1, 0), (1, 0), (0, 1)$
Symmetry: y-axis

65.



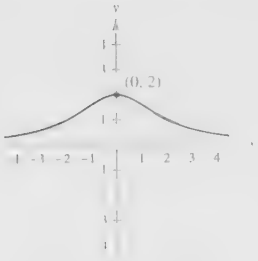
Intercepts:
$(3, 0), (1, 0), (0, 3)$
Symmetry: none

67.



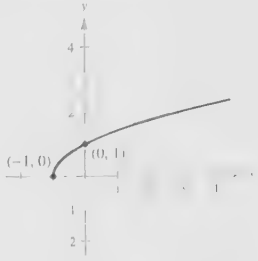
Intercepts:
$\left(-\sqrt[3]{2}, 0\right), (0, 2)$
Symmetry: none

69.



Intercept: (0, 2)
Symmetry: y-axis

71.



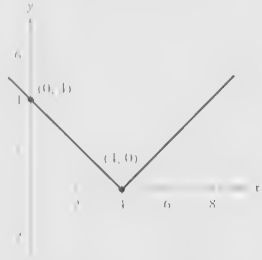
Intercepts: $(-1, 0), (0, 1)$
Symmetry: none

73.



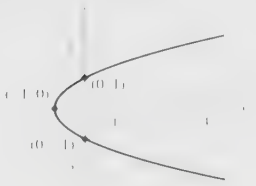
Intercept: (0, 0)
Symmetry: origin

75.



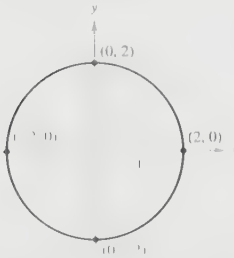
Intercepts: (4, 0), (0, 4)
Symmetry: none

77.



Intercepts: $(-1, 0), (0, 1), (0, -1)$
Symmetry: x-axis

79.



Intercepts: $(-2, 0), (2, 0), (0, 2), (0, -2)$
Symmetry: x-axis, y-axis, origin

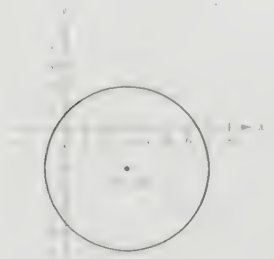
81. Radius: 2 **83.** Radius: $\sqrt{5}$

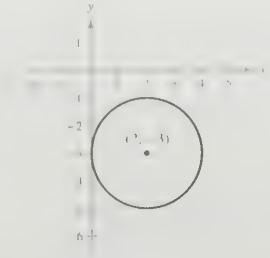
85. Center: (3, 2); Radius: $\sqrt{5}$

87. $x^2 + y^2 = 9$ **89.** $(x + 4)^2 + (y - 1)^2 = 2$

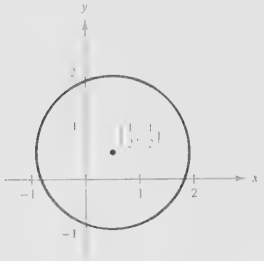
91. $(x + 1)^2 + (y - 2)^2 = 5$ **93.** $(x - 1)^2 + (y - 1)^2 = 25$

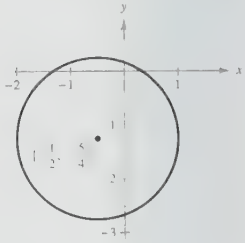
95. $(x - 3)^2 + (y + 2)^2 = 16$ **97.** $(x - 2)^2 + (y + 3)^2 = 4$





99. $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 2$ **101.** $\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{9}{4}$





103. Center: (3, −1); Radius: 5
$x^2 + y^2 - 6x + 2y - 15 = 0$

105. (a) ≈ 17% (b) ≈ 8%

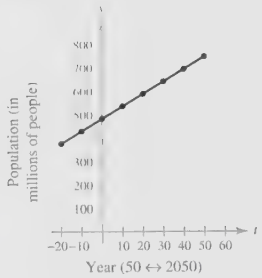
107. (a) (0, 482); It represents the population (in millions of people) of North America in 2000.

(b)

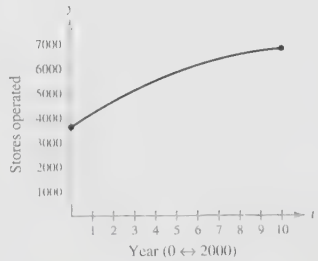
x	−20	−10	0	10
y	376	429	482	535

x	20	30	40	50
y	588	641	694	747

(c)



109. (a)

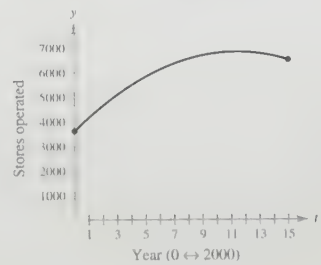


(b) 2011: about 6842; The model's prediction for 2011 is not close to the plan of Family Dollar Stores.

(c)

Year, t	12	13	14	15
Number of stores	6836	6780	6674	6518

(d)



(e) The model does not support the company's expectations.
2013: 6780
2014: 6674
2015: 6518
The graph of the model begins to decrease in 2011.

111. (a) $x^2 + y^2 = 7656.25$
(b) Quadrant II, Quadrant III, Quadrant II, Quadrant I, Quadrant IV
(c) 100 cm

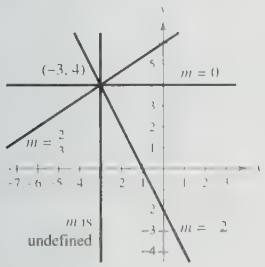
Section 2.2 *(page 180)*

Skills Warm Up *(page 180)*

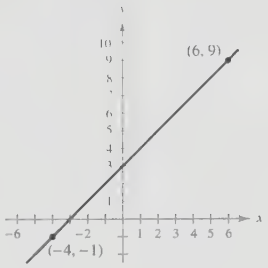
1. -4 **2.** $-\frac{13}{3}$ **3.** $\frac{5}{4}$ **4.** $\frac{1}{3}$ **5.** $y = \frac{2}{3}x - 2$
6. $y = -2x$ **7.** $y = 3x - 1$ **8.** $y = \frac{2}{3}x + 5$
9. $y = -2x + 7$ **10.** $y = x + 3$

1. $\frac{1}{2}$ **3.** -2 **5.** (a) L_2 (b) L_3 (c) L_1

7.

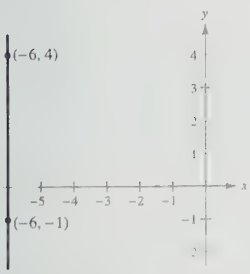


9.



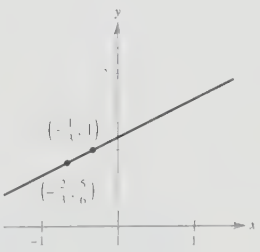
$m = 1$

11.



m is undefined.

13.



$m = \frac{1}{2}$

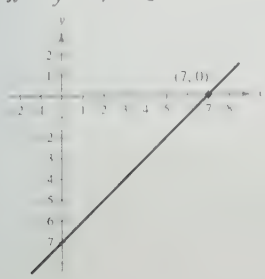
15. Answers will vary.
Sample answer: $(3, -2), (-1, -2), (0, -2)$

17. Answers will vary.
Sample answer: $(2, -3), (2, -7), (2, 9)$

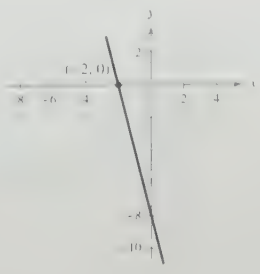
19. Answers will vary.
Sample answer: $(6, -5), (7, -4), (8, -3)$

21. Answers will vary.
Sample answer: $(2, 3), (-4, 0), (4, 4)$

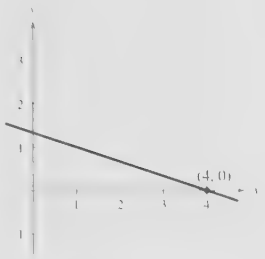
23. $x - y - 7 = 0$



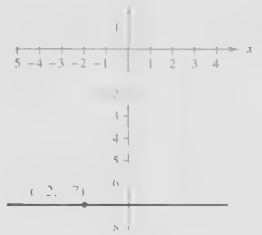
25. $4x + y + 8 = 0$



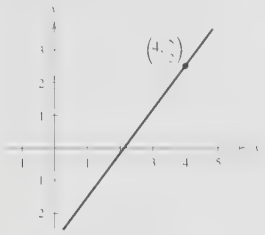
27. $x + 3y - 4 = 0$



29. $y + 7 = 0$

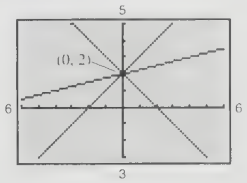


31. $8x - 6y - 17 = 0$

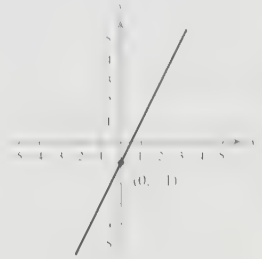


33. $y = 4x$

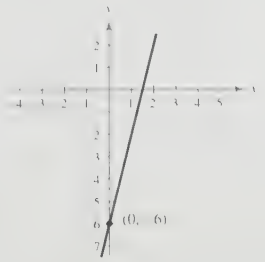
35. $x + 2y + 1 = 0$ **37.** $x + 9 = 0$ **39.** $y - 7 = 0$
41. $x + 2y - 3 = 0$ **43.** $2x - 5y + 1 = 0$
45. $(0, 2)$



47. $m = 2, (0, -1)$



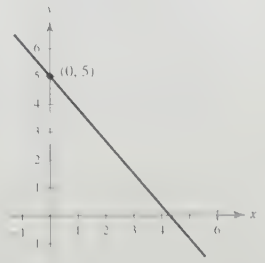
49. $m = 4, (0, -6)$



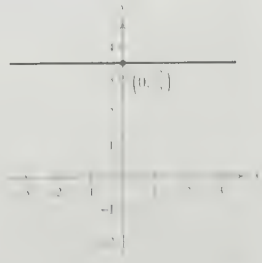
51. m is undefined; no y-intercept



53. $m = -\frac{7}{6}$, $(0, 5)$



55. $m = 0, (0, \frac{7}{2})$



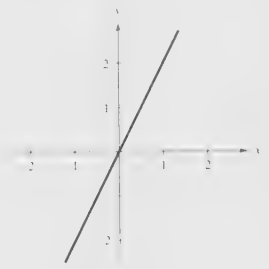
57. Answers will vary. Sample answer: You could graph a vertical line and pick two convenient points on the line to find the slope. Regardless of the points selected, the slope will have zero in the denominator. Division by zero is not possible, so the slope does not exist.

59. $4x - y - 4 = 0$ **61.** $x + y + 2 = 0$
63. $12x + 3y + 2 = 0$
65. Changing the viewing window will affect the appearance of the slope of the line. Answers will vary.
67. Neither **69.** Perpendicular **71.** Parallel
73. Perpendicular **75.** Parallel **77.** Perpendicular
79. Neither **81.** Parallel
83. (a) $2x - y - 10 = 0$ (b) $x + 2y - 10 = 0$
85. (a) $4x - 6y - 5 = 0$ (b) $36x + 24y + 7 = 0$
87. (a) $y = 0$ (b) $x + 1 = 0$ **89.** Yes $\left(m = \frac{17}{180}\right)$
91. $F = \frac{9}{5}C + 32$ **93.** $y = 7000x + 151{,}000$; \$179,000; No
95. $p = \frac{1}{33}d + 1$; $\frac{1}{33}$ atmosphere per foot

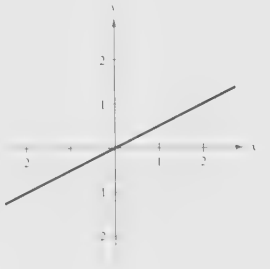
Section 2.3 *(page 190)*

Skills Warm Up *(page 190)*

1.



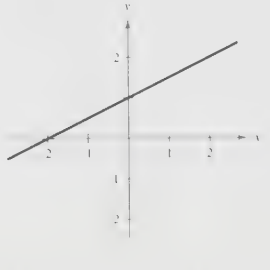
2.



3.

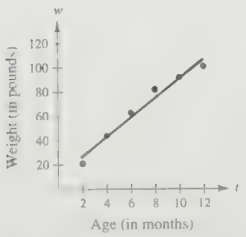


4.



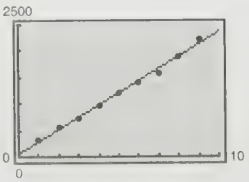
5. $y = x + 2$ **6.** $y = \frac{3}{2}x + 3$ **7.** $y = x + 2$
8. $y = \frac{6}{7}x + 4$ **9.** $5x + 40y - 213 = 0$
10. $29x + 60y - 448 = 0$

1.



The model is a good fit for the actual data.

3.



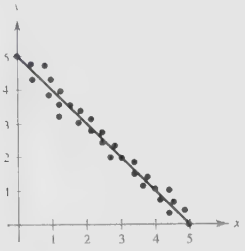
$y = 231.1x + 53.256$
The model is a good fit for the actual data.

5. $y = \frac{3}{8}x$ **7.** $y = 20x$ **9.** $y = \frac{3.2}{7}x$ or $y = \frac{16}{35}x$
11. $H = 3p$ **13.** $c = \frac{3}{5}d$ **15.** $I = 0.06P$

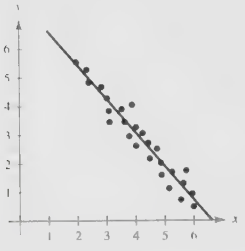
17. (a) $y = 0.0134x$ (b) \$2479
19. (a) $M = \frac{64}{103}K$
(b)

km/h	40	60	80	100	120
mi/h	24.85	37.28	49.71	62.14	74.56

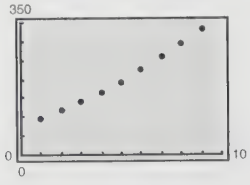
21. $V = 140t + 860,\ 12 \le t \le 17$
23. $V = -2142t + 46{,}104,\ 12 \le t \le 17$
25. $V = 10{,}780t + 24{,}640,\ 12 \le t \le 17$
27. (a) $h = 7000 - 20t$ (b) 2:13:50 P.M.
29. $V = 875 - 175t,\ 0 \le t \le 5$
31. $W = 0.75x + 11.50$ **33.** (a) $H = 0.5t + 4$ (b) 13 in.
35. Yes; Answers will vary. **37.** No
Sample answer:



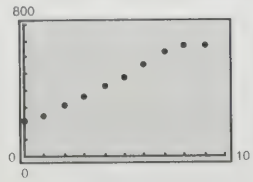
$y = 5 - x$
39. Yes; Answers will vary.
Sample answer:



$y = -1.18x + 7.76$
41. (a)



(b) $C = 29.19t + 57.1$
Yes
(c) Slope: 29.19
The number of autistic children receiving disability services is increasing by about 29 thousand children per year.
(d) 2011: 378 thousand children
2012: 407 thousand children
The estimates are reasonable.
43. (a)



(b) Answers will vary.
Sample answer:
$y = 58.1t + 210.8$

(c) $y = 56.69t + 200.7$

Using linear regression equation:

2010: \$767.6 million

2011: \$824.29 million

Using equation from part (b):

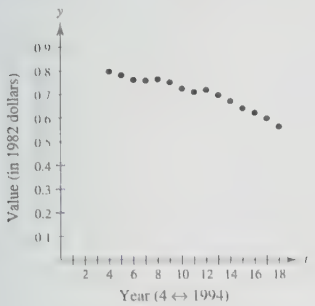
2010: \$791.8 million

2011: \$849.9 million

(d) The projections made by California Pizza Kitchen are lower than the predictions given by the models.

(e) Yes; Using the linear regression equation, the yearly revenue is expected to reach \$937.67 million by 2013. Using the equation from part (b), the yearly revenue is expected to reach \$966.1 million by 2013.

45. (a)



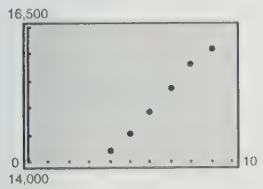
The data appear to be approximately linear.

(b) $y = -0.0152t + 0.873$

(c) 2010: 0.569; 2011: 0.5538

Because the data follow a linear pattern from 1994 to 2008, you can assume that the estimates for 2010 and 2011 are reliable.

47. (a)



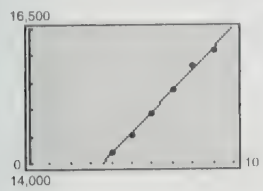
The data appear to be approximately linear.

(b) $E = 394.17t + 12{,}592.9$

(c) 2011: 16,928,770 employees

2013: 17,717,110 employees

(d)



The predictions are most likely going to be just about right because the model is a good fit for the actual data.

49. The model for population, because population tends to change at a consistent rate, whereas snowfall can be quite different from year to year. You should use more than three data points to ensure that the data can be represented accurately by a linear model.

51. Yes; When you start, you have zero wrong moves ($x = 0$) and 800 points ($y = 800$). So, 800 is the y-intercept. Then you lose 50 points for each wrong move. This is a slope of -50.

53. A known rate of change corresponds to the slope of a linear model, so determine whether the rate of change of y with respect to x is constant. The data points show a 10-unit increase in y for each 1-unit increase in x, so the rate of change is constant and the data points are linear.

Section 2.4 *(page 203)*

Skills Warm Up *(page 203)*

1. -73 **2.** 13 **3.** $2(x + 2)$ **4.** $-8(x - 2)$
5. $y = \frac{7}{5} - \frac{2}{5}x$ **6.** $y = 4x$ **7.** $x \neq -2, x \neq 2$
8. $-3 \leq x \leq 3$ **9.** All real numbers
10. $x \neq 1, x \neq 2$

1. This is a function from A to B because each element of A is matched with an element of B.

3. Not a function; The relationship does not match the element b of A with an element of B.

5. This is a function from A to B because each element of A is matched with an element of B.

7. This is a function from A to B because each element of A is matched with an element of B.

9. Not a function; The relationship assigns two elements of B to the element c of A.

11. Not a function from A to B; The relationship defines a function from B to A.

13. This is a function from A to B because each element of A is matched with an element of B.

15. Not a function **17.** Function **19.** Function

21. Not a function **23.** Function

25. (a) -6 (b) 34 (c) $6 - 4t$ (d) $2 - 4c$

27. (a) -1 (b) $\frac{1}{15}$ (c) $\frac{1}{t^2 - 2t}$ (d) $\frac{1}{t^2 - 1}$

29. (a) -1 (b) -9 (c) $2x - 5$ (d) $-\frac{5}{2}$

31. (a) 0 (b) 3 (c) $x^2 + 2x$ (d) -0.75

33. (a) 36π (b) 0 (c) $\frac{9\pi}{2}$ (d) $\frac{32\pi r^3}{3}$

35. (a) 1 (b) -7 (c) $3 - 2|x|$ (d) 2.5

37. (a) Undefined (b) $-\frac{1}{16}$

(c) $\frac{1}{y^2 + 4y - 12}$, $y \neq -6, 2$

(d) $\frac{1}{y^2 - 4y - 12}$, $y \neq -2, 6$

39. (a) 1 (b) -1 (c) 1 (d) $\frac{|x - 1|}{x - 1}$

41. (a) -4 (b) 3 (c) -7 (d) 7

43. (a) 2 (b) -1 (c) 3 (d) 3 **45.** 5 **47.** ± 3

49. 0, 1, -1 **51.** $\frac{10}{7}$ **53.** $\{0, 3, 5, 8, 9\}$

55. $\{-2, -1, 0, 1, 2\}$ **57.** All real numbers x

59. All real numbers except $t = 0$ **61.** All real numbers y

63. $-1 \leq x \leq 1$ **65.** All real numbers except $x = 0, -2$

67. All real numbers $x \geq -1$ except $x = 2$ **69.** $x > 0$

71. $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$

73. $\{(-2, 0), (-1, 1), (0, \sqrt{2}), (1, \sqrt{3}), (2, 2)\}$

75. The domain of $f(x) = \sqrt{x-2}$ is all real numbers $x \geq 2$ because an even root of a negative number is not a real number. The domain of $g(x) = \sqrt[3]{x-2}$ is all real numbers. f and g have different domains because an odd root of a negative number is a real number, but an even root of a negative number is not a real number.

77. n ordered pairs

79. (a) $V = x(18-2x)^2$ (b) Domain: $0 < x < 9$
(c) 400 cubic inches

81. Yes $[y(30) = 6]$

83. 2000: 945 million acres; 2009: 917.98 million acres

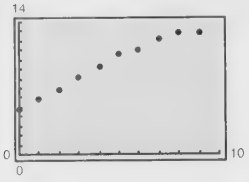
85. (a) $C = 57{,}000 + 2.05x$ (b) $\overline{C} = \dfrac{57{,}000}{x} + 2.05$

(c)

x	100	1000	10,000	100,000
$\overline{C}$	572.05	59.05	7.75	2.62

(d) Answers will vary. Sample answer: The average cost per unit decreases as x gets larger.

87. (a)



(b) Linear model: $P = 0.965t + 5.11$
Quadratic model: $P = -0.0456t^2 + 1.375t + 4.56$

(c)

Year	P (Actual)	P (Linear)	P (Quadratic)
2000	4.80	5.11	4.56
2001	5.89	6.08	5.89
2002	6.81	7.04	7.13
2003	8.10	8.01	8.27
2004	9.30	8.97	9.33
2005	10.55	9.94	10.30
2006	11.13	10.90	11.17
2007	12.09	11.87	11.95
2008	12.95	12.83	12.64
2009	12.91	13.80	13.24

The quadratic model is a better fit because its values of P tend to be closer to the actual values than those of the linear model.

89. (a) $r(t) = 0.75t$ (b) $A = 0.5625\pi t^2$

Time, t	1	2	3	4	5
Radius, r (in feet)	0.75	1.5	2.25	3	3.75
Area, A (in square feet)	1.767	7.069	15.904	28.274	44.179

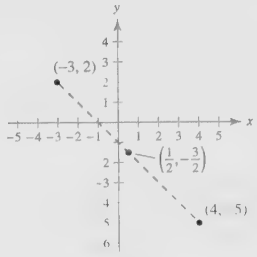
(c) $\dfrac{A(2)}{A(1)} = \dfrac{7.069}{1.767} = 4.000$, $\dfrac{A(4)}{A(2)} = \dfrac{28.274}{7.069} = 4.000$
Predicted area when $t = 8$: 113.096 square feet
Calculated area when $t = 8$: 113.097 square feet

91. (a) Correct; Each price has exactly one tax amount.
(b) Incorrect; For any given value of education, there is not exactly one specific value of intelligence.

93. Answers will vary.

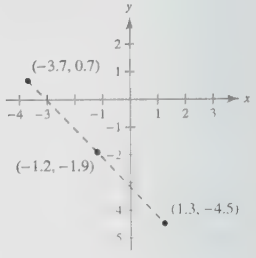
Quiz Yourself *(page 208)*

1. (a)



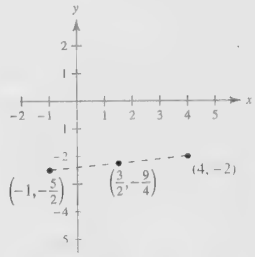
(b) $7\sqrt{2}$
(c) $\left(\frac{1}{2}, -\frac{3}{2}\right)$

2. (a)



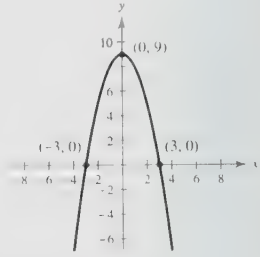
(b) $\sqrt{52.04}$
(c) $(-1.2, -1.9)$

3. (a)



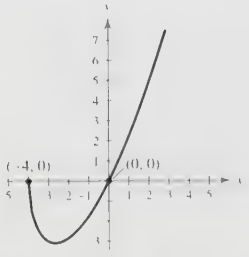
(b) $\dfrac{\sqrt{101}}{2}$ (c) $\left(\dfrac{3}{2}, -\dfrac{9}{4}\right)$

4.



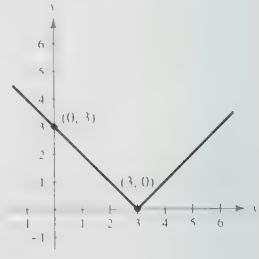
Intercepts:
$(-3, 0), (3, 0), (0, 9)$
Symmetry: y-axis

5.



Intercepts: $(-4, 0), (0, 0)$
Symmetry: none

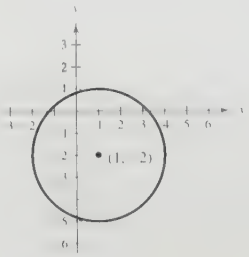
6.



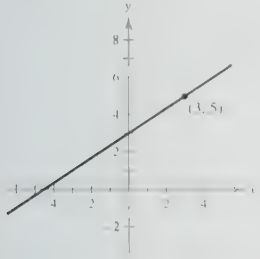
Intercepts: $(3, 0), (0, 3)$
Symmetry: none

7. $(x-4)^2 + (y-5)^2 = 20$

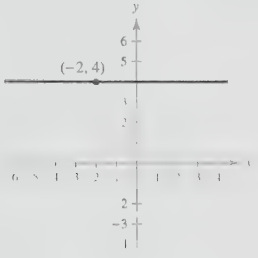
8. $(x-1)^2 + (y+2)^2 = 9$



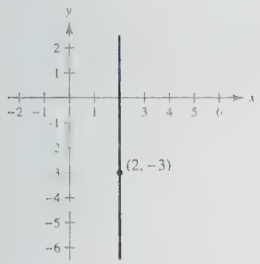
9. $2x - 3y + 9 = 0$



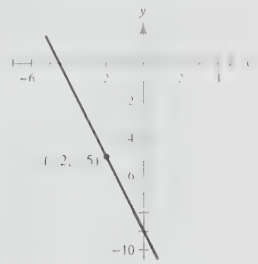
10. $y - 4 = 0$



11. $x - 2 = 0$



12. $2x + y + 9 = 0$



13. 261,019 **14.** y is a function of x.

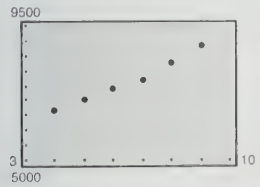
15. The collection of ordered pairs does represent a function from A to B.

16. The collection of ordered pairs does not represent a function from A to B.

17. (a) 2 (b) -7 **18.** (a) 1 (b) -20

19. $x \geq 4$ **20.** All real numbers except $x = -2$

21.



Linear: $y = 428.5t + 4873$

Quadratic: $y = 35.29t^2 - 30.2t + 6261$

22.

Linear:	Quadratic:
2010: \$9158 million	2010: \$9488 million
2011: \$9586.5 million	2011: \$10,198.89 million

23. $A = \dfrac{C^2}{4\pi}$

Section 2.5 *(page 216)*

Skills Warm Up *(page 216)*

1. 2 **2.** 0 **3.** $-\dfrac{3}{x}$ **4.** $x^2 + 3$ **5.** $0, \pm 4$

6. $\frac{1}{2}, 1$ **7.** All real numbers except $x = 4$

8. All real numbers except $x = 4, 5$ **9.** $t \leq \frac{5}{3}$

10. All real numbers

1. Domain: $[1, \infty)$; Range: $[0, \infty)$; 0

3. Domain: $(-\infty, \infty)$; Range: $(-\infty, 4]$; 4

5. Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$; -1

7. Domain: $[-5, 5]$; Range: $[0, 5]$; 5

9. Function **11.** Not a function **13.** Function

15. Not a function **17.** Increasing on $(-\infty, \infty)$; No change

19. Increasing on $(-\infty, 0)$ and $(2, \infty)$, decreasing on $(0, 2)$; behavior changes at $(0, 0)$ and $(2, -4)$.

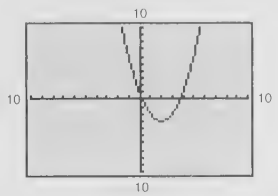
21. Increasing on $(-1, 0)$ and $(1, \infty)$, decreasing on $(-\infty, -1)$ and $(0, 1)$; behavior changes at $(-1, -3)$, $(0, 0)$, and $(1, -3)$.

23. Increasing on $(-2, \infty)$, decreasing on $(-3, -2)$; behavior changes at $(-2, -2)$.

25. Increasing on $(2, \infty)$, decreasing on $(-\infty, 2)$; behavior changes at $(2, -4)$.

27. Increasing on $(-\infty, 0)$ and $(2, \infty)$, decreasing on $(0, 2)$; behavior changes at $(0, 2)$ and $(2, -2)$.

29.

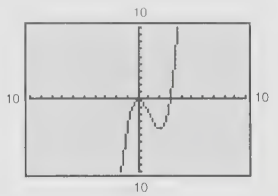


Minimum: $(2, -3)$

Increasing: $(2, \infty)$

Decreasing: $(-\infty, 2)$

31.



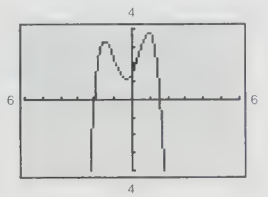
Relative maximum: $(0, 0)$

Relative minimum: $(2, -4)$

Increasing: $(-\infty, 0), (2, \infty)$

Decreasing: $(0, 2)$

33.



Relative maxima: $(-1.54, 3.29)$, $(0.95, 3.77)$

Relative minimum: $(-0.34, 1.14)$

Increasing: $(-\infty, -1.54)$, $(-0.34, 0.95)$

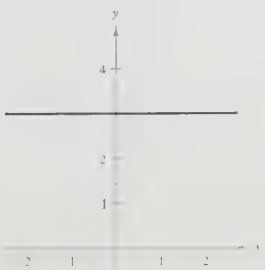
Decreasing: $(-1.54, -0.34)$, $(0.95, \infty)$

35. (a) 2 (b) 2 (c) -3 (d) -4

37. (a) 2 (b) 1 (c) -8 (d) -9

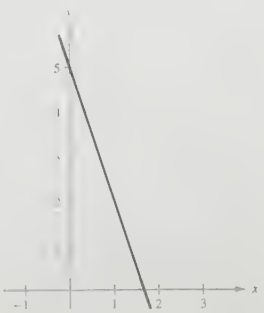
39. Even **41.** Odd **43.** Odd

45.



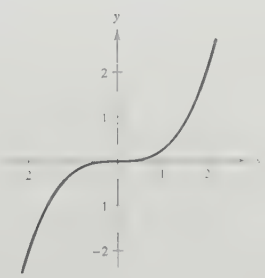
Even

47.



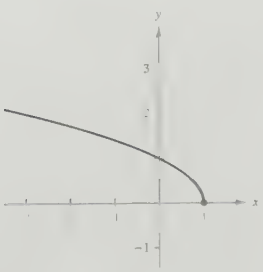
Neither even nor odd

49.



Odd

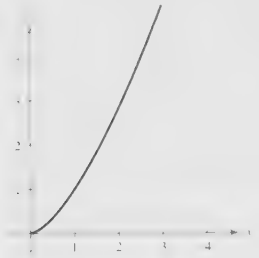
51.



Neither even nor odd

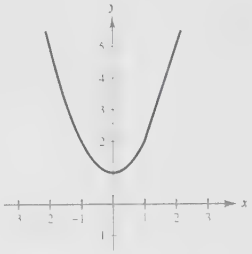
CHAPTER 2

53.



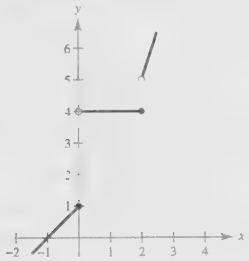
Neither even nor odd

55.



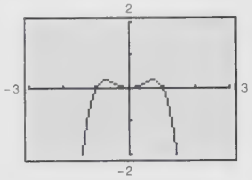
Neither even nor odd

57.



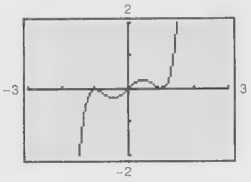
Neither even nor odd

59.



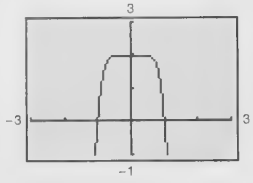
Even

61.



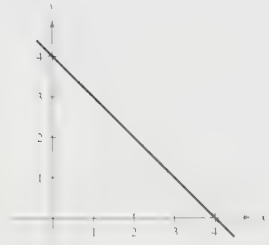
Odd

63.

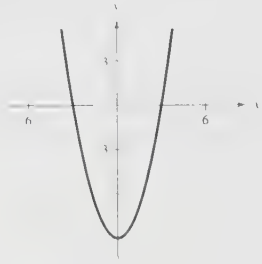


Even

65.



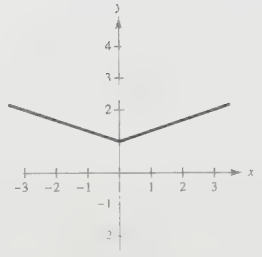
67.



69.



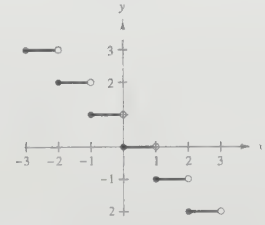
71.



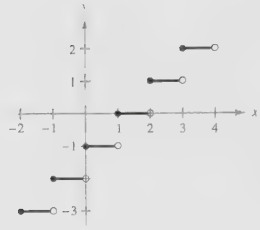
73.



75.



77.

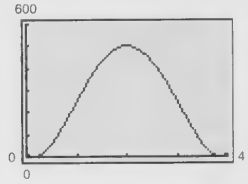


79. (a) Maximum: \$1212.62

(b) Decreasing: Price did not decrease during 2000–2010.
Increasing: 2000–2010

(c) The slope of the graph seems to increase quite rapidly in future years. The price of gold probably will not increase that quickly.

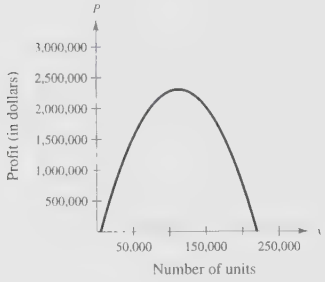
81. (a)



Increasing: 0 seconds to 2 seconds
Decreasing: 2 seconds to 4 seconds

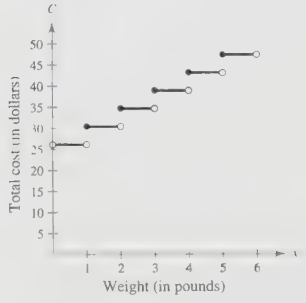
(b) Maximum change in volume: $\approx$ 501.9 milliliters

83.

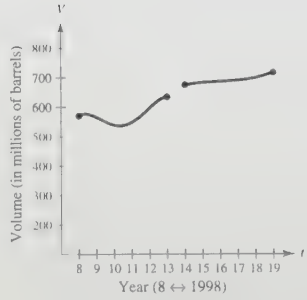


Approximately 112,500 units

85.



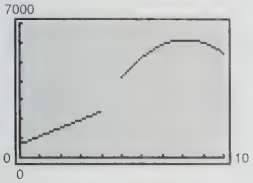
87.



89. (a) $y = 425.71t + 705.54$
Domain: (0, 4)
Range: (705.54, 2408.38)

(b) $y = -202.95t^2 + 3304.67t - 7311.8$
Domain: (5, 9)
Range: (4137.8, 6140.3)

(c) $y = \begin{cases} 425.71t + 705.54, & 0 \le t \le 4 \\ -202.95t^2 + 3304.67t - 7311.8, & 5 \le t \le 9 \end{cases}$



(d) The revenues increased from 2000 to 2008 and decreased from 2008 to 2009.

91. (a) $\left(\frac{3}{2}, 4\right)$ (b) $\left(\frac{3}{2}, -4\right)$

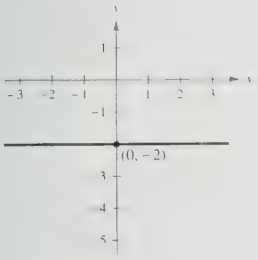
93. (a) $(-4, 9)$ (b) $(-4, -9)$

Section 2.6 *(page 226)*

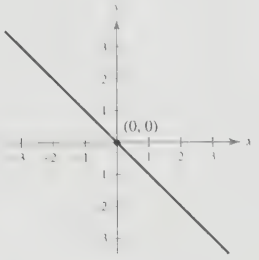
Skills Warm Up *(page 226)*

1. 12 **2.** $\frac{-2x}{x-3}$ **3.** $0, \pm\sqrt{10}$ **4.** $\frac{4}{3}, -2$

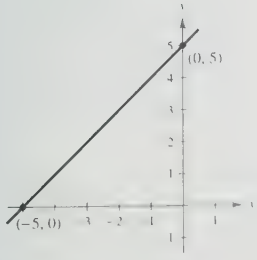
5. $f(x) = -2$



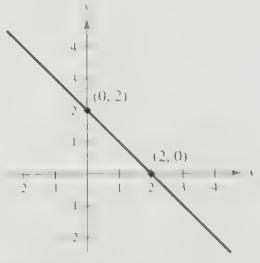
6. $f(x) = -x$



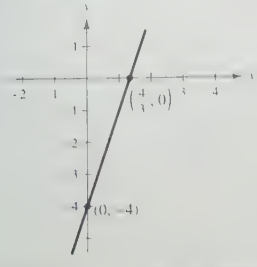
7. $f(x) = x + 5$



8. $f(x) = 2 - x$



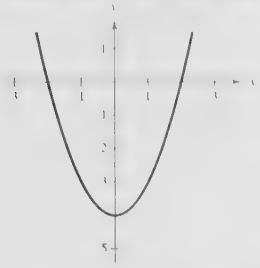
9. $f(x) = 3x - 4$



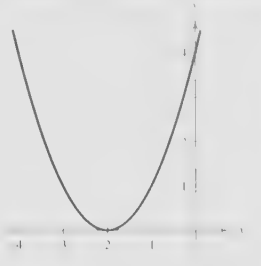
10. $f(x) = 9x + 10$



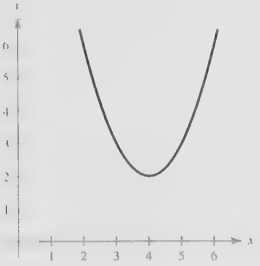
1. Shifted four units downward



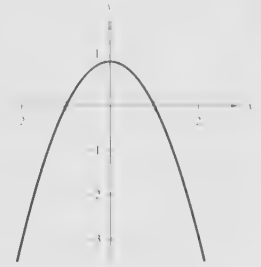
3. Shifted two units to the left



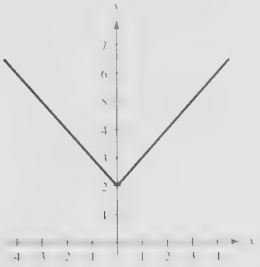
5. Shifted two units upward and four units to the right



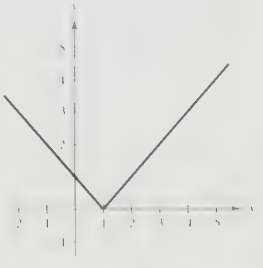
7. Reflected about the x-axis and shifted one unit upward



9. Shifted two units upward



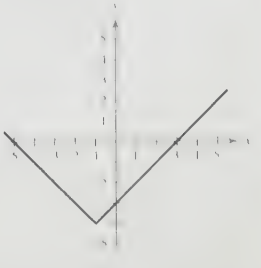
11. Shifted one unit to the right



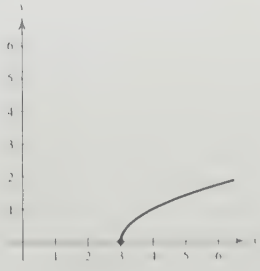
13. Reflected about the x-axis and shifted three units upward



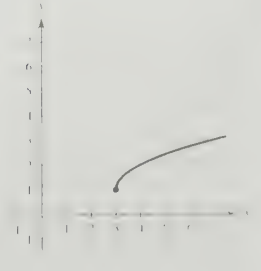
15. Shifted one unit to the left and four units downward



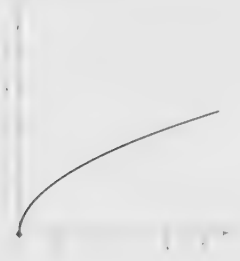
17. Shifted three units to the right



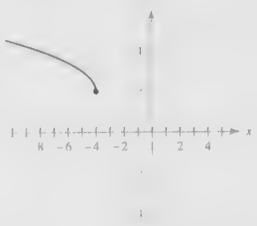
19. Shifted three units to the right and one unit upward



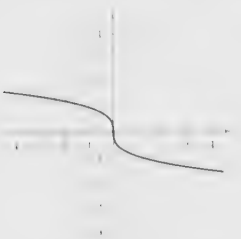
21. Vertically stretched by a factor of $\sqrt{2}$



23. Reflected about the y-axis and shifted four units to the left and two units upward



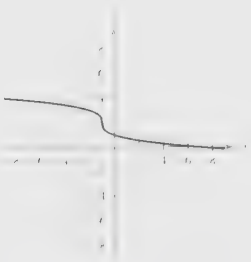
25. Reflected about the y-axis



27. Shifted two units downward



29. Reflected about the x-axis, shifted one unit to the left, and two units upward



31. Vertically stretched by a factor of 2, shifted one unit to the left, and one unit downward



33. Vertically shrunk by a factor of $\frac{1}{2}$



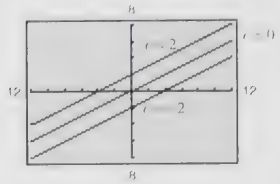
35. Common function: $y = x^3$
Transformation: shifted two units to the right
Equation: $y = (x - 2)^3$

37. Common function: $y = x^2$
Transformation: reflected about the x-axis
Equation: $y = -x^2$

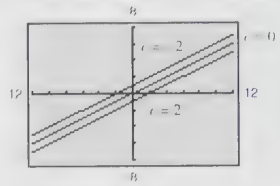
39. Common function: $y = \sqrt{x}$
Transformation: reflected about the x-axis and shifted one unit upward
Equation: $y = -\sqrt{x} + 1$

41. Common function: $y = x$
Transformation: vertically stretched by a factor of 2
Equation: $y = 2x$

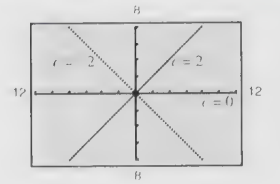
43. (a) Vertical shift of two units



(b) Horizontal shift of two units

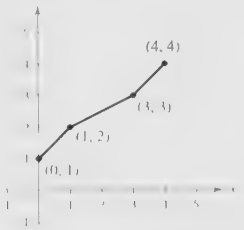


(c) Slope of the function changes

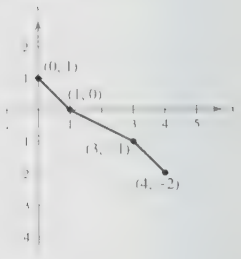


45. (a) $g(x) = -(x - 1)^2 + 1$ (b) $g(x) = -(x + 1)^2$

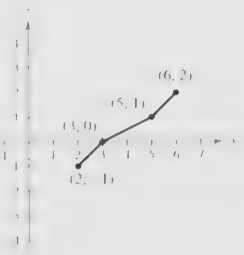
47. (a)



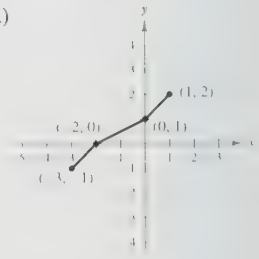
(b)



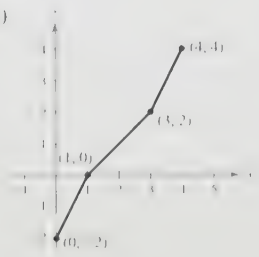
(c)



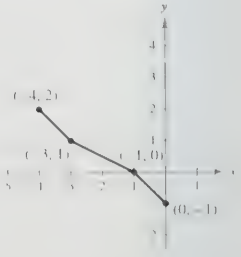
(d)



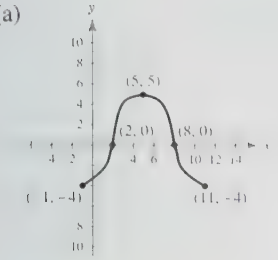
(e)



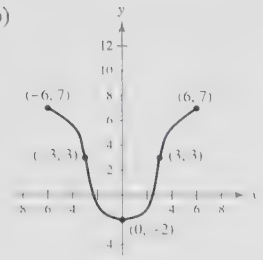
(f)



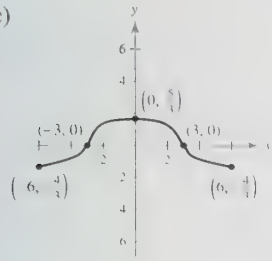
49. (a)



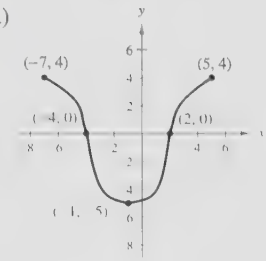
(b)



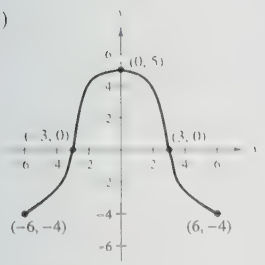
(c)



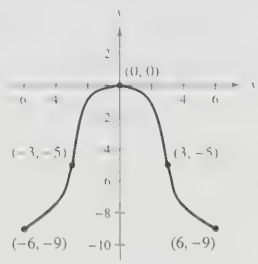
(d)



(e)



(f)



51. $y = x^3 - 2$ **53.** $y = 4x^3$

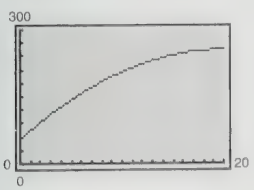
55. $h(x) = \sqrt{x - 4} - 3$ **57.** $h(x) = \frac{1}{2}\sqrt{x - 3}$

59. $g(x) = x^3 - 3x^2 + 2$ **61.** $g(x) = (x + 1)^3 - 3(x + 1)^2$

63. Shifted one unit to the right and two units downward
$g(x) = (x - 1)^2 - 2$

65. (a) $P(x) = 80 + 20x - 0.5x^2 - 25$
$P(x) = 55 + 20x - 0.5x^2, \quad 0 \le x \le 20$
Shifted 25 units down

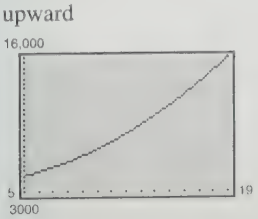
(b)



(c) $P\left(\frac{x}{100}\right) = 80 + \frac{x}{5} - 0.00005x^2,$
$0 \le x \le 2000$ (x in dollars)
Horizontal stretch

67. (a) Vertically stretched by a factor of 34, shifted 3705 units upward

(b)



(c) 2007

(d) $M(t) = 34.0(t + 10)^2 + 3705$;
To make a horizontal shift 10 years backward (10 units to the left), add 10 to t.

69. $y = x^7$ will be an odd function and its graph will be similar in shape to the graphs of $y = x^3$ and $y = x^5$. $y = x^8$ will be an even function and its graph will be similar in shape to the graphs of $y = x^2$, $y = x^4$, and $y = x^6$.

Section 2.7 *(page 236)*

Skills Warm Up *(page 236)*

1. $\frac{1}{x(1 - x)}$ **2.** $-\frac{12}{(x + 3)(x - 3)}$ **3.** $\frac{3x - 2}{x(x - 2)}$

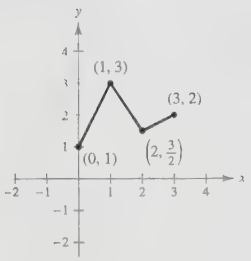
4. $\frac{4x - 5}{3(x - 5)}$ **5.** $\frac{\sqrt{x^2 - 1}}{x + 1}, \quad x \ne 1$

6. $\frac{x + 1}{x(x + 2)}, \quad x \ne 2$ **7.** $5(x - 2), \quad x \ne -2$

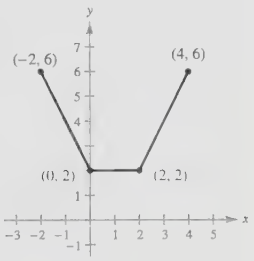
8. $\frac{x + 1}{(x - 2)(x + 3)}, \quad x \ne -5, -1, 0$

9. $\frac{1 + 5x}{3x - 1}, \quad x \ne 0$ **10.** $\frac{x + 4}{4x}, \quad x \ne 4$

1.



3.



5. (a) $2x$ (b) 2 (c) $x^2 - 1$
(d) $\frac{x + 1}{x - 1}$; Domain: $(-\infty, 1) \cup (1, \infty)$

7. (a) $3x - x^2$ (b) $x^2 + 3x - 2$ (c) $-3x^3 + x^2 + 3x - 1$
(d) $\frac{3x - 1}{1 - x^2}$; Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

9. (a) $x^2 + \sqrt{1 - x} + 5$ (b) $x^2 - \sqrt{1 - x} + 5$
(c) $x^2\sqrt{1 - x} + 5\sqrt{1 - x}$
(d) $\frac{x^2 + 5}{\sqrt{1 - x}}$; Domain: $(-\infty, 1)$

11. (a) $\frac{x + 1}{x^2}$ (b) $\frac{x - 1}{x^2}$ (c) $\frac{1}{x^3}$
(d) $x, x \ne 0$; Domain: $(-\infty, 0) \cup (0, \infty)$

13. 14 **15.** $-4t^2 + 4t + 3$ **17.** -6 **19.** $\frac{11}{23}$ **21.** 3

23. -8 **25.** (a) $6x + 15$ (b) $6x + 5$ (c) $9x$

27. (a) $9x^2 + 6x + 1$ (b) $3x^2 + 1$ (c) x^4

29. (a) $x - \frac{8}{3}$ (b) $x - 8$

31. (a) $x, x \ge 0$ (b) $x, x \ge -4$ **33.** (a) $\sqrt[4]{x}$ (b) $\sqrt[4]{x}$

35. (a) $|x + 6|$ (b) $|x| + 6$

37. (a) All real numbers, or $(-\infty, \infty)$
(b) $x \ge 0$, or $[0, \infty)$ (c) $x \ge 0$, or $[0, \infty)$

39. (a) All real numbers except $x = 0$, or $(-\infty, 0) \cup (0, \infty)$
(b) All real numbers, or $(-\infty, \infty)$
(c) All real numbers except $x = 2$, or $(-\infty, 2) \cup (2, \infty)$

41. (a) 3 (b) 0 **43.** (a) 0 (b) 4

45. Answers will vary.
Sample answer:
$f(x) = x^2, g(x) = 2x + 1$

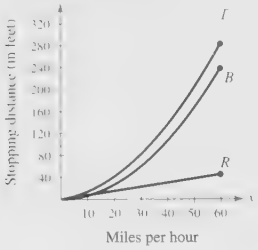
47. Answers will vary.
Sample answer:
$f(x) = \sqrt[3]{x}, g(x) = x^2 - 4$

CHAPTER 2

49. Answers will vary.
Sample answer:
$f(x) = \frac{1}{x}, g(x) = x + 2$

51. Answers will vary.
Sample answer:
$f(x) = x^2 + 2x$,
$g(x) = x + 4$

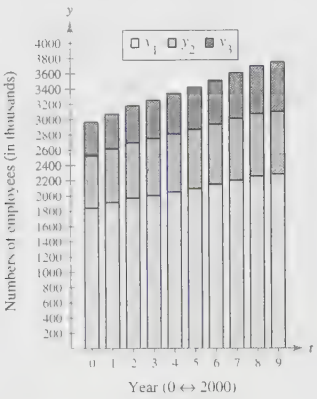
53. $T = \frac{3}{4}x + \frac{1}{15}x^2$



55. $(C \circ x)(t) = 1500t + 495$
$C \circ x$ represents the cost of producing x units in t hours.

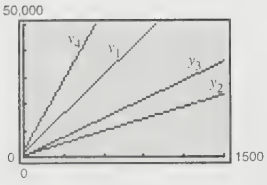
57. (a) Total sales decreased.
(b) $R_1 + R_2 = 917 - 6.7t$
The function is linear with a negative slope, so its value decreases for increasing values of t.

59. (a)



(b) $y_1 = 48.5t + 1856$
$y_2 = 15.2t + 694$
$y_3 = 23.7t + 434$

(c)



2011: 3,945,400 employees
2013: 4,120,200 employees

61. $(A \circ r)(t) = \pi(0.6t)^2 = 0.36\pi t^2$
$A \circ r$ represents the area of the circle at time t.

63.

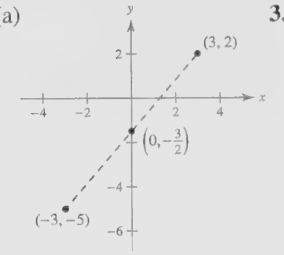
Year	2005	2006	2007	2008	2009
P/E	16.7	21.0	25.9	12.7	11.3

65. Domain of $(f/g)(x)$: all real numbers $0 \le x < 3$, or $[0, 3)$
Domain of $(g/f)(x)$: all real numbers $0 < x \le 3$, or $(0, 3]$
The two domains differ because if $x = 3$, $(f/g)(x)$ is undefined (division by zero), and if $x = 0$, $(g/f)(x)$ is undefined (division by zero).

67. True. The range of g must be a subset of the domain of f for $(f \circ g)(x)$ to be defined.

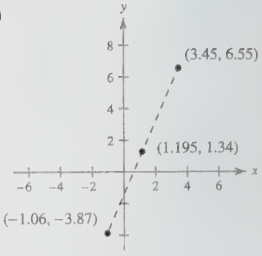
Review Exercises *(page 244)*

1. (a)



(b) $\sqrt{85} \approx 9.22$
(c) $\left(0, -\frac{3}{2}\right)$

3. (a)



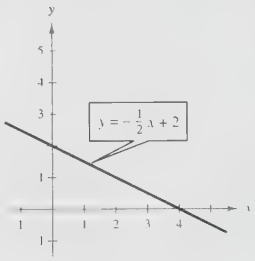
(b) $\sqrt{128.9165} \approx 11.35$
(c) (1.195, 1.34)

5. $x = -10$ or 30

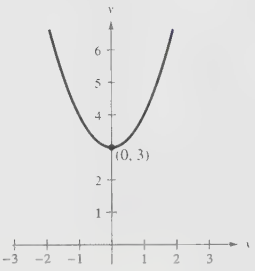
7. (a) Yes (b) Yes

9.

x	y
-2	3
0	2
2	1
3	$\frac{1}{2}$
4	0

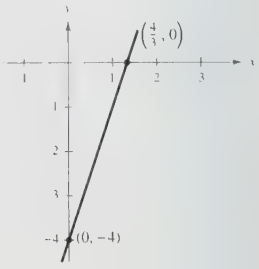


11.



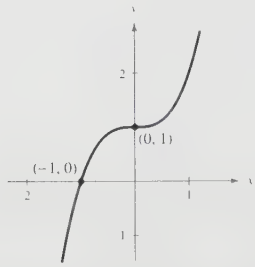
y-intercept: (0, 3)
Symmetry: y-axis

13.



x-intercept: $\left(\frac{4}{3}, 0\right)$
y-intercept: (0, −4)
Symmetry: none

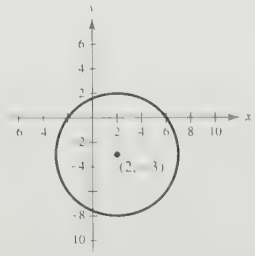
15.



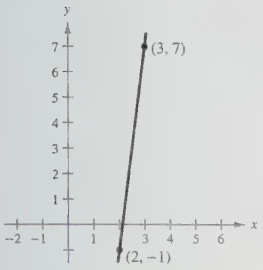
x-intercept: $(-1, 0)$
y-intercept: (0, 1)
Symmetry: none

17. $(x + 1)^2 + (y - 2)^2 = 36$

19. $(x - 2)^2 + (y + 3)^2 = 25$

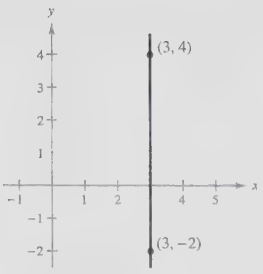


21.



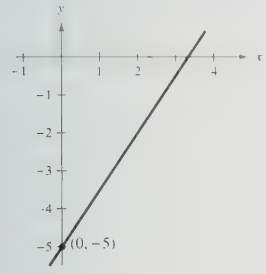
$m = 8$

23.

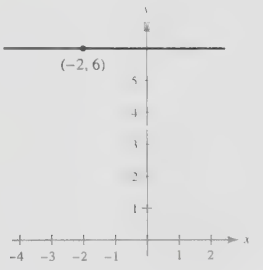


m is undefined.

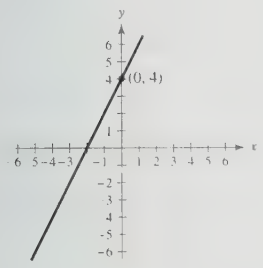
25. $3x - 2y - 10 = 0$



27. $y = 6$



29. Slope: 2
y-intercept: $(0, 4)$



31. Slope: undefined
y-intercept: none



33. Parallel **35.** Neither

37. (a) $y = \frac{1}{2}x + 1$ (b) $y = -2x + 11$

39. (a) $5x - 4y - 23 = 0$ (b) $4x + 5y - 2 = 0$

41. (a) $y = -2$ (b) $x = -1$

43. $y = \frac{7}{3}x$ **45.** $y = 348x$ **47.** $A = 5r$ **49.** $a = \frac{3}{4}b$

51. $y = 0.0084x$; \$1470 **53.** \$3,350,000

55. $V = -12{,}950t + 135{,}000,\ 0 \le t \le 10$

57. y is a function of x. **59.** y *is not* a function of x.

61. Function; Every element of A is assigned to an element of B.

63. (a) -2 (b) -3 (c) -5 (d) $\sqrt{x+7} - 5$

65. All real numbers x **67.** $x \ge -5$

69. $1 \le t < 4, t > 4$

71. The domain of $h(x)$ is all real numbers except $x = 0$, because division by zero is undefined. The domain of $k(x)$ is all real numbers except $x = -2$ and $x = 2$, because if $x = 2$ or $x = -2$, then $x^2 - 4$ equals zero, and division by zero is undefined. When a graphing utility and the *table* feature are used, $h(0)$ results in an error, and $k(-2)$ and $k(2)$ also result in errors.

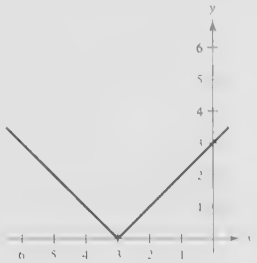
73. (a) $B = 6500\left(1 + \dfrac{0.0575}{4}\right)^{4t}$ (b) $t \ge 0$

75. (a) Domain: all real numbers
Range: $[1, \infty)$
(b) Decreasing: $(-\infty, 0)$
Increasing: $(0, \infty)$
(c) Even (d) Minimum: $(0, 1)$

77. (a) Domain: all real numbers
Range: all real numbers
(b) Decreasing: $\left(0, \frac{8}{3}\right)$
Increasing: $(-\infty, 0) \cup \left(\frac{8}{3}, \infty\right)$
(c) Neither
(d) Relative minimum: $\left(\frac{8}{3}, -\frac{256}{27}\right)$
Relative maximum: $(0, 0)$

79. y is a function of x. **81.** y is not a function of x.

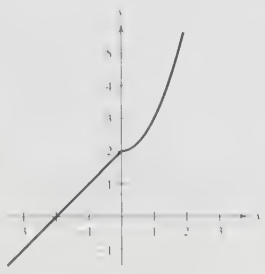
83.



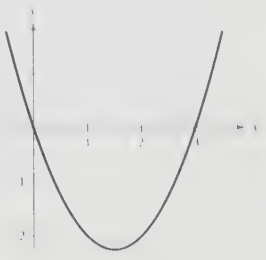
85.



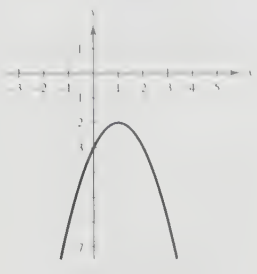
87.



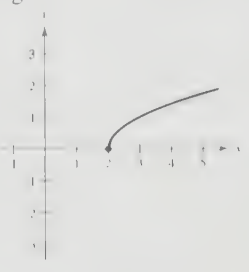
89.



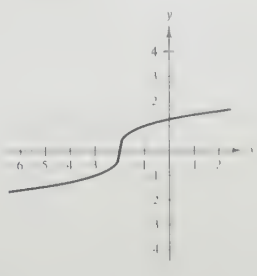
91. Reflected about the x-axis and shifted two units downward and one unit to the right



93. Shifted two units to the right



95. Shifted two units to the left



97. Common function: $y = \sqrt{x}$
Transformation: reflected about the y-axis and shifted three units to the right
Equation: $y = \sqrt{3 - x}$ or $y = \sqrt{-(x - 3)}$

99. (a) $(f + g)(x) = x^2 + 5x - 1$
(b) $(f - g)(x) = -x^2 + x - 1$
(c) $(fg)(x) = 3x^3 + 5x^2 - 2x$
(d) $(f/g)(x) = \dfrac{3x - 1}{x^2 + 2x}$
Domain of f/g: $x < -2, -2 < x < 0, x > 0$, or $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$

101. 9 **103.** 18

105. (a) $x^2 + 6x + 9$

Domain: All real numbers

(b) $x^2 + 3$

Domain: All real numbers

107. (a) $\frac{1}{3x + x^2}$

Domain: All real numbers except $x = 0$ and $x = -3$

(b) $\frac{3}{x} + \frac{1}{x^2}$, or $\frac{3x + 1}{x^2}$

Domain: All real numbers except $x = 0$

109. Answers will vary.

Sample answer: $f(x) = x^2$, $g(x) = 6x - 5$

111. Answers will vary.

Sample answer: $f(x) = \frac{1}{x}$, $g(x) = x - 1$

113. (a) $r = \frac{x}{2}$ (b) $A = \pi r^2$ (c) $(A \circ r)(x) = \frac{\pi x^2}{4}$

$(A \circ r)(x)$ is the area of the base of the tank in terms of x, where x is the length of a side of the square base.

115. $(N \circ T)(t) = 32t^2 + 36t + 204$

$(N \circ T)(t)$ represents the number of bacteria in the petri dish at time t.

Test Yourself *(page 248)*

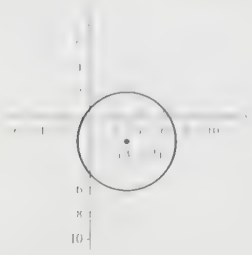
1. Distance: $4\sqrt{5}$
Midpoint: $(1, 0)$

2. Distance: ≈ 7.81
Midpoint: $(0.44, 4.335)$

3. x-intercepts: $(-5, 0), (3, 0)$
y-intercept: $(0, -15)$

4. Symmetric with respect to the origin

5. $(x - 3)^2 + (y + 2)^2 = 16$



6. $2x - 3y + 21 = 0$

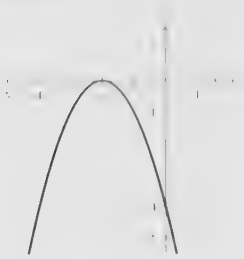
7. True. Each value of x corresponds to exactly one value of y.

8. False. The element -9 is not included in set B.

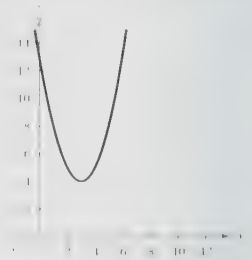
9. (a) Domain: All real numbers
Range: $(-\infty, 2]$
(b) Decreasing: $(0, \infty)$
Increasing: $(-\infty, 0)$
(c) Even
(d) Maximum: $(0, 2)$

10. (a) Domain: $(-\infty, -2] \cup [2, \infty)$
Range: $[0, \infty)$
(b) Decreasing: $(-\infty, -2)$
Increasing: $(2, \infty)$
(c) Even
(d) Minima: $(-2, 0), (2, 0)$

11. Reflected about the x-axis and shifted two units to the left



12. Shifted three units to the right and four units upward



13. $(f - g)(x) = x^2 - 2x + 3$

14. $(fg)(x) = 2x^4 - 2x^3 + 2x^2 - 2$

15. $(f \circ g)(x) = 4x^2 - 8x + 5$

16. $(g \circ f)(x) = 2x^2$

17. $V = 30{,}000 - 5300t$

18.



$P = 0.17t + 19.3$

Chapter 3

Section 3.1 *(page 258)*

Skills Warm Up *(page 258)*

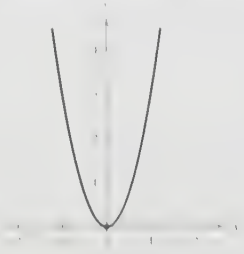
1. $\frac{1}{5}, 6$ **2.** $\frac{1}{5}, 3$ **3.** $\frac{3}{5}, 1$ **4.** 10

5. $3 \pm \sqrt{5}$ **6.** $2 \pm \sqrt{3}$ **7.** $4 \pm \frac{\sqrt{14}}{2}$

8. $5 \pm \frac{\sqrt{3}}{3}$ **9.** $\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ **10.** $\frac{3}{2} \pm \frac{\sqrt{21}}{2}$

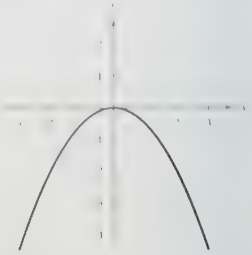
1. Compared with the graph of $y = x^2$, each output of $f(x) = 5x^2$ vertically stretches the graph by a factor of 5.

3.



The graph of $f(x) = 3x^2$ is a vertical stretch of the graph of $y = x^2$ by a factor of 3.

5.

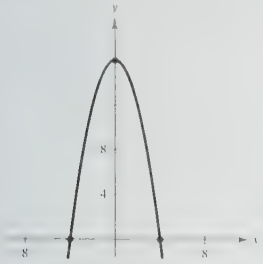


The graph of $g(x) = -\frac{1}{2}x^2$ is a reflection in the x-axis of the graph of $y = x^2$ and a vertical shrink by a factor of $\frac{1}{2}$.

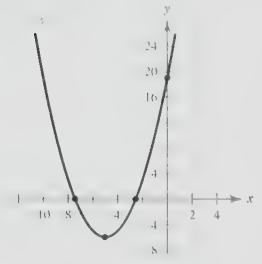
7. The graph of f is the graph of $y = x^2$ reflected in the x-axis, shifted to the left 1 unit and shifted upward 1 unit.

9. e **10.** c **11.** a **12.** d **13.** f **14.** b

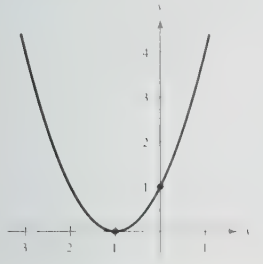
15. $f(x) = -x^2 + 16$
Vertex: $(0, 16)$



17. Vertex: $(-5, -6)$



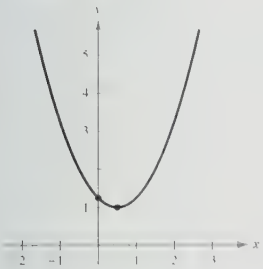
19. $g(x) = (x + 1)^2$
Vertex: $(-1, 0)$



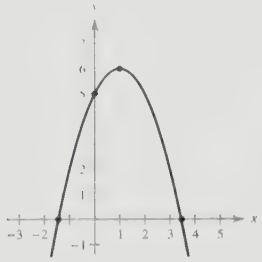
21. $f(x) = -(x + 1)^2 + 4$
Vertex: $(-1, 4)$



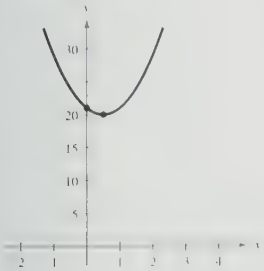
23. $f(x) = \left(x - \frac{1}{2}\right)^2 + 1$
Vertex: $\left(\frac{1}{2}, 1\right)$



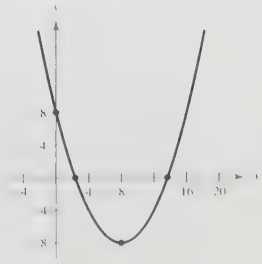
25. $f(x) = -(x - 1)^2 + 6$
Vertex: $(1, 6)$



27. $h(x) = 4\left(x - \frac{1}{2}\right)^2 + 20$
Vertex: $\left(\frac{1}{2}, 20\right)$



29. $f(x) = \frac{1}{4}(x - 8)^2 - 8$
Vertex: $(8, -8)$



31. $y = -(x + 2)^2$ **33.** $y = (x - 3)^2 - 9$

35. $y = -2(x + 3)^2 + 3$

37. $y = -\frac{1}{2}(x - 2)^2 - 1$

39. $y = \frac{3}{4}(x - 5)^2 + 12$

41. Answers will vary.
Sample answer:
$f(x) = x^2 - x - 2$
$g(x) = -x^2 + x + 2$

43. Answers will vary.
Sample answer:
$f(x) = x^2 - 10x$
$g(x) = -x^2 + 10x$

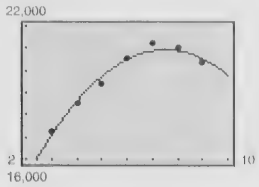
45. Answers will vary.
Sample answer:
$f(x) = 2x^2 + 7x + 3$
$g(x) = -2x^2 - 7x - 3$

47. $A = 100x - x^2$; $(50, 2500)$; The rectangle has the greatest area ($A = 2500$ square feet) when its width is 50 feet.

49. $x = 150$ feet, $y = 200$ feet; 300 feet × 200 feet

51. 25,000 units **53.** 20 fixtures **55.** 14 feet

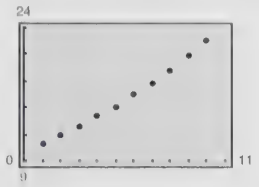
57. (a) and (b)



$R = -186.43t^2 + 2811.9t + 10,329$

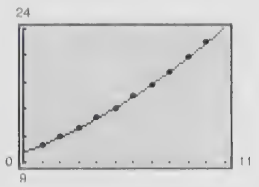
(c) 2007 ($t \approx 7.5$); Yes, this agrees with the actual data.

59. (a)



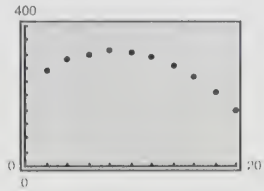
(b) $N = 0.047t^2 + 0.76t + 10.1$

(c)



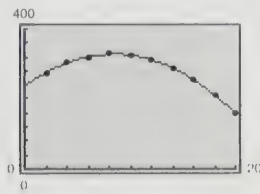
(d) 25,988 physicians

61. (a)



(b) $p(x) = -1.276x^2 + 21.63x + 236.5$

(c)



(d) $\approx (8.5, 328.2)$; Producing about 85,000 units yields the maximum profit, about $32,820,000.

(e) Sample answer: Production costs may be growing faster than revenue, so profit decreases.

63. $f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$

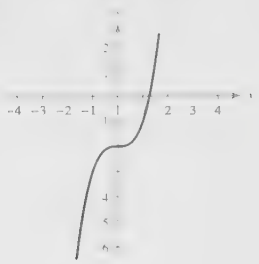
Section 3.2 *(page 269)*

Skills Warm Up *(page 269)*

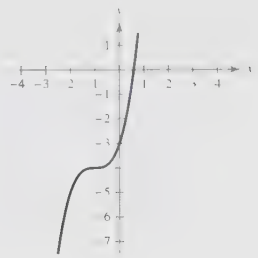
1. $(3x - 2)(4x + 5)$ **2.** $x(5x - 6)^2$
3. $z^2(12z + 5)(z + 1)$ **4.** $(y + 5)(y^2 - 5y + 25)$
5. $(x + 3)(x + 2)(x - 2)$ **6.** $(x + 2)(x^2 + 3)$
7. No real solution **8.** $3 \pm \sqrt{5}$ **9.** $\frac{1}{2} \pm \sqrt{3}$
10. ± 3

1. b **2.** d **3.** c **4.** a

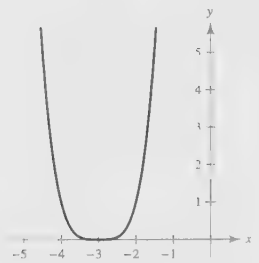
5.



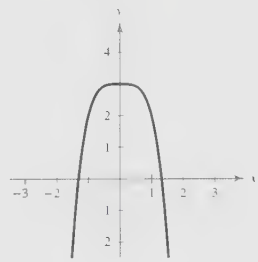
7.



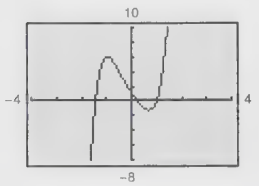
9.



11.

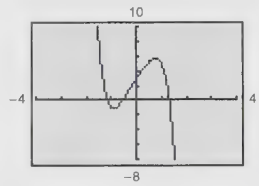


13. (a) 5; odd
(b) 2; positive
(c)



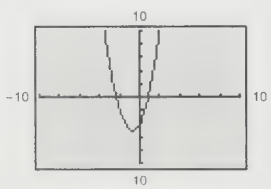
(d) Falls to the left, rises to the right

15. (a) 5; odd
(b) -2; negative
(c)



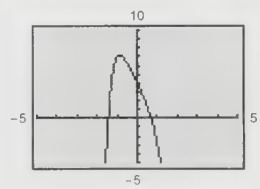
(d) Rises to the left, falls to the right

17. (a) 2; even
(b) 2; positive
(c)



(d) Rises to the left and to the right

19. (a) 6; even
(b) -1; negative
(c)



(d) Falls to the left and to the right

21. When the leading coefficient is positive, the graph falls to the left and rises to the right. When the leading coefficient is negative, the graph rises to the left and falls to the right.

23. Rises to the left
Falls to the right

25. Rises to the left
Falls to the right

27. Rises to the left
Rises to the right

29. Rises to the left
Rises to the right

31. Falls to the left; Falls to the right

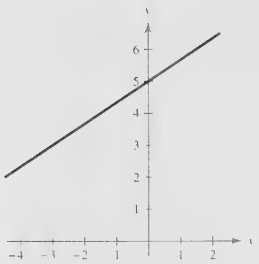
33. (a) 1 (b) 2 **35.** (a) 4 (b) 5 **37.** ± 3

39. -4 **41.** No real zeros **43.** $-1 \pm \sqrt{5}$ **45.** 2, 0

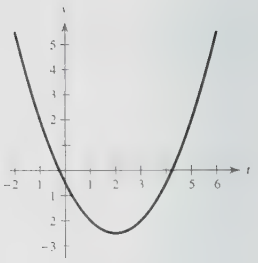
47. ± 1 **49.** $\pm\sqrt{5}$ **51.** 3 **53.** e **54.** c

55. g **56.** d **57.** f **58.** h **59.** a **60.** b

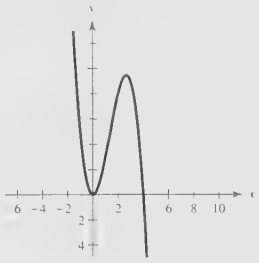
61.



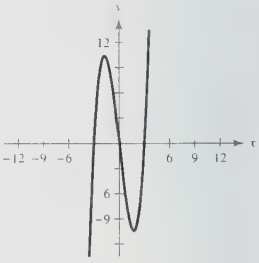
63.



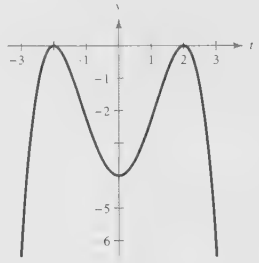
65.



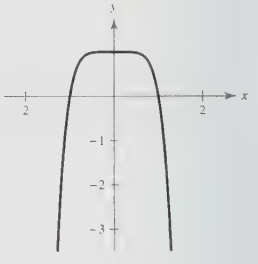
67.



69.

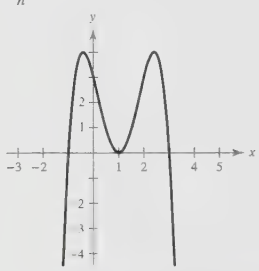


71.

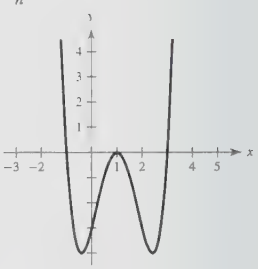


73. Answers will vary. Sample answers:

$a_n < 0$

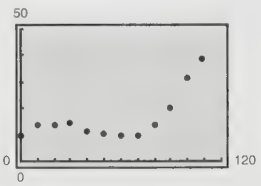


$a_n > 0$



75. $f(x) = \frac{1}{2}(x^4 - 11x^3 + 28x^2)$

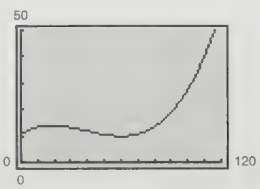
77. (a)



(b) Positive

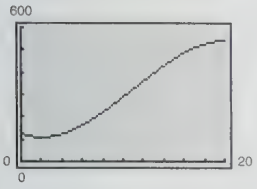
(c) $P = 0.00011t^3 - 0.013t^2 + 0.37t + 10.8$
The model agrees with the prediction from part (b).

(d)



2012 ($t \approx 112.8$); The prediction seems reasonable.

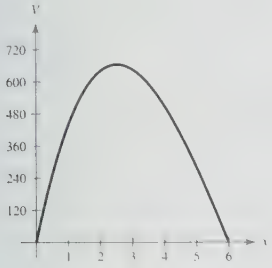
79.



(11, 323.9)

81. (a) Answers will vary. (b) Domain: $0 < x < 6$

(c)



$x \approx 2.5$

Section 3.3 *(page 279)*

Skills Warm Up *(page 279)*

1. $x^3 - x^2 + 2x + 3$ **2.** $2x^3 + 4x^2 - 6x - 4$
3. $x^4 - 2x^3 + 4x^2 - 2x - 7$
4. $2x^4 + 12x^3 - 3x^2 - 18x - 5$
5. $(x - 3)(x - 1)$ **6.** $8(x + 2)(x - 5)$
7. $(3x + 5)(x - 1)$ **8.** $(3x - 4)^2$
9. $2x(x - 1)(2x - 3)$ **10.** $x(3x + 2)(2x + 1)$

1. $x + 2$ **3.** $3x - 4$ **5.** $2x + 4$ **7.** $x^2 + 2x + 3$

9. $x + 3$ **11.** $x^3 + 3x^2 - 1$ **13.** $7 - \dfrac{25}{x + 4}$

15. $3x + 5 - \dfrac{2x - 3}{2x^2 + 1}$ **17.** $x + \dfrac{x - 27}{x^2 - 1}$

19. $x^2 - 6x + 17 - \dfrac{36}{x + 2}$

21. $2x^3 + 4x^2 - 2x - 8 - \dfrac{10x - 7}{x^2 - 2x + 1}$ **23.** $x + 7$

25. $2x^2 - 3x + 5$ **27.** $4x^2 - 9$ **29.** $-x^2 + 10x - 25$

31. $x^3 - 7x^2 + 14x - 20 + \dfrac{84}{x + 3}$

33. $10x^3 + 10x^2 + 60x + 360 + \dfrac{1360}{x - 6}$

35. $x^4 - x^3 + x^2 - x + 1 - \dfrac{1}{x + 1}$

37. $-3x^3 - 6x^2 - 12x - 24 - \dfrac{48}{x - 2}$

39. $-x^2 + 3x - 6 + \dfrac{11}{x + 1}$ **41.** $4x^2 + 14x - 30$

43. $f(x) = (x - 2)(x^2 + 3x - 6) + 8$; $f(2) = 8$

45. $f(x) = \left(x - \frac{1}{3}\right)(3x^2 + 3x + 6)$; $f\left(\frac{1}{3}\right) = 0$

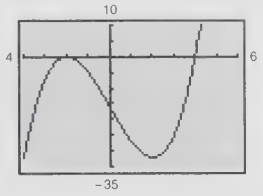
47. $f(x) = \left(x - \sqrt{3}\right)\left[x^2 + \left(2 + \sqrt{3}\right)x + 2\sqrt{3}\right] - 6$;
$f\left(\sqrt{3}\right) = -6$

49. $f(x) = \left(x - 1 - \sqrt{3}\right)\left[2x^2 + \left(3 + 2\sqrt{3}\right)x + \left(-5 + 5\sqrt{3}\right)\right]$; $f\left(1 + \sqrt{3}\right) = 0$

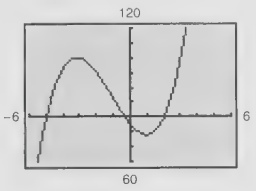
51. (a) -69 (b) -2081 (c) -6 (d) 446

53. (a) 30 (b) -21 (c) 1.092 (d) -9

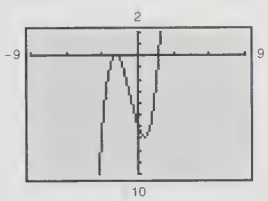
55. (a) Proof (b) $x + 2$
(c) $f(x) = (x - 4)(x + 2)^2$ (d) $4, -2$
(e)



57. (a) Proof (b) $x + 5$
(c) $f(x) = (3x + 1)(x - 2)(x + 5)$ (d) $-\frac{1}{3}, 2, -5$
(e)



59. (a) Proof (b) $x + \sqrt{3}$
(c) $f(x) = \left(x - \sqrt{3}\right)(x + 2)\left(x + \sqrt{3}\right)$
(d) $\sqrt{3}, -2, -\sqrt{3}$
(e)



61. The second polynomial is a factor of the first polynomial.

63. e; $3, \dfrac{-1 \pm \sqrt{17}}{2}$ **64.** d; $-2, \dfrac{3 \pm \sqrt{5}}{2}$

65. a; $-1, -2 \pm \sqrt{2}$ **66.** c; $3, 1 \pm \sqrt{5}$

67. b; $-3, \pm\sqrt{5}$ **68.** f; $2, \pm\sqrt{5}$

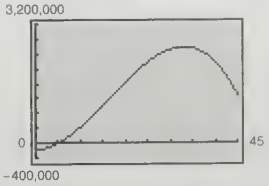
69. Answers will vary. Sample answer:
$f(x) = 3x^3 - 13x^2 + 4x + 20$
$f(x) = -3x^3 + 13x^2 - 4x - 20$
Infinitely many polynomial functions

71. $x^2 - 7x + 10$ **73.** $3x^2 - x - 10$ **75.** $x^2 - 6x$

77. $x^2 + 4x + 3$ **79.** $x^2 + 9x + 16$ ft^2

81. Multiply the divisor and the quotient, then add the remainder to obtain the dividend.

83. (a)



(b) Another nonnegative real solution is $x \approx 25.0366$, representing an advertising expense of \$250,366.
(c) The answer is the same as part (b).

85. $c = 42$

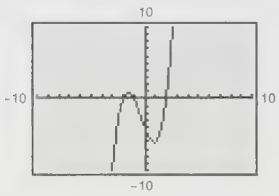
CHAPTER 3

Section 3.4 *(page 291)*

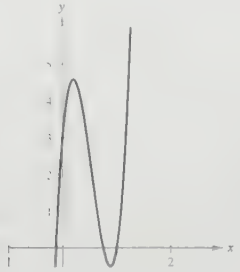
Skills Warm Up *(page 291)*

1. Answers will vary. Sample answer: $f(x) = 3x^3 - 8x^2 - 5x + 6$
2. Answers will vary. Sample answer: $f(x) = 4x^4 - 3x^3 - 16x^2 + 12x$
3. $x^4 - 3x^3 + 5 + \frac{3}{x + 3}$
4. $3x^3 + 15x^2 - 9 - \frac{2}{x + (2/3)}$
5. $\frac{1}{2}, -3 \pm \sqrt{5}$ **6.** $10, -\frac{2}{3}, -\frac{3}{2}$ **7.** $-\frac{3}{4}, 2 \pm \sqrt{2}$
8. $\frac{2}{5}, -\frac{7}{2}, -2$ **9.** $\pm\sqrt{2}, \pm 1$ **10.** $\pm 2, \pm\sqrt{3}$

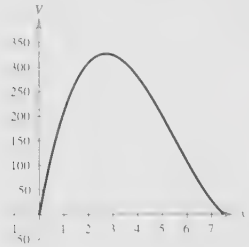
1. $\pm 1; f(1) = 2, f(-1) = 0$
   Actual zero of f: -1
3. $\pm 1; f(1) = -2, f(-1) = -4$
   Actual zero of f: none
5. $\pm 1, \pm 2, \pm 3, \pm 6;$
   $f(1) = -12, f(-1) = 0, f(2) = -12, f(-2) = 0,$
   $f(3) = 0, f(-3) = -12, f(6) = 168, f(-6) = -180$
   Actual zeros of f: $-2, -1, 3$
7. Possible: $\pm 1, \pm 2, \pm 4$
   Actual: $-1, \pm 2$



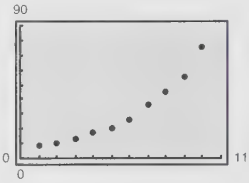
9. Possible: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$
   Actual: $1, 2, 6$
11. Possible: $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$
    Actual: $-2, 3, \frac{1}{2}$
13. $-3, \frac{1}{2}, 4$ **15.** $\pm\frac{1}{2}, \pm 2$ **17.** $1, 2, 3$ **19.** $-1, -10$
21. $\frac{1}{2}, -1$ **23.** $\pm 3, \pm\sqrt{2}$ **25.** $-1, 2$ **27.** $-6, \frac{1}{2}, 1$
29. $-2, 0, 1$
31. (a) $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{1}{16}, \pm\frac{3}{16}, \pm\frac{1}{32}, \pm\frac{3}{32}$
    (b)



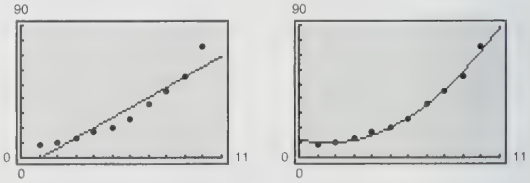
    (c) $1, \frac{3}{4}, -\frac{1}{8}$
33. $f(1) = -2, f(2) = 7$ **35.** $f(2) = -6, f(3) = 44$
37. Real zero ≈ 0.7 **39.** Real zero ≈ 3.3
41. e; -1.769 **42.** c; 0.755 **43.** d; 0.206
44. a; $0.266, 1.175, 2.559$ **45.** f; 2.769
46. b; $-1.675, -0.539, 2.214$ **47.** $-1.164, 1.453$
49. $0.900, 1.100, 1.900$ **51.** $-1.453, 1.164$
53. $-2.177, 1.563$ **55.** d **56.** a **57.** b **58.** c
59. (a) $V = x(18 - 2x)(15 - 2x)$
    Domain: $0 < x < 7.5$
    (b)



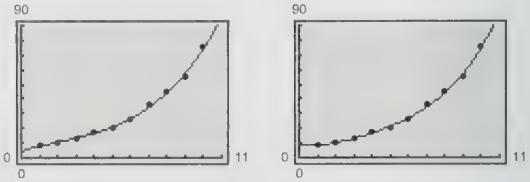
    Approximate dimensions:
    2.72 inches $\times$ 12.56 inches $\times$ 9.56 inches
    (c) $x \approx 0.448, 6, \approx 10.052$
    A value of $x \approx 10.052$ inches is impossible because it would yield a negative length and width.
    (d) $x = 6$
61. 18 inches $\times$ 18 inches $\times$ 36 inches
63. (a) $V = x^3 + 9x^2 + 26x + 24 = 120$
    (b) 4 feet by 5 feet by 6 feet
65. 4.49 hours
67. (a)



    (b) Linear: $R = 6.923t - 7.43$
    Quadratic: $R = 0.8884t^2 - 2.849t + 12.12$
    Cubic: $R = 0.08806t^3 - 0.5646t^2 + 3.852t + 4.56$
    Quartic:
    $R = 0.012630t^4 - 0.18979t^3 + 1.4688t^2 - 1.705t + 8.90$
    (c) Linear Quadratic

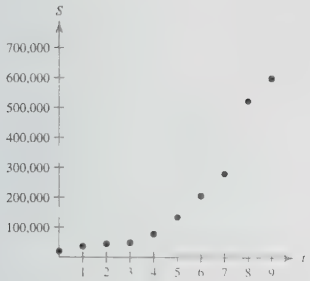


    Cubic Quartic



    The nonlinear models fit the data well.
    (d) Linear: 2015 ($t \approx 15.52$)
    Quadratic: 2011 ($t \approx 11.68$)
    Cubic: 2011 ($t \approx 11.18$)
    Quartic: 2010 ($t \approx 10.99$)
    Answers will vary. Sample answer: The higher the degree of the model, the faster the graph climbs after 2010, so the quartic model makes the earliest prediction of when the revenue per share will reach \$100. Based on the increasing trend of the last few data points, the year (2015) predicted by the linear model seems too late.

69. (a)

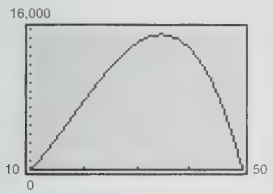


(b) Quadratic: $S = 11{,}001.49t^2 - 36{,}269.6t + 47{,}202$
Cubic: $S = 735.402t^3 + 1073.57t^2 - 2367.6t + 28{,}670$

(c) Quadratic: 2011 ($t \approx 11.1$)
Cubic: 2010 ($t \approx 10.6$)

Answers will vary. Sample answer: Based on the trend of the data, both models seem appropriate for predicting future values.

71. (a)



30 and ≈ 38.91

(b) You can solve

$$-x^3 + 54x^2 - 140x - 3000 = 14{,}400$$

by rewriting the equation as

$$y = -x^3 + 54x^2 - 140x - 17{,}400.$$

Using the *table* feature of a graphing utility, you can approximate the solutions to be $x \approx -14.91$, $x = 30$, and $x \approx 38.91$. The company should charge \$38.91 to generate greater revenue.

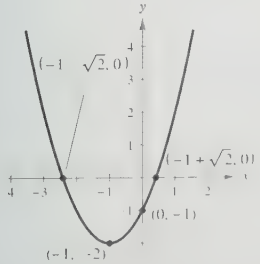
73. No; setting $h = 64$ and solving the resulting equation yields imaginary roots.

75. No **77.** Yes. Answers will vary. Sample answer: $f(x) = (x + 1)(x^2 - 5)$

79. Answers will vary.

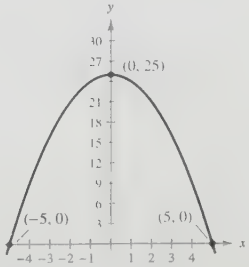
Quiz Yourself *(page 296)*

1.



Vertex: $(-1, -2)$
Intercepts: $\left(-1 \pm \sqrt{2}, 0\right), (0, -1)$

2.



Vertex: $(0, 25)$
Intercepts: $(\pm 5, 0), (0, 25)$

3. Rises to the left, Falls to the right **4.** Rises to the left, Rises to the right

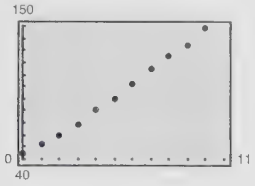
5. 293 **6.** $f(x) = (x + 1)(x^3 - x^2 - 4x + 4) + 0$; $f(-1) = 0$

7. $f(x) = (x - 2)(x^2 + 7x + 12) + 0$; $f(2) = 0$

8. $2x^2 + 5x - 12$ **9.** $\pm\sqrt{5}, -\frac{7}{2}$ **10.** $\pm 3, \pm\frac{1}{2}$

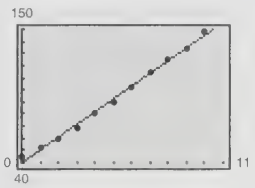
11. $1, -\frac{4}{3}$ **12.** $\frac{3}{2}$ **13.** $P = \$2{,}534{,}375$; $\$337{,}600$

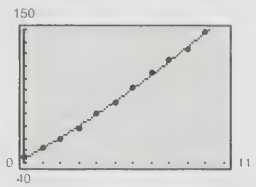
14. (a)



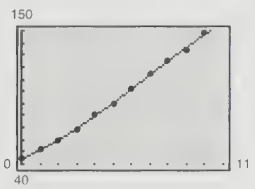
(b) Linear: $A = 10.53t + 39.9$
Quadratic: $A = 0.210t^2 + 8.43t + 43.0$
Cubic: $A = -0.0386t^3 + 0.789t^2 + 6.22t + 44.4$
Quartic: $A = 0.0062t^4 - 0.1633t^3 + 1.568t^2 + 4.66t + 44.9$

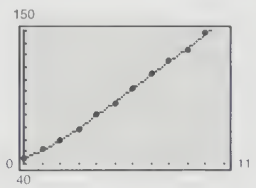
(c) Linear Quadratic





Cubic Quartic





Each model could be considered a good fit, but the cubic and quartic models appear to fit the data a little better than the linear and quadratic models.

(d) Linear: 2012 ($t \approx 12.36$)
Quadratic: 2011 ($t \approx 11.67$)
Cubic: 2012 ($t \approx 12.49$)
Quartic: 2011 ($t \approx 11.79$)

Answers will vary. Sample answer: The linear and cubic predictions are close to each other and the quadratic and quartic predictions are close to each other. The quadratic and quartic models give earlier predictions because their graphs rise faster in the years after 2010 than the graphs of the linear and cubic models.

Section 3.5 *(page 304)*

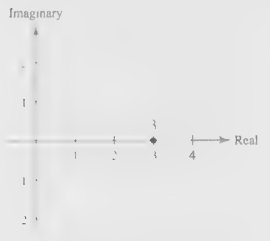
Skills Warm Up *(page 304)*

1. $2\sqrt{3}$ **2.** $10\sqrt{5}$ **3.** $\sqrt{5}$ **4.** $-6\sqrt{3}$

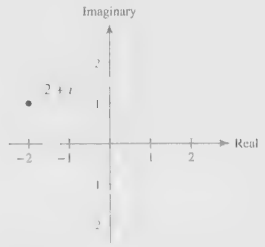
5. 12 **6.** 48 **7.** $\frac{\sqrt{7}}{7}$ **8.** $\sqrt{5}$

9. $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ **10.** $-1 \pm \sqrt{2}$

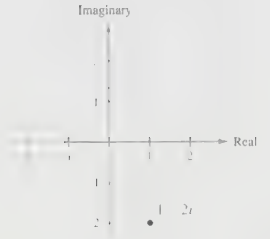
1. $i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, 1$
$i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i$, n is an integer.
3. $a = 7, b = 12$ **5.** $a = 4, b = -2$ **7.** $2 + i$
9. $5 + 6i$ **11.** $\frac{5}{3} + \frac{7}{6}i$ **13.** $1.3 + 9i$ **15.** 25
17. $5 + i$ **19.** $30 + 20i$ **21.** $-11 + 60i$ **23.** -14
25. 8 **27.** $9 + 4i; 9 - 4i$ **29.** $-3 - 2\sqrt{3}i; -3 + 2\sqrt{3}i$
31. $-21; -21$ **33.** $-1 - 6i; -1 + 6i$ **35.** $-5i; 5i$
37. $-3; -3$ **39.** $\frac{4}{5} - \frac{3}{5}i$ **41.** $1 + \frac{1}{2}i$ **43.** $5 - \frac{7}{2}i$
45. $\frac{1}{8}i$ **47.** $-\frac{44}{125} - \frac{8}{125}i$ **49.** $\frac{35}{29} + \frac{595}{29}i$
51. $-7 + 7\sqrt{2}i$ **53.** $-2\sqrt{6}$ **55.** -10
57. $(16 + 4\sqrt{3}) + (-16\sqrt{2} + 2\sqrt{6})i$
59. Error: $(3 - 2i)(3 + 2i) = 9 - 4i^2 = 9 + 4 = 13$
(not $9 - 4 = 5$)
61. $\pm 3i$ **63.** $1 \pm i$ **65.** $-2 \pm \frac{1}{2}i$ **67.** $-2 \pm i$
69. $\frac{1}{8} + \frac{\sqrt{11}}{8}i$
71. No. Sample answer: When the discriminant of the Quadratic Formula is negative, there are two complex imaginary solutions to the equation. Otherwise, the solutions are real.
73.



75.



77.



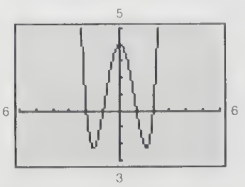
79. The complex number 0 is in the Mandelbrot Set because, for $c = 0$, the corresponding Mandelbrot sequence is 0, 0, 0, 0, 0, 0, which is bounded.
81. The complex number 1 is not in the Mandelbrot Set because, for $c = 1$, the corresponding Mandelbrot sequence is 1, 2, 5, 26, 677, 458,330, . . . , which is unbounded.
83. The complex number $\frac{1}{2}i$ is in the Mandelbrot Set because, for $c = \frac{1}{2}i$, the corresponding Mandelbrot sequence is $\frac{1}{2}i$, $-\frac{1}{4} + \frac{1}{2}i, -\frac{3}{16} + \frac{1}{4}i, -\frac{7}{256} + \frac{13}{32}i, -\frac{10,767}{65,536} + \frac{1957}{4096}i$, $-\frac{864,513,055}{4,294,967,296} + \frac{46,037,845}{134,217,728}i$, which is bounded.
85. False. If the complex number is real, it equals its conjugate.
87. Answers will vary.

Section 3.6 *(page 312)*

Skills Warm Up *(page 312)*

1. $4 - \sqrt{29}i, 4 + \sqrt{29}i$ **2.** $-5 - 12i, -5 + 12i$
3. $-1 + 4\sqrt{2}i, -1 - 4\sqrt{2}i$ **4.** $6 + \frac{1}{2}i, 6 - \frac{1}{2}i$
5. $-13 + 9i$ **6.** $12 + 16i$ **7.** $26 + 22i$
8. 29 **9.** i **10.** $-9 + 46i$

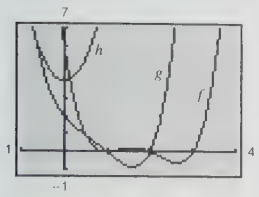
1. 1 **3.** 3 **5.** 4 **7.** $\pm 5i; (x + 5i)(x - 5i)$
9. $6 \pm \sqrt{10}; (x - 6 - \sqrt{10})(x - 6 + \sqrt{10})$
11. $2 \pm \sqrt{3}; (x - 2 - \sqrt{3})(x - 2 + \sqrt{3})$
13. $\pm 3, \pm 3i; (x - 3)(x + 3)(x - 3i)(x + 3i)$
15. $0, \pm\sqrt{5}i; x(x - \sqrt{5}i)(x + \sqrt{5}i)$
17. $-5, 8 \pm i; (x + 5)(x - 8 + i)(x - 8 - i)$
19. $2, 2 \pm i; (x - 2)(x - 2 + i)(x - 2 - i)$
21. $-5, 4 \pm 3i; (t + 5)(t - 4 + 3i)(t - 4 - 3i)$
23. $-10, -7 \pm 5i; (x + 10)(x + 7 - 5i)(x + 7 + 5i)$
25. $-5, -2 \pm \sqrt{3}i; (x + 5)(x + 2 - \sqrt{3}i)(x + 2 + \sqrt{3}i)$
27. $-\frac{3}{4}, 1 \pm \frac{1}{2}i; (4x + 3)(2x - 2 + i)(2x - 2 - i)$
29. $-\frac{1}{5}, 1 \pm \sqrt{5}i; (5x + 1)(x - 1 + \sqrt{5}i)(x - 1 - \sqrt{5}i)$
31. $2, \pm 2i; (x - 2)^2(x + 2i)(x - 2i)$
33. $\pm i, \pm 3i; (x + i)(x - i)(x + 3i)(x - 3i)$
35. $-4, 3, \pm i; (t + 4)^2(t - 3)(t + i)(t - i)$
37. Answers will vary. Sample answer: $x^3 - 2x^2 + x - 2$
39. Answers will vary. Sample answer: $x^3 - 5x^2 + 9x - 5$
41. Answers will vary.
Sample answer: $x^5 + 4x^4 + 13x^3 + 52x^2 + 36x + 144$
43. Answers will vary.
Sample answer: $x^4 + 8x^3 + 9x^2 - 10x + 100$
45. Answers will vary.
Sample answer: $3x^4 - 17x^3 + 25x^2 + 23x - 22$
47. (a) $(x^2 - 8)(x^2 + 1)$ (b) $(x - 2\sqrt{2})(x + 2\sqrt{2})(x^2 + 1)$
(c) $(x - 2\sqrt{2})(x + 2\sqrt{2})(x - i)(x + i)$
49. (a) $(x^2 - 2x + 3)(x^2 - 3x - 5)$
(b) $(x^2 - 2x + 3)\left(x - \frac{3 + \sqrt{29}}{2}\right)\left(x - \frac{3 - \sqrt{29}}{2}\right)$
(c) $(x - 1 + \sqrt{2}i)(x - 1 - \sqrt{2}i)$
$\left(x - \frac{3 + \sqrt{29}}{2}\right)\left(x - \frac{3 - \sqrt{29}}{2}\right)$
51. $\pm 5i, \frac{3}{2}$ **53.** $\pm 6i, 1$ **55.** $-3 \pm i, \frac{1}{4}$
57. $1, 2, 3 \pm \sqrt{2}i$ **59.** $\frac{3}{4}, \frac{1}{2} \pm \frac{\sqrt{5}}{2}i$
61. (a) 1, 2 (b) $-3 \pm \sqrt{2}i$
63. (a) 0.75 (b) $\frac{1 \pm \sqrt{5}i}{2}$
65. $\pm 1, \pm 2$. The x-intercepts occur at the solutions of the equation.



67. Answers will vary. Sample answer:
(a) $f(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$
Zeros: 1, 1, 2, 3
(b) $g(x) = x^4 - 3x^3 + 3x^2 - 3x + 2$
Zeros: $1, 2, \pm i$

(c) $h(x) = x^4 + 5x^2 + 4$

Zeros: $\pm i, \pm 2i$



Similarities: f, g, and h all rise to the left and rise to the right.

Differences: f, g, and h have different numbers of x-intercepts.

69. $-0.0001x^2 + 140x = 50{,}000{,}000$

$x = 700{,}000 \pm 100{,}000i$

The solutions are imaginary, so there is no quantity of units x or price p that will yield a revenue of \$50 million.

71. Imaginary zeros of polynomials with real coefficients occur in conjugate pairs. So, a fourth-degree polynomial with real coefficients cannot have exactly three imaginary zeros.

73. Imaginary zeros occur only in conjugate pairs. Because f has three zeros, one or all of them must be real numbers.

Section 3.7 *(page 322)*

Skills Warm Up *(page 322)*

1. $x(x - 4)$ **2.** $2x(x^2 - 3)$ **3.** $(x - 5)(x + 2)$

4. $(x - 5)(x - 2)$ **5.** $x(x + 1)(x + 3)$

6. $(x^2 - 2)(x - 4)$

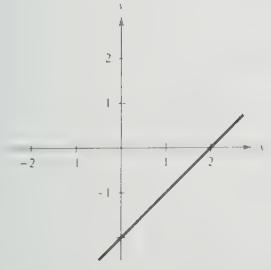
7.



8.



9.



10.



1. Domain: All $x \neq 2$

$f(x) \to -\infty$ as $x \to 2^-$

$f(x) \to \infty$ as $x \to 2^+$

3. Domain: All $x \neq -1$

$f(x) \to \infty$ as $x \to -1^-$

$f(x) \to -\infty$ as $x \to -1^+$

5. Domain: All $x \neq \pm 1$

$f(x) \to -\infty$ as $x \to -1^-$ or $x \to 1^-$

$f(x) \to \infty$ as $x \to -1^+$ or $x \to 1^+$

7. Horizontal asymptote: $y = 3$; Vertical asymptote: $x = -1$

9. Horizontal asymptote: $y = -1$; Vertical asymptote: $x = 5$

11. Horizontal asymptote: $y = 3$; Vertical asymptote: None

13. Horizontal asymptote: $y = 0$; Vertical asymptote: $x = -4$

15. f **16.** e **17.** a **18.** b **19.** c **20.** d

21. $g(x)$ shifts downward two units.

23. $g(x)$ is a reflection about the x-axis.

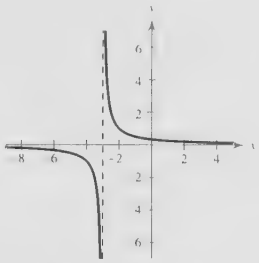
25. $g(x)$ shifts upward three units.

27. $g(x)$ is a reflection about the x-axis.

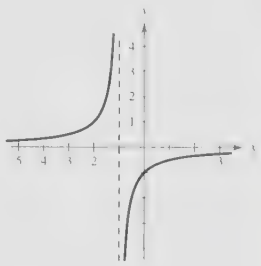
29. $g(x)$ shifts upward five units.

31. $g(x)$ is a reflection about the x-axis.

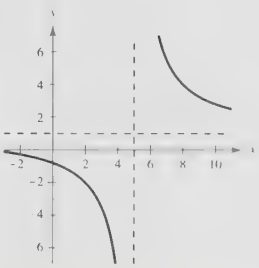
33.



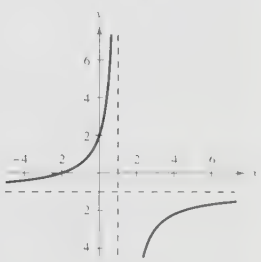
35.



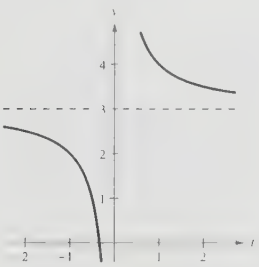
37.



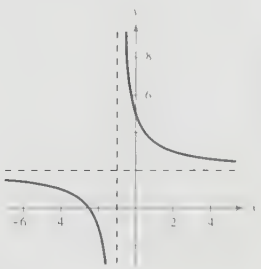
39.



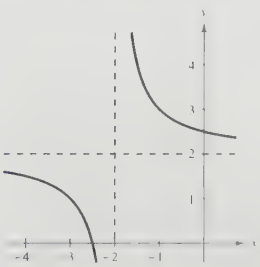
41.



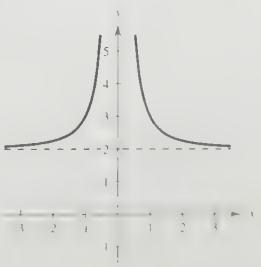
43.



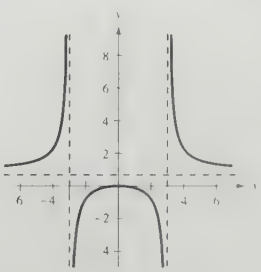
45.



47.



49.



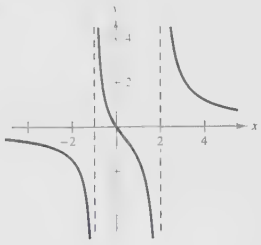
51.



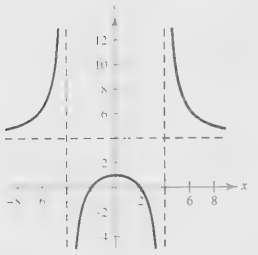
53.



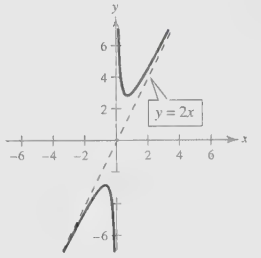
55.



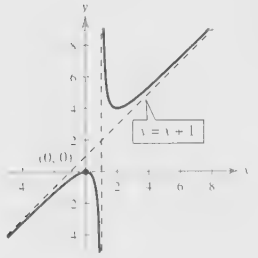
57.



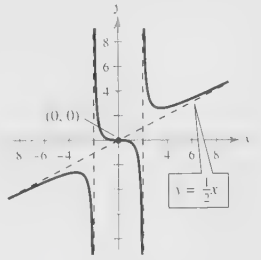
59.



61.



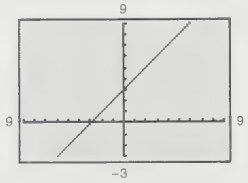
63.



65. Answers will vary. Sample answer: $f(x) = \dfrac{2x^2}{x^2 + 1}$

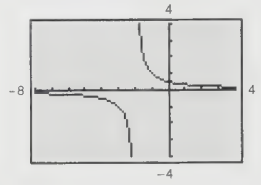
67. Answers will vary. Sample answer: $f(x) = \dfrac{x^3}{x^2 + x - 2}$

69.



$f(x)$ has a hole at $x = 3$.

71.



$f(x)$ has a hole at $x = 2$.

73. Exclude $x = 7$ from the domain of the new function.

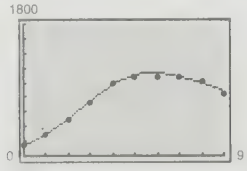
75. No. Given

$$f(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$$

if $n > m$, there is no horizontal asymptote, and n must be greater than m for a slant asymptote to occur.

77. (a) \$176 million (b) \$528 million (c) \$1584 million
(d) No. The model has a vertical asymptote at $p = 100$.

79. (a)



(b) 2012: $N \approx 624.5$
2015: $N \approx 459.7$
2020: $N \approx 308.5$
Answers will vary.

The model fits the data well.

(c) $N = 0$; Over time, the number of DVDs sold will decrease and approach 0.

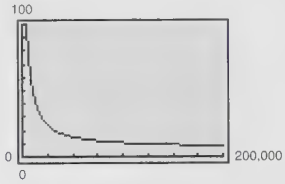
81. (a)

n	1	2	3	4	5
P	0.60	0.79	0.86	0.90	0.92

n	6	7	8	9	10
P	0.93	0.94	0.95	0.95	0.96

(b) 100%

83. (a)



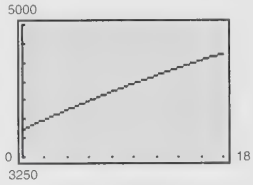
(b)

x	10,000	100,000	1,000,000	10,000,000
$\overline{C}$	\$51	\$10.50	\$6.45	\$6.05

Eventually, the average recycling cost per pound will approach the horizontal asymptote of \$6.

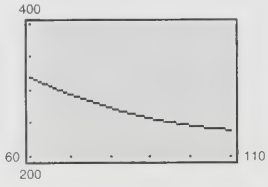
85. (a) $E = \dfrac{27{,}904t + 906{,}950}{3.023t + 250.90}$

(b)



(c) About 4615.6 kWh/person

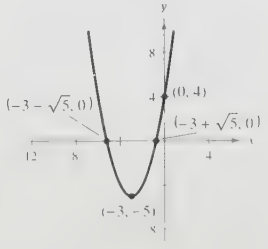
87. (a)



(b) 2014: 238.54 seconds; 2018: 238.04 seconds
(c) No; Answers will vary.

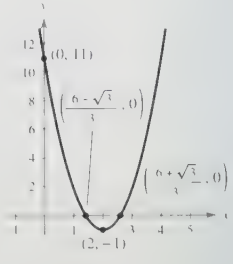
Review Exercises *(page 330)*

1.



Vertex: $(-3, -5)$
Intercepts:
$(0, 4), \left(-3 \pm \sqrt{5}, 0\right)$

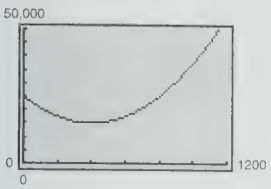
3. $h(x) = 3(x - 2)^2 - 1$



Vertex: $(2, -1)$
Intercepts:
$(0, 11), \left(\dfrac{6 \pm \sqrt{3}}{3}, 0\right)$

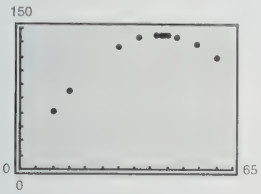
5. $f(x) = \frac{7}{9}(x + 5)^2 - 1$

7. $A(x) = 250x - x^2$; 125 feet × 125 feet

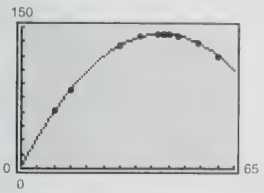
9. (a) 

(b) About 385 units

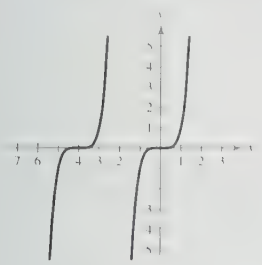
(c) Write the equation of the quadratic function in standard form. The vertex is the minimum cost.

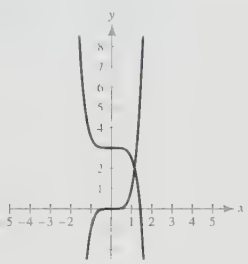
11. (a) 

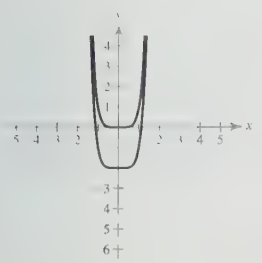
(b) $d(x) = -0.077x^2 + 6.59x + 2.4$

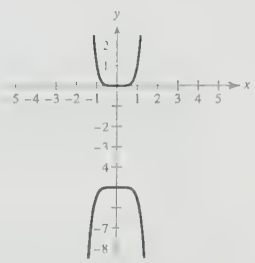
(c) 

(d) (42.8, 143.4); The vertex gives the angle 42.8°, which results in the greatest distance of 143.4 meters.

13. 

15. 

17. 

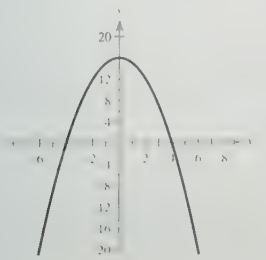
19. 

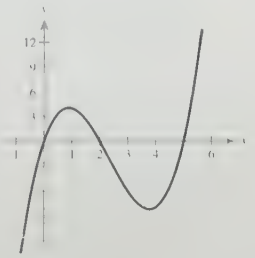
21. Falls to the left
Rises to the right

23. Falls to the left
Falls to the right

25. ± 4

27. 0, 2, 5





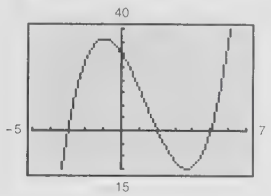
29. $x^2 - 3x + 1 - \dfrac{1}{2x + 1}$

31. $x^2 - 3x + 3$

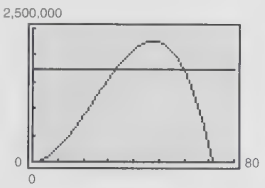
33. (a) -10 (b) 11

35. (a) Proof (b) $x - 2$

(c) $(x - 5)(x + 3)(x - 2)$ (d) $-3, 2, 5$

(e) 

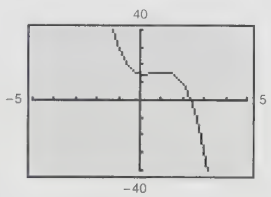
37. $x^2 + 11x + 24$

39. (a) 

(b) Answers will vary.

$\approx$ \$325,167

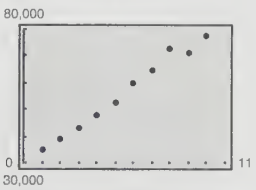
41. $\pm 1, \pm 3, \pm 5, \pm 15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}, \pm\frac{15}{4}$



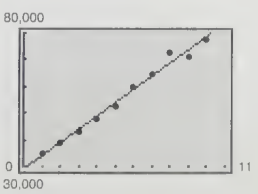
From the graph: $x \approx 2.357$, so the zero is not rational.

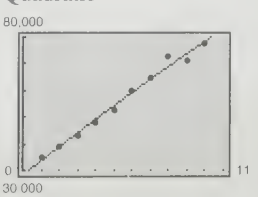
43. $-3, -1, 2$ **45.** $\pm 2, \pm\sqrt{5}$ **47.** $-\frac{1}{3}, \frac{3}{2}, 2$

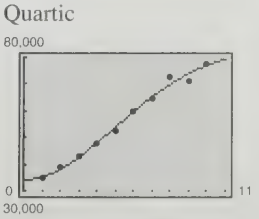
49. $-1, 2$ **51.** $x \approx -2.3$ **53.** $-1.321, -0.283, 1.604$

55. (a) 

(b) Linear: $R = 4996.7t + 28{,}927$
Quadratic: $R = -36.14t^2 + 5394.3t + 28{,}132$
Quartic: $R = 3.4725t^4 - 134.788t^3 + 1486.42t^2 - 577.4t + 34{,}333$

(c) Linear 

Quadratic 

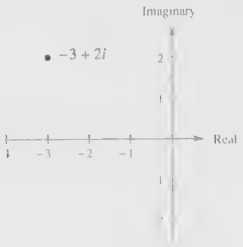
Quartic 

Each model could be considered a good fit for the data.

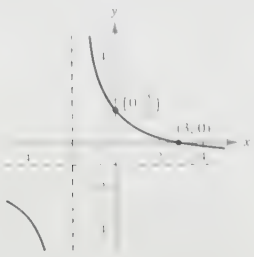
CHAPTER 3

(d) Linear: 2011 (t 11.6)
Quadratic: 2011 ($t \approx 11.9$)
Quartic: 2018 (t 18.8)
Answers will vary. Sample answer: The linear and quadratic predictions are around 2011, while the quartic model levels off for a few years before increasing.

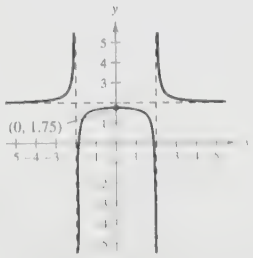
57. $4\sqrt{2}i; -4\sqrt{2}i$ **59.** $-3 + 4i; -3 - 4i$ **61.** $5 + 2i$
63. 89 **65.** $-10 - 8i$ **67.** $-7 + 24i$ **69.** 10
71. $3 - 2i$ **73.** $-3 - 4i$
75. $\frac{1 \pm \sqrt{23}i}{4}$ **77.** $\frac{-11 \pm \sqrt{73}}{8}$
79.



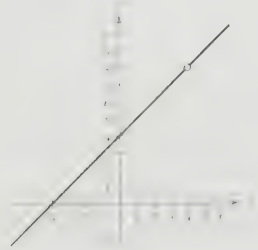
81. $\pm 3, \pm 3i; (x - 3)(x + 3)(x - 3i)(x + 3i)$
83. $-5, \pm\sqrt{3}i; (t + 5)(t - \sqrt{3}i)(t + \sqrt{3}i)$
85. $2, \pm\frac{3i}{2}; (x - 2)(2x - 3i)(2x + 3i)$
87. Answers will vary. Sample answer: $x^3 - x^2 + 9x - 9$
89. (a) $(x^2 + 8)(x^2 - 3)$ (b) $(x^2 + 8)(x - \sqrt{3})(x + \sqrt{3})$
(c) $(x + 2\sqrt{2}i)(x - 2\sqrt{2}i)(x - \sqrt{3})(x + \sqrt{3})$
91. $\pm 4i, \frac{1}{4}$ **93.** $-1 \pm 3i, -1, -4$
95. Domain: All $x \neq -2$
Vertical asymptote: $x = -2$
Horizontal asymptote: $y = 0$
97. Domain: All $x \neq \pm 3$
Vertical asymptotes: $x = -3, x = 3$
Horizontal asymptote: $y = 2$
99.



101.



103. None

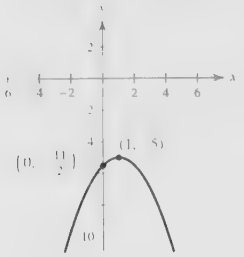


105. (a) 430,769 fish; 662,500 fish; 1,024,000 fish
(b) 1,666,667 fish

107. (a) $35,000 (b) $157,500

Test Yourself *(page 334)*

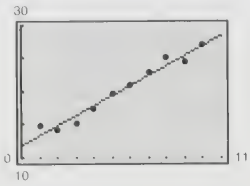
1.

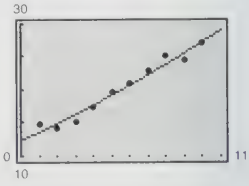


Vertex: $(1, -5)$
Intercept: $\left(0, -\frac{11}{2}\right)$
2. (a) Falls to the left (b) Rises to the left
Rises to the right Rises to the right
3. $x^2 + 7x + 12$
4. $\pm 1, \pm 3, \pm 9, \pm 27, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm\frac{3}{2}, \pm\frac{9}{4}, \pm\frac{9}{2}, \pm\frac{27}{4}, \pm\frac{27}{2}$
$f\left(\frac{3}{2}\right) = 0$
$f\left(-\frac{3}{2}\right) = 0$
$f(x) = (x - \frac{3}{2})(x + \frac{3}{2})(x - 1)(x - 3)$
5. (a)

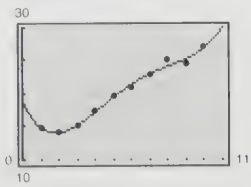


(b) Linear: $S = 1.517t + 11.81$
Quadratic: $S = 0.024t^2 + 1.253t + 12.34$
Quartic:
$S = 0.010739t^4 - 0.26789t^3 + 2.2749t^2 - 5.879t + 18.74$
(c) Linear Quadratic





Quartic



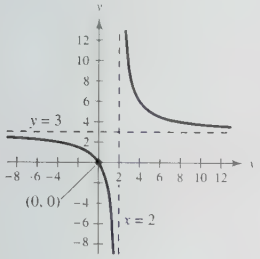
Each model could be considered a good fit for the data.
(d) Linear: 2015 ($t \approx 15.29$)
Quadratic: 2014 ($t \approx 14.21$)
Quartic: 2011 ($t \approx 11.9$)
Answers will vary. Sample answer: All three models give predictions that reflect the increasing data. The quartic model increases most rapidly, while the linear and quadratic models increase at slower rates.
6. $16 - 2i$ **7.** $7 - 10i$ **8.** $27 - 4\sqrt{3}i$ **9.** $23 - 14i$

10. i **11.** $x = \dfrac{-5 \pm \sqrt{3}i}{2}$ **12.** $x = \dfrac{5 \pm 3\sqrt{7}i}{4}$

13. Answers will vary. Sample answer:
$x^4 - 7x^3 + 19x^2 - 63x + 90$

14. $\pm\sqrt{5}i, -2$

15.



Domain: All $x \neq 2$

Chapter 4

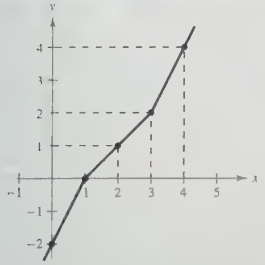
Section 4.1 *(page 343)*

Skills Warm Up *(page 343)*

1. All real numbers **2.** $[-1, \infty)$
3. All real numbers except $x = 0, 2$
4. All real numbers except $x = -\frac{5}{3}$ **5.** x **6.** x
7. x **8.** x **9.** $x = \dfrac{3}{2}y + 3$ **10.** $x = \dfrac{y^3}{2} + 2$

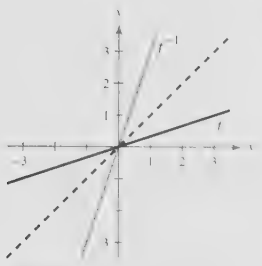
1. $f^{-1} = \{(4, 1), (5, 2), (6, 3), (7, 4)\}$
3. $f^{-1} = \{(1, -1), (2, -2), (3, -3), (4, -4)\}$
5. $f^{-1}(x) = \frac{1}{2}x$ **7.** $f^{-1}(x) = x + 5$
9. $f(g(x)) = 5\left(\dfrac{x-1}{5}\right) + 1 = x$; $g(f(x)) = \dfrac{(5x+1)-1}{5} = x$
11. $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$; $g(f(x)) = \sqrt[3]{x^3} = x$
13. $f(g(x)) = \sqrt{(x^2+4)-4} = x$; $g(f(x)) = \left(\sqrt{x-4}\right)^2 + 4 = x$
15. $f(g(x)) = 1 - \left(\sqrt[3]{1-x}\right)^3 = x$; $g(f(x)) = \sqrt[3]{1-(1-x^3)} = x$
17.

x	0	1	2	3	4
$f^{-1}(x)$	-2	0	1	2	4



19. f doesn't have an inverse function. **21.** $g^{-1}(x) = 8x$
23. p doesn't have an inverse function.
25. $f^{-1}(x) = \sqrt{x} - 3,\ x \geq 0$
27. $h^{-1}(x) = \dfrac{1}{x}$ **29.** $f^{-1}(x) = \dfrac{x^2-3}{2},\ x \geq 0$
31. g doesn't have an inverse function.
33. $f^{-1}(x) = -\sqrt{25 - x},\ x \leq 25$

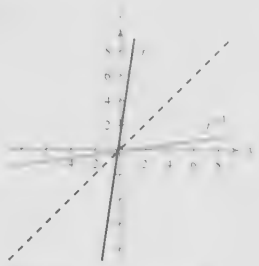
35.



x	-3	0	3
$f(x)$	-1	0	1

x	-1	0	1
$f^{-1}(x)$	-3	0	3

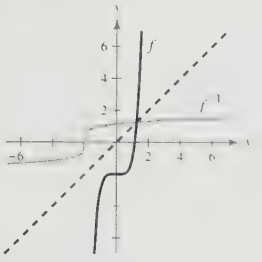
37.



x	-1	0	1
$f(x)$	-6	1	8

x	-6	1	8
$f^{-1}(x)$	-1	0	1

39.

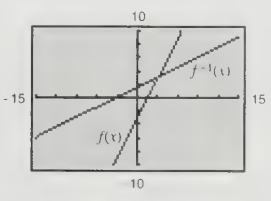


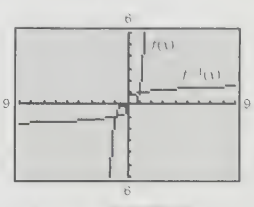
x	-1	0	1
$f(x)$	-3	-2	-1

x	-3	-2	-1
$f^{-1}(x)$	-1	0	1

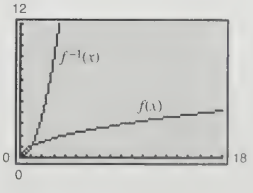
41. Error: f^{-1} does not mean to take the reciprocal of $f(x)$.
$f^{-1}(x) = \dfrac{x^2+5}{2}$

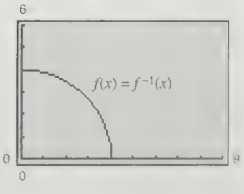
43. $f^{-1}(x) = \dfrac{x+3}{2}$ **45.** $f^{-1}(x) = \sqrt[5]{x}$



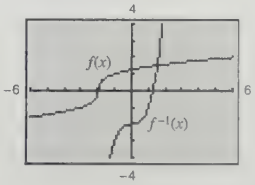


47. $f^{-1}(x) = x^2$, $x \geq 0$ **49.** $f^{-1}(x) = \sqrt{16 - x^2}$, $0 \leq x \leq 4$



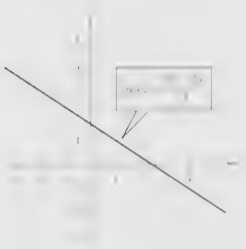


51. $f^{-1}(x) = x^3 - 2$

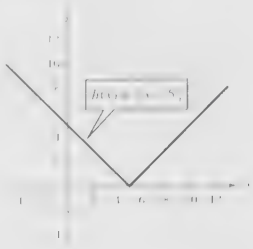


53. Because f is one-to-one, f has an inverse function.
55. f doesn't have an inverse function because two x-values share the same y-value.

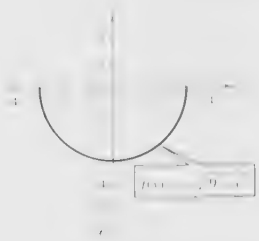
57. g has an inverse function.



59. h doesn't have an inverse function.



61. f doesn't have an inverse function.



63. $(g^{-1} \circ f^{-1})(x) = \dfrac{x+2}{2}$ **65.** $(f \circ g)^{-1}(x) = \dfrac{x+2}{2}$

67. 32 **69.** 600

71. $P^{-1}(x) = \dfrac{x + 5736}{47}$; $P^{-1}(x)$ computes the number of units that must be sold to obtain a profit of x.
Domain of P: $[0, \infty)$
Domain of P^{-1}: $[-5736, \infty)$

73. (a)



(b) $y = 0.263t + 5.12$

(c) $y^{-1} = \dfrac{t - 5.12}{0.263}$; y^{-1} represents the year in which the average price is t dollars.

(d) 2014 ($y^{-1} \approx 14.75$)

75. $f^{-1}(x) = \sqrt[3]{\dfrac{x^2 + 0.021}{0.0209}}$; \$3.49

Section 4.2 *(page 354)*

Skills Warm Up *(page 354)*

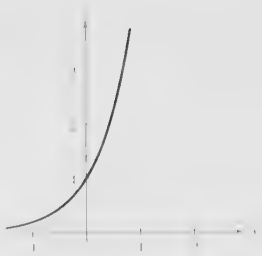
1. 5^x **2.** 3^{2x} **3.** 4^{3x} **4.** 10^x **5.** 4^{2x}

6. 4^{-x} **7.** $\left(\dfrac{3}{2}\right)^x$ **8.** 4^{3x} **9.** $\dfrac{1}{2^x}$ **10.** $\dfrac{3^x}{5^x}$

11. $2^x \cdot$ **12.** 3^x

1. 3.463 **3.** 94.818 **5.** 0.079 **7.** g **8.** e
9. b **10.** h **11.** d **12.** a **13.** f **14.** c

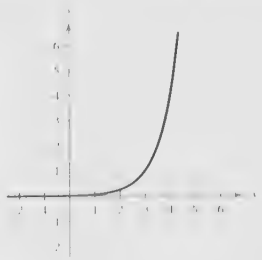
15.



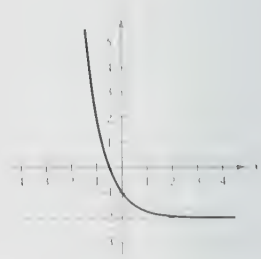
17.



19.



21.



23.

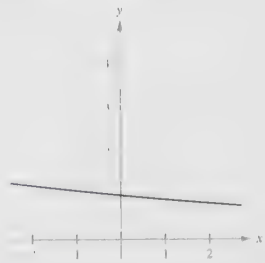


25.

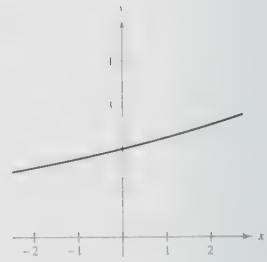


27. 54.598 **29.** 0.067

31.



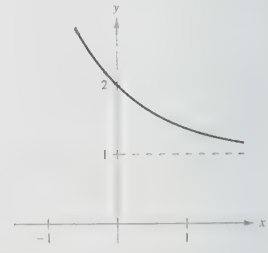
33.



35.



37.



39. Yes. For the natural exponential function, $f(x) = e^x$, the base is $a = e$.

41.

n	1	2	4
A	\$7346.64	\$7401.22	\$7429.74

n	12	365	Continuous
A	\$7449.23	\$7458.80	\$7459.12

43.

n	1	2	4
A	\$6633.24	\$6712.66	\$6753.71

n	12	365	Continuous
A	\$6781.60	\$6795.24	\$6795.70

45. The account paying 5% interest compounded quarterly earns more money. Even though the interest is compounded less frequently, the higher interest rate yields a higher return.

47. \$7424.70 **49.** \$12,434.43

51.

t	1	10	20
P	\$90,521.24	\$36,940.70	\$13,646.15

t	30	40	50
P	\$5040.98	\$1862.17	\$687.90

53.

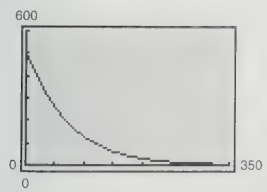
t	1	10	20
P	\$91,393.12	\$40,656.97	\$16,529.89

t	30	40	50
P	\$6720.55	\$2732.37	\$1110.90

55. \$19,691.17 **57.** \$147,683.76

59. \$20,700.76 **61.** \$155,255.66

63. (a) \$182.91 (b) \$29.58

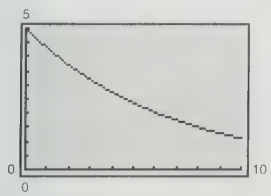
(c) 

(d) $\approx$ 117 books

65. (a) 100 (b) $\approx$ 110 (c) $\approx$ 121 (d) $\approx$ 158

67. (a) 5907 (b) 6237 (c) 6767 (d) 7753

69. $\approx$ 30.42 kilograms

71. (a) 

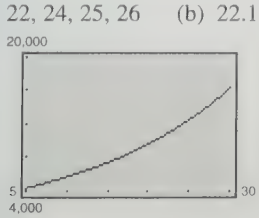
(b) $\approx$ 1.11 pounds

(c) On the graph, when $P = 2.5$, $t \approx 4.6$ months.

73. (a) 9000 screens, 39,000 screens

(b) 9059 screens, 39,186 screens

75. (a) 22, 24, 25, 26 (b) 22.1, 24.0, 25.1, 25.9

77. (a) 

(b) \$5682.8 billion, \$9624.6 billion, \$15,463.8 billion

79. Answers will vary.

Section 4.3 *(page 365)*

Skills Warm Up *(page 365)*

1. 3 **2.** 0 **3.** -1 **4.** 1

5. 7.389 **6.** 0.368

7. The graph of g is the graph of f shifted two units to the left.

8. The graph of g is the graph of f reflected about the x-axis.

9. The graph of g is the graph of f shifted downward one unit.

10. The graph of g is the graph of f reflected about the y-axis.

1. c **2.** f **3.** b **4.** d **5.** a **6.** e

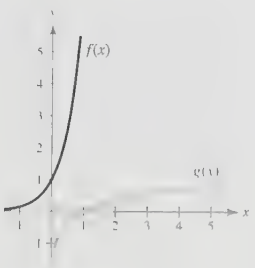
7. $\log_4 256 = 4$ **9.** $\log_{81} 3 = \frac{1}{4}$ **11.** $\log_6 \frac{1}{36} = -2$

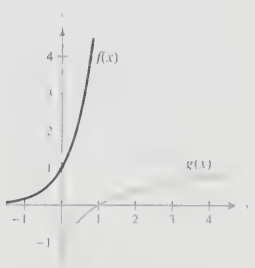
13. $4^2 = 16$ **15.** $2^{-1} = \frac{1}{2}$ **17.** $5^{-1} = 0.2$

19. $27^{1/3} = 3$ **21.** 2 **23.** -4 **25.** $\frac{1}{3}$ **27.** 1

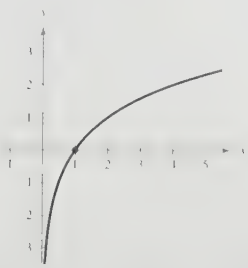
29. -4 **31.** 2.538 **33.** -0.097 **35.** 0.452 **37.** 1

39. -1 **41.** 3 **43.** 7

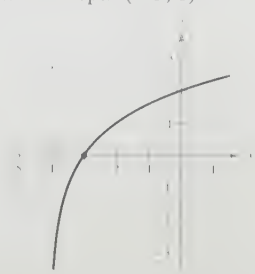
45. 

47. 

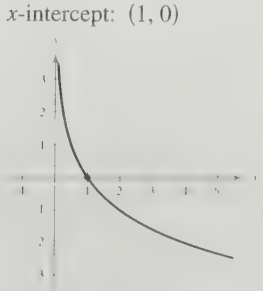
49. Domain: $(0, \infty)$
Asymptote: $x = 0$
x-intercept: $(1, 0)$



51. Domain: $(-4, \infty)$
Asymptote: $x = -4$
x-intercept: $(-3, 0)$



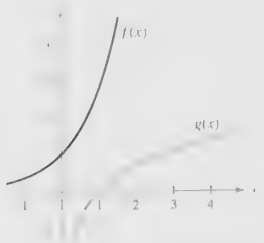
53. Domain: $(0, \infty)$
Asymptote: $x = 0$
x-intercept: $(1, 0)$



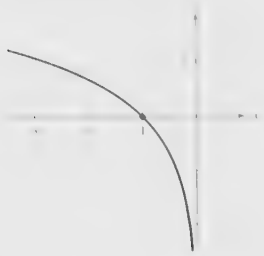
55. $\ln e = 1$

CHAPTER 4

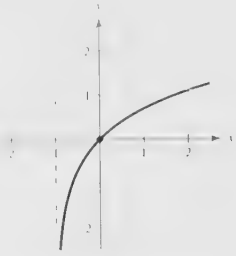
57. $\ln 4 = x$ **59.** $e^1 = e$ **61.** $e^0 = 1$ **63.** 1 **65.** -4
67. 1.946 **69.** 2.913 **71.** 0.896 **73.** 4 **75.** ± 5
77.



79. Domain: $(-\infty, 0)$
Asymptote: $x = 0$
x-intercept: $(-1, 0)$

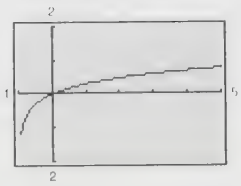


81. Domain: $(-1, \infty)$
Asymptote: $x = -1$
x-intercept: $(0, 0)$

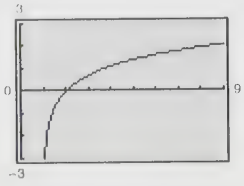


83. c **84.** d **85.** a **86.** b

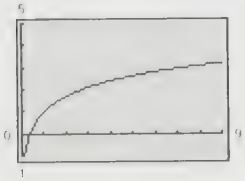
87.



89.

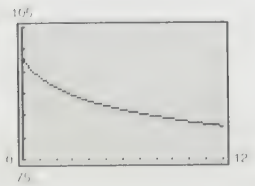


91.



93. $t \approx 26.1$ years

95.



The domain $0 \le t \le 12$ covers the period of 12 months or 1 year. The scores range from the average score on the original exam to the average score after 12 months, so the range is $82.40 \le f(t) \le 98$

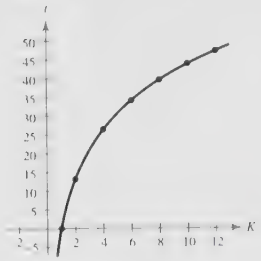
97. (a) 87 (b) ≈ 73.5 (c) In 11 months ($t \approx 10.6$)

99. (a) 30 years; 20 years
(b) \$396,234; \$301,123
(c) \$246,234; \$151,123
(d) $x = 1000$; As the monthly payment approaches \$1000, the length of time it takes to pay off the loan increases without bound.

101. (a)

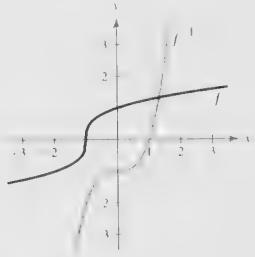
K	1	2	4	6	8	10	12
t	0	13.2	26.4	34.1	39.6	43.9	47.3

(b)

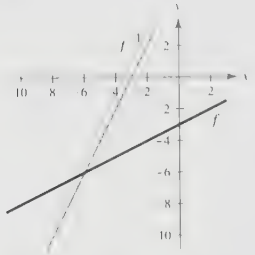


Quiz Yourself *(page 368)*

1. f does not have an inverse function.
2. Yes; $f^{-1}(x) = x^3 - 1$



3. (a) $f^{-1}(x) = 2(x + 3)$
(b)



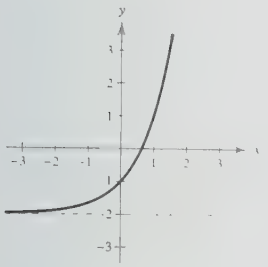
(c) $f^{-1}(f(x)) = 2\left(\frac{1}{2}x - 3 + 3\right) = x$
$f(f^{-1}(x)) = \frac{1}{2}[2(x + 3)] - 3 = x$

4. (a) $f^{-1}(x) = x^2 - 1,\ x \ge 0$
(b)

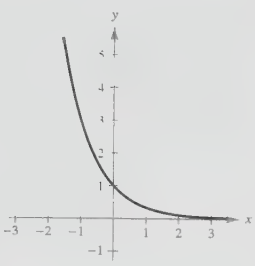


(c) $f^{-1}(f(x)) = \left(\sqrt{x + 1}\right)^2 - 1 = x$
$f(f^{-1}(x)) = \sqrt{x^2 - 1 + 1} = x,\ x \ge 0$

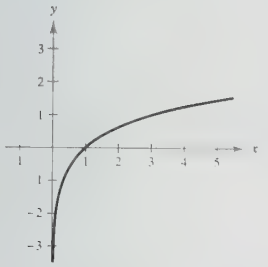
5.



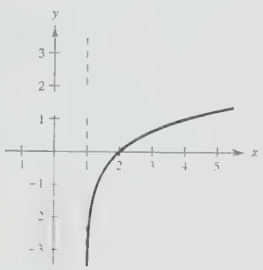
6.



7.



8.



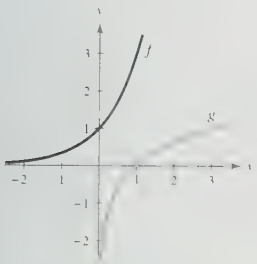
9. (a) Monthly: \$15,085.04 (b) Continuously: \$15,098.34

10. (a) 2006: 9.49 million; 2008: 10.26 million
(b) 2014: 12.99 million; 2015: 13.51 million

11. (a) 100 (b) ≈ 364 (c) ≈ 1326

12. \$108.54 **13.** 2 **14.** 4 **15.** -2 **16.** 0

17.



Domain of $f(x)$: $(-\infty, \infty)$
Domain of $g(x)$: $(0, \infty)$
The graphs are reflections of each other about the line $y = x$.

Section 4.4 *(page 374)*

Skills Warm Up *(page 374)*

1. 2 **2.** -5 **3.** -2 **4.** -3 **5.** e^5
6. $\frac{1}{e}$ **7.** e^6 **8.** 1 **9.** $y = x^{-2}$
10. $y = x^{1/2}$ **11.** $4^3 = 64$ **12.** $16^{1/2} = 4$

1. $\frac{\log_{10} 8}{\log_{10} 5} \approx 1.292$ **3.** $\frac{\log_{10} 30}{\log_{10} 4} \approx 2.453$

5. $\frac{\ln 8}{\ln 5} \approx 1.292$ **7.** $\frac{\ln 14}{\ln 6} \approx 1.473$ **9.** 2.585

11. 1.079 **13.** 2.633 **15.** -0.683 **17.** -1.661

19. $\ln 5 + \ln 2$ **21.** $\ln 5 - \ln 2$ **23.** $2(\ln 5 + \ln 2)$

25. $-\log_{10} \frac{11}{8} = \log_{10}\left(\frac{11}{8}\right)^{-1} = \log_{10} \frac{8}{11}$

27. $-2 \ln \frac{e}{7} = \ln\left(\frac{e}{7}\right)^{-2} = \ln\left(\frac{7}{e}\right)^{2} = 2(\ln 7 - \ln e) = 2 \ln 7 - 2$

29. 1.1833 **31.** -0.2084 **33.** 1.0686 **35.** 0.1781

37. 1.8957 **39.** -2.7124 **41.** 0.5708

43. $-1 - \log_9 2$ **45.** $\frac{1}{2} + \frac{1}{2} \log_7 10$

47. $-3 - \log_5 2$ **49.** $6 + \ln 5$

51. The change-of-base formula was used incorrectly. The formula is $\log_a x = \frac{\ln x}{\ln a}$. So, $\log_2 5 = \frac{\ln 5}{\ln 2}$.

53. The Quotient Property was used incorrectly. The Quotient Property is $\ln \frac{u}{v} = \ln u - \ln v$. So, $\ln 8 - \ln 2 = \ln \frac{8}{2} = \ln 4$.

55. $6 + 5 \log_2 3$ **57.** $\log_3 4 + \log_3 n$ **59.** $\log_5 x - 2$

61. $4 \log_2 x$ **63.** $\frac{1}{2} \ln z$ **65.** $\ln x + \ln y + \ln z$

67. $\frac{1}{2} \ln(a - 1)$ **69.** $2 \ln(z - 1) - \ln z$

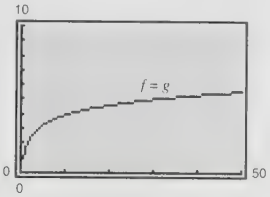
71. $\ln y + \ln z - \frac{1}{3} \ln(z + 3)$ **73.** $\ln 5 + \frac{1}{3}(\ln x - \ln y)$

75. $\frac{3}{4} \ln x + \frac{1}{4} \ln(x^2 + 3)$ **77.** $\log_3 5x$ **79.** $\log_4 \frac{8}{x}$

81. $\log_{10}(x + 4)^2$ **83.** $\ln \frac{1}{216x}$ **85.** $\ln \frac{\sqrt[3]{5x}}{x + 1}$

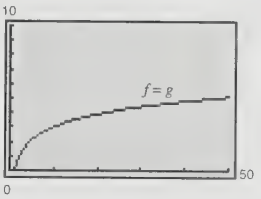
87. $\log_8 \frac{x - 2}{x + 2}$ **89.** $\ln \frac{144}{\sqrt{x^2 + 1}}$ **91.** $\ln \frac{x^2}{(x + 2)(x - 2)}$

93.



The two graphs are the same. The property is $\log_a(uv) = \log_a u + \log_a v$.

95.



The two graphs are the same. The graphs demonstrate the Power Property for logarithms.

97. $\frac{1}{3}$ **99.** $-\frac{1}{2}$ **101.** -3 **103.** $\ln y = \frac{1}{4} \ln x$

105. $\ln y = \frac{2}{3} \ln x + \ln 0.070$ **107.** ≈ 26 decibels

109. (a) $a^m a^n = a^{m+n}$ (b) $\frac{a^m}{a^n} = a^{m-n}$ (c) $(a^m)^n = a^{mn}$

111. Let $\log_a u = x$ and $\log_a v = y$.
$a^x = u$ and $a^y = v$
$\frac{u}{v} = \frac{a^x}{a^y} = a^{x-y}$
$\log_a a^{x-y} = x - y$
$\log_a \frac{u}{v} = \log_a u - \log_a v$

Section 4.5 *(page 384)*

Skills Warm Up *(page 384)*

1. $\frac{\ln 3}{\ln 2}$ **2.** $1 + \frac{2}{\ln 4}$ **3.** $\frac{e}{2}$ **4.** $2e$
5. $2 \pm i$ **6.** $\frac{1}{2}, 1$ **7.** x **8.** $2x$
9. $2x$ **10.** $-x^2$

1. 2 **3.** -2 **5.** 4 **7.** 64 **9.** $\frac{2}{5}$ **11.** x^2

13. $x + 1$ **15.** $x^3 - 7$ **17.** $8 + x^3$ **19.** $x + 5$

21. x^2 **23.** $\ln 3 \approx 1.099$ **25.** $\log_3 8 \approx 1.893$

27. $\ln 28 \approx 3.332$ **29.** $\frac{1}{2} \log_3 80 \approx 1.994$ **31.** 2

CHAPTER 4

33. $\log_3 28 + 1 \approx 4.033$ **35.** $3 - \log_2 565 \approx -6.142$

37. $\frac{1}{3}\log_{10}\frac{3}{2} \approx 0.059$ **39.** $\log_5 7 + 1 \approx 2.209$

41. $\frac{1}{3}\ln 12 \approx 0.828$ **43.** $\ln\frac{5}{3} \approx 0.511$

45. $\ln\frac{1}{2} \approx -0.693$ **47.** $\frac{1}{3} + \frac{1}{3}\log_2\frac{8}{3} \approx 0.805$

49. $\ln 6 \approx 1.792$, $\ln 2 \approx 0.693$ **51.** $\ln 4 \approx 1.386$

53. $2 \ln 75 \approx 8.635$ **55.** $\frac{1}{2}\ln 1498 \approx 3.656$

57. $\dfrac{\ln 4}{365 \ln\left(1 + \dfrac{0.065}{365}\right)} \approx 21.330$

59. $\dfrac{\ln 2}{12 \ln\left(1 + \dfrac{0.10}{12}\right)} \approx 6.960$ **61.** 10,000

63. $e^{-3} \approx 0.050$ **65.** $\dfrac{e^{2.4}}{2} \approx 5.512$ **67.** 5,000,000

69. 26 **71.** $\frac{1}{5}e^{10/3} \approx 5.606$ **73.** $e^2 - 2 \approx 5.389$

75. $e^{-2/3} \approx 0.513$ **77.** No solution **79.** No solution

81. 7 **83.** $\dfrac{-1 + \sqrt{17}}{2} \approx 1.562$ **85.** 2

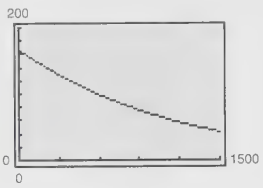
87. $y = 2x + 1, x > -\dfrac{1}{2}$ **89.** $y = \dfrac{(x-1)^2}{x+2}, x \geq 1$

91. 2.807 **93.** 20.086 **95.** $\approx$ 9.56 years

97. $\approx$ 13.32 years **99.** 30 months

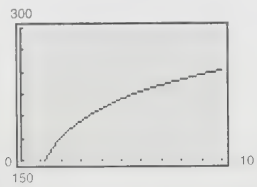
101. (a) $\approx$ 210 coin sets (b) $\approx$ 588 coin sets

(c)



103. (a) $\approx$ 29.3 years (b) $\approx$ 39.8 years

105. (a)

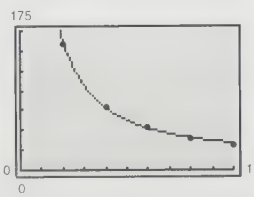


(b) and (c) 2003

107. (a)

x	y
0.2	162.6
0.4	78.5
0.6	52.5
0.8	40.5
1.0	33.9

(b)



The model is a good fit for the data.

(c) 1.197 meters

(d) No. To reduce the g's to fewer than 23 requires a crumple zone of more than 2.27 meters, a length that exceeds the front width of most cars.

109. Yes. See Exercise 79.

111. $\log_b uv = \log_b u + \log_b v$

True by the Product Property in Section 4.4.

113. $\log_b(u - v) = \log_b u - \log_b v$

False.

$1.95 \approx \log(100 - 10) \neq \log 100 - \log 10 = 1$

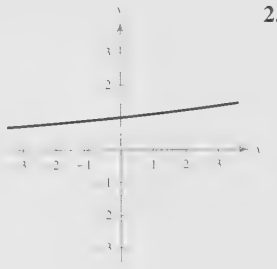
115. Yes. Time to double: $t = \dfrac{\ln 2}{r}$.

Time to quadruple: $t = \dfrac{\ln 4}{r} = 2\left(\dfrac{\ln 2}{r}\right)$

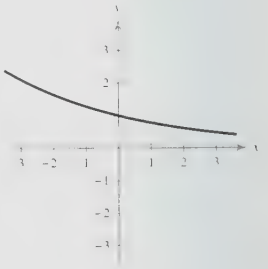
Section 4.6 *(page 395)*

Skills Warm Up *(page 395)*

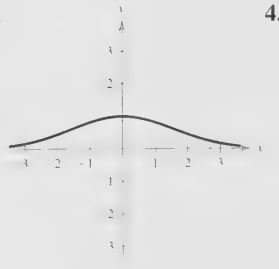
1.



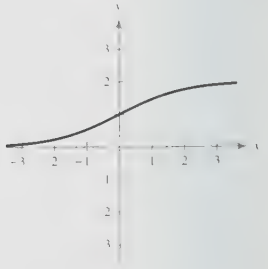
2.



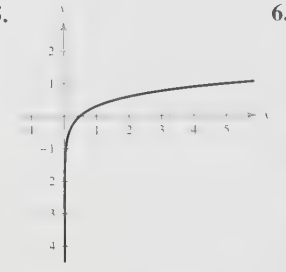
3.



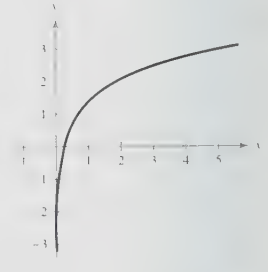
4.



5.



6.

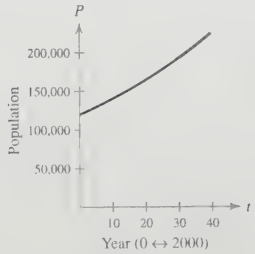


7. $\frac{1}{2}\ln\frac{7}{3}$ **8.** $\frac{1}{5}e^{7/2}$ **9.** $\approx$ 34.539 **10.** $\approx$ 3.695

1. c **2.** e **3.** b **4.** a **5.** d **6.** f

7. Exponential growth **9.** Exponential decay

11.



2025

13. $C = 1, k = \frac{1}{4}\ln 10$ **15.** $C = 1, k = \frac{1}{4}\ln\frac{1}{4}$

	Initial Investment	Annual % Rate	Time to Double	Amount After 10 Years
17.	$5000	7%	9.90 years	$10,068.76
19.	$500	6.93%	10 years	$1000.00
21.	$1000	8.25%	8.40 years	$2281.88
23.	$6392.79	11%	6.30 years	$19,205.00
25.	$5000	8%	8.66 years	$11,127.70

	Isotope	Half-Life (Years)	Initial Quantity	Amount After 1000 Years
27.	^{226}Ra	1599	4 g	2.59 g
29.	^{14}C	5715	3.95 g	3.5 g
31.	^{239}Pu	24,100	1.65 g	1.6 g

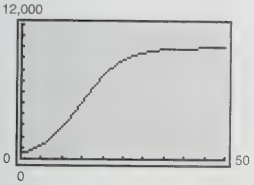
33. $k = -\frac{1}{20}\ln\frac{3}{5}$; ≈36,671 people

35. ≈ 12.36 hours to double
≈ 19.59 hours to triple

37. 12,180 years **39.** ≈ 9.92%

41. (a) $N = 40(1 - e^{-0.049t})$ (b) ≈ 42 days **43.** 100

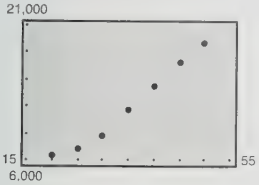
45. (a)



(b) ≈ 1252 fish
(c) ≈ 7.8 months

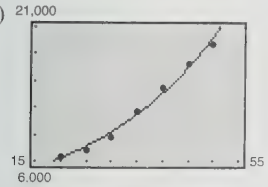
47. No, because the function is always increasing.

49. (a)



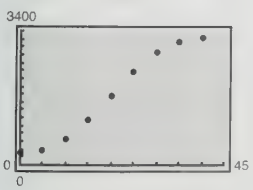
(b) $P = 2922.32(1.0391)^t$; $P = 2922.32e^{0.0384t}$

(c)



(d) 2022: ≈6,801,736 people
2042: ≈14,660,808 people

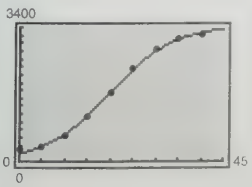
51. (a)



(b) Logistic growth model; The data points seem to be leveling off in the later years as though approaching an asymptote.

(c) $P = \dfrac{3455}{1 + 16.51e^{-0.1415t}}$

(d)



(e) 3407

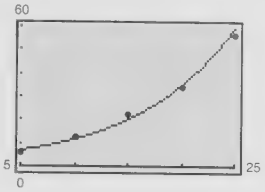
The model is a good fit.

53. (a) ≈ 7.906 (b) ≈ 7.684

55. (a) 20 decibels (b) 70 decibels

57. ≈ 1.585×10^{-6} **59.** ≈ 31,623 **61.** 3:00 A.M.

63.



The data fit an exponential model.
Exponential model: $n = 3.9405e^{0.1086t}$
A logistic growth model would be more appropriate for this data because after the initial rapid growth in productivity, the worker's production rate will eventually level off.

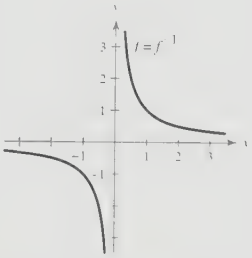
Review Exercises *(page 404)*

1. $f(g(x)) = 3\left(\dfrac{x-5}{3}\right) + 5 = x$

$g(f(x)) = \dfrac{3x + 5 - 5}{3} = x$

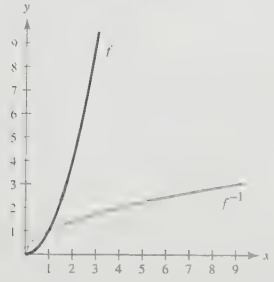
$f(g(x)) = x = g(f(x))$, so f and g are inverse functions of each other.

3. $f^{-1}(x) = \dfrac{1}{x}$ (f is its own inverse function.)



5. (a) $f^{-1}(x) = \sqrt{x}, x \geq 0$

(b)

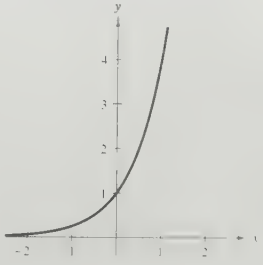


(c) $f^{-1}(f(x)) = \sqrt{x^2} = x$
$f(f^{-1}(x)) = (\sqrt{x})^2 = x$

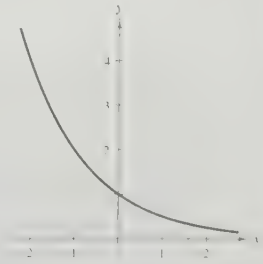
7. 16,777,216 **9.** 0.431 **11.** 2.117

13. c **14.** d **15.** a **16.** b

17.

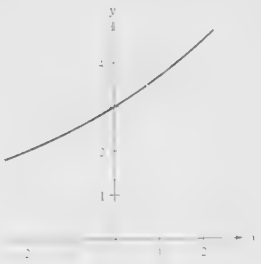


19.

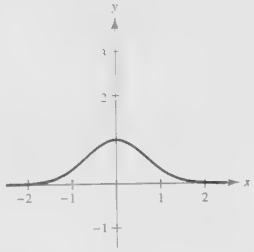


CHAPTER 4

21.



23.



25.

n	1	2	4
A	\$13,308.43	\$13,576.74	\$13,718.59

n	12	365	Continuous
A	\$13,816.21	\$13,864.33	\$13,865.97

27.

t	1	10	20
P	\$185,085.40	\$92,140.76	\$42,449.59

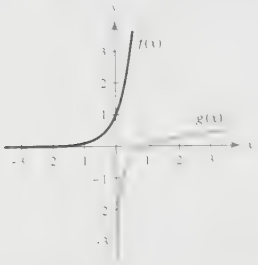
t	30	40	50
P	\$19,556.69	\$9009.84	\$4150.87

29. \$8471.94 **31.** $\log_4 64 = 3$ **33.** $\ln 7.3890\ldots = 2$

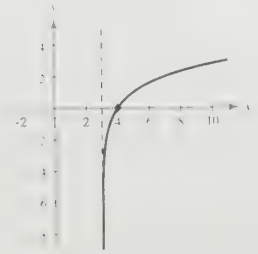
35. $3^4 = 81$ **37.** $e^0 = 1$ **39.** 5 **41.** 7 **43.** $-\frac{1}{2}$

45. d **46.** b **47.** c **48.** a

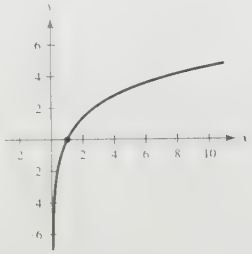
49.



51. Domain: $(3, \infty)$
Vertical asymptote: $x = 3$
x-intercept: $(4, 0)$



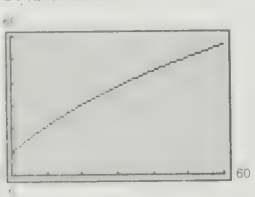
53. Domain: $(0, \infty)$
Vertical asymptote: $x = 0$
x-intercept: $(1, 0)$



55. The average score decreased from 82 to about 68.

57. (a) 53.42 inches

(b)



59. 2.096 **61.** 2.132 **63.** 0.9208 **65.** 0.2823

67. 2 **69.** 3.2 **71.** $\log_{10} x - \log_{10} y$

73. $\ln x + \frac{1}{2}\ln(x - 3)$ **75.** $4\log_5(y - 3)$ **77.** $\log_4 6$

79. $\ln\sqrt{x}$ **81.** $\ln\dfrac{x}{(x-3)(x+1)}$ **83.** $\ln y = \dfrac{4}{3}\ln x$

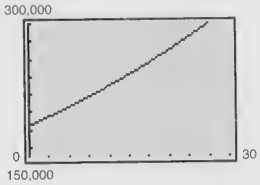
85. $\ln 8 \approx 2.079$ **87.** 1 **89.** $\ln 4 \approx 1.386$

91. $\frac{1}{3}e^{8.2} \approx 1213.650$ **93.** $\frac{1}{5}e^2 \approx 1.478$

95. $3e^2 \approx 22.167$ **97.** 7

99. (a) $\approx$ 197 desks (b) $\approx$ 257 desks **101.** 10.63 g

103. (a)



(b) 2021 ($t \approx 21.00$)

105. $\approx$ 9.93 hours to double
$\approx$ 15.74 hours to triple

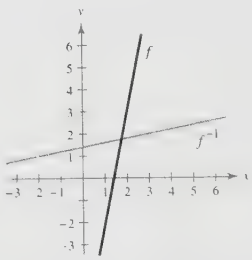
107. (a) $N = 50(1 - e^{-0.04838t})$ (b) 48 days

109. (a) $k \approx 0.2121$
(b) $\approx$ 196 deer; $\approx$ 294 deer; $\approx$ 383 deer

111. Yes

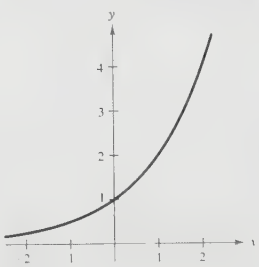
Test Yourself *(page 408)*

1. Yes; $f^{-1}(x) = \dfrac{x + 7}{5}$

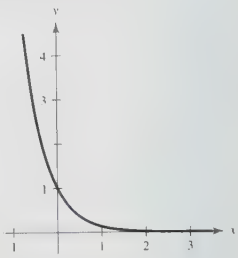


2. The function does not have an inverse function.

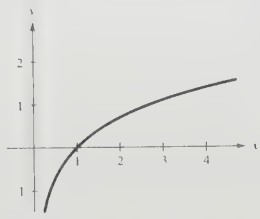
3.



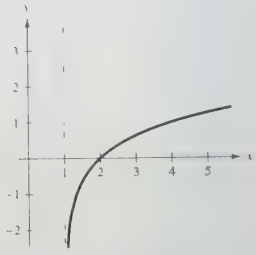
4.



5.



6.



7. 87; $\approx$ 79.8; $\approx$ 76.5

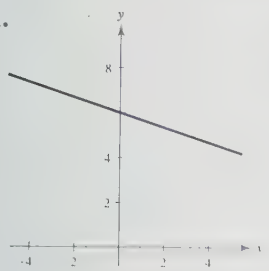
8. After 6 months: ≈ 74.3
After 12 months: ≈ 70.3
After 18 months: ≈ 67.8
Human memory diminishes slowly over time.
9. $2 \ln x + 3 \ln y - \ln z$
10. $\log_{10} 3 + \log_{10} x + \log_{10} y + 2 \log_{10} z$
11. $\log_2 x + \frac{1}{3} \log_2 (x - 2)$ **12.** $\frac{1}{5} \log_8 (x^2 + 1)$
13. $\ln \dfrac{x^2 y^3}{z}$ **14.** $\log_{10} \sqrt[3]{x^2 y^2}$ **15.** $\dfrac{\log_2 21}{4} \approx 1.098$
16. $\ln 6 \approx 1.792$, $\ln 2 \approx 0.693$
17. 127 **18.** $e^6 - 2 \approx 401.429$ **19.** ≈ 16.3 years
20. About 2015 ($t \approx 15.4$). Because the exponent on e is positive, the population of the city is growing.
21. ≈ 9.9 hours **22.** ≈ 2.97 grams; ≈ 0.88 gram
23. Answers will vary. Sample answer:
The growth of the bear population will slow down as the population approaches the carrying capacity of the island. So, the population will grow logistically.

Chapter 5

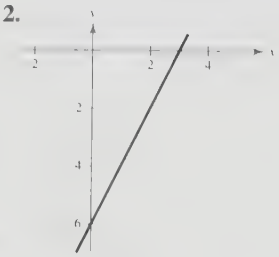
Section 5.1 *(page 417)*

Skills Warm Up *(page 417)*

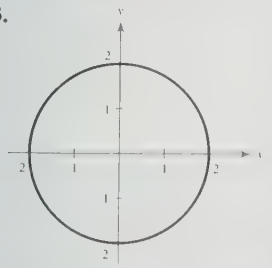
1.



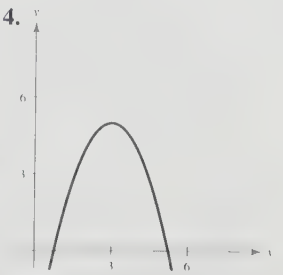
2.



3.



4.



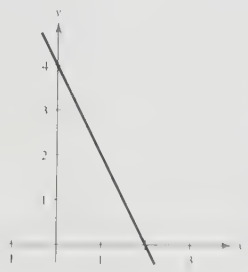
5. x **6.** $-37v$ **7.** $2x^2 + 9$ **8.** -1
9. $x = 6$ **10.** $y = 1$

1. (a) No (b) Yes **3.** (a) Yes (b) No
5. (a) No (b) No **7.** $(-1, 1)$ **9.** $\left(\frac{1}{2}, 3\right)$ **11.** $(10, 3)$
13. $(1.5, 0.3)$ **15.** $\left(\frac{20}{3}, \frac{40}{3}\right)$ **17.** No solution
19. \$5000 at 3%, \$7000 at 5%
21. \$6000 at 2.8%, \$6000 at 3.8%
23. $\left(1 + \sqrt{2}, 2 + 2\sqrt{2}\right), \left(1 - \sqrt{2}, 2 - 2\sqrt{2}\right)$
25. $\left(\frac{29}{10}, \frac{21}{10}\right), (-2, 0)$ **27.** No solution **29.** $(0, 1), (\pm 1, 0)$
31. $(2, 1)$ **33.** $(2, -3)$ **35.** $(-1, 2), (2, 5)$
37. $(0, 4), \left(\frac{12}{5}, -\frac{16}{5}\right)$ **39.** $(4, 3)$ **41.** $\left(\frac{5}{2}, \frac{3}{2}\right)$
43. $(2, 2), (4, 0)$ **45.** $(1, 4), (4, 7)$ **47.** $\left(4, -\frac{1}{2}\right)$
49. No solution **51.** One solution **53.** Two solutions
55. No solution **57.** $\left(\frac{1}{2}, \frac{3}{4}\right), (-3, -1)$ **59.** $(1, 4), (4, 7)$
61. $(0, 1)$ **63.** $(1, 4), (3, 12)$ **65.** $(-3, 4), (5, 0)$
67. Answers will vary. Sample answers:
(a) $\begin{cases} x + 2y = 0 \\ 2x - 4y = 3 \end{cases}$ (b) $\begin{cases} 2x^2 - y = -1 \\ x + y = 4 \end{cases}$
(c) $\begin{cases} 3x + y = 8 \\ 3x + y = -2 \end{cases}$
69. 192 units **71.** 233,333 units
73. 2000 software packages **75.** 2002 ($t \approx 2.03$)
77. Yes, at age 20. **79.** \$100,000
81. According to the graphs, sales of general merchandise stores first exceeded sales of food and beverage stores in 2003 ($t \approx 13.37$). Both models eventually decrease and become negative. So, it is unlikely that these models will continue to be accurate.

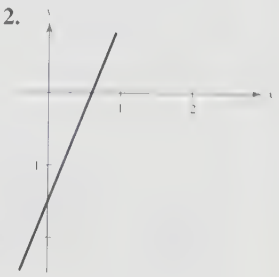
Section 5.2 *(page 428)*

Skills Warm Up *(page 428)*

1.



2.



3. $x - y + 4 = 0$ **4.** $5x + 3y - 28 = 0$
5. $-\frac{1}{2}$ **6.** $\frac{7}{4}$ **7.** Perpendicular **8.** Parallel
9. Neither parallel nor perpendicular
10. Perpendicular

1. $(2, 2)$ **3.** $(2, 0)$ **5.** Inconsistent
7. $(2a, 3a - 3)$, where a is any real number
9. $\left(2, \frac{1}{2}\right)$; consistent **11.** $(-7, -13)$; consistent
13. $(4, -5)$; consistent **15.** $(4, -1)$; consistent
17. $(40, 40)$; consistent **19.** $\left(-\frac{6}{35}, \frac{43}{35}\right)$; consistent
21. $\left(\frac{18}{5}, \frac{3}{5}\right)$; consistent **23.** $\left(\frac{19}{7}, -\frac{2}{7}\right)$; consistent
25. $\begin{cases} x + y = 13 \\ x - y = 3 \end{cases}$; $(8, 5)$ **27.** $\begin{cases} 2r + s = 8 \\ r - s = 7 \end{cases}$; $(5, -2)$
29. No solution; inconsistent
31. $\left(a, \frac{5}{6}a - \frac{1}{2}\right)$, where a is any real number; consistent
33. $\left(\frac{90}{31}, -\frac{67}{31}\right)$; consistent **35.** $\left(-\frac{1}{2}, 1\right)$ **37.** $(-2, -9)$
39. $(-0.874, 0.417), (0, 1)$ **41.** $(20, 20)$
43. Answers will vary. Sample answers:
(a) $\begin{cases} 3x + y = 18 \\ x + y = 6 \end{cases}$ (b) $\begin{cases} -2x + 8y = 38 \\ 4x + y = -8 \end{cases}$
(c) $\begin{cases} 8x - y = 1 \\ -4x + 4y = -25 \end{cases}$

45. No. The solution of the system is (79,400, 398).

47. 550 miles per hour; 50 miles per hour

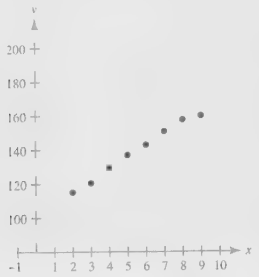
49. $6\frac{2}{3}$ gallons of 20% solution
$3\frac{1}{3}$ gallons of 50% solution

51. \$8000 at 4.75%, \$17,000 at 5.5%

53. $x \approx 309{,}091$ units; $p \approx \$25.09$

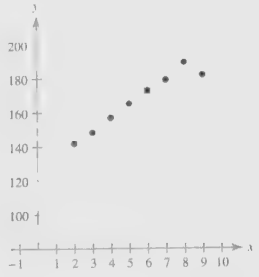
55. $x = 2{,}000{,}000$ units; $p = \$100.00$

57. (a) Fast-food



$y = 6.79x + 101.8$

Full-service



$y = 6.67x + 130.3$

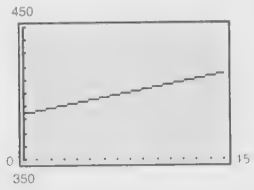
(b) In the near future, fast-food sales will not equal full-service sales. According to the models, fast-food sales will equal full-service sales in the year 2237. These models, therefore, are not appropriate to use for predicting when fast-food sales will equal full-service sales.

59. 1380 units at \$810.60 **61.** $y = 0.97x + 2.1$

63. $y = 0.318x + 4.061$

65. (a) $y = 2.031t + 384.59$

(b)



404.9

(c) You obtain the same model: $y = 2.031t + 384.59$.

67. Answers will vary.

Section 5.3 *(page 439)*

Skills Warm Up *(page 439)*

1. (10, 15) **2.** $\left(-3, -\frac{10}{3}\right)$ **3.** (28, 4) **4.** (4, 3)
5. Not a solution **6.** Not a solution **7.** Solution
8. Solution **9.** $5a + 2$ **10.** $a + 13$

1. c **2.** a **3.** b **4.** d

5. Yes. The system has a "stair-step" pattern with leading coefficients of 1.

7. No. The system has a "stair-step" pattern, but not all of its leading coefficients are 1.

9. $(4, -2, -2)$ **11.** $(2, -3, -2)$ **13.** $(-1, -6, 8)$

15. Inconsistent **17.** $\left(1, -\frac{3}{2}, \frac{1}{2}\right)$

19. $(-3a + 10, 5a - 7, a)$ **21.** $\left(-4a + 13, -\frac{15}{2}a + \frac{45}{2}, a\right)$

23. Inconsistent **25.** $(-3, 4, 2)$ **27.** $(3, -1, 2)$

29. $\left(\frac{3}{4}a, -2a, a\right)$ **31.** $(-5a + 3, -a - 5, a)$

33. Inconsistent **35.** $(1, 1, 1, 1)$

37. Answers will vary. Sample answer:

$$\begin{cases} 2x + y + z = 9 \\ y + z = 1, \\ z = 2 \end{cases} \quad \begin{cases} x + 2y + 4z = 13 \\ x + y + z = 4 \\ x + z = 5 \end{cases}$$

39. Answers will vary. Sample answer:

$$\begin{cases} x - y + z = 3 \\ y - z = -2, \\ z = -3 \end{cases} \quad \begin{cases} x + 3y + 4z = -26 \\ 4x - y - 5z = 24 \\ x + 2y = -9 \end{cases}$$

41. Answers will vary. Sample answer:
$a = 1$: $\left(1, 3, \frac{1}{3}\right)$
$a = 2$: $\left(2, 4, \frac{2}{3}\right)$
$a = 0$: $(0, 2, 0)$

43. Answers will vary. Sample answer:
$a = -4$: $(-2, -16, 7)$
$a = 2$: $(1, 8, 7)$
$a = 6$: $(3, 24, 7)$

45. $y = 2x^2 + 3x - 4$ **47.** $y = -4x^2 + 2x + 1$

49. $x^2 + y^2 - 4x = 0$ **51.** $x^2 + y^2 - 6x + 6y + 9 = 0$

53. \$1,100,000 at 3%
\$200,000 at 4%
\$200,000 at 6%

55. 15,000 units of \$15 candles
30,000 units of \$10 candles
5000 units of \$5 candles

57. 18 gallons of spray X
1 gallon of spray Y
6 gallons of spray Z

59. Invest $\$33{,}333.33 + 0.8a$ in certificates of deposit, $\$341{,}666.67 - 0.8a$ in municipal bonds, $\$125{,}000.00 - a$ in blue-chip stocks, and a in growth or speculative stocks, where $0 \le a \le 125{,}000$.

61. $y = 0.079x^2 + 0.63x + 2.9$

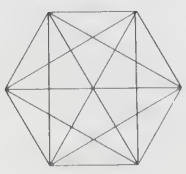
63. $y = -0.207x^2 - 0.89x + 5.1$

65. (a) $y = 0.114x^2 - 0.40x + 4.0$ (b) Same model

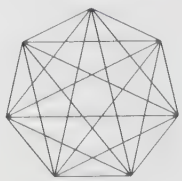
67. (a) $y = 1.07x^2 - 1.6x + 22$ (b) 32.72%

69. (a) $y = 0.125x^2 - 2.55x + 18$
(b) $y = 0.125x^2 - 2.55x + 18$ (c) ≈ 530 ft

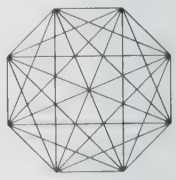
71. $y = \frac{1}{2}x^2 - \frac{1}{2}x$



$x = 6, y = 15$



$x = 7, y = 21$



$x = 8, y = 28$

Yes. The relationship between x and y is represented by the equation for regular polygons with 6, 7, and 8 sides.

73. Yes. A system of linear equations can have three possible types of solutions: exactly one solution, infinitely many solutions, or no solution.

75. Answers will vary. Sample answer:
(a) $a = 4, b = -2, c = -2$
(b) $a = 3, b = 2, c = 1$
(c) Not possible

77. Answers will vary.

Quiz Yourself *(page 444)*

1. (2, 5) **2.** No solution **3.** (1, 3)

4. $\left(-\frac{2}{5} \pm \frac{2\sqrt{11}}{5}, \frac{1}{5} \pm \frac{4\sqrt{11}}{5}\right)$ **5.** 1500 units

6. 500,000 units **7.** $(2, -1)$ **8.** $\left(1, \frac{3}{2}\right)$

9. $x = 5000$ units
$p = \$40$

10. $y = 0.84t + 42.6$ **11.** $(1, -2, 3)$

12. Answers will vary.
Sample answer: $(a + 6, a + 6, a)$, a is any real number.

13. Inconsistent

14. (a) $y = 0.1400x^2 + 3.204x + 69.83$
(b) 2010: \$80.70
2011: \$84.89

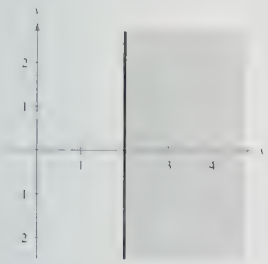
Section 5.4 *(page 451)*

Skills Warm Up *(page 451)*

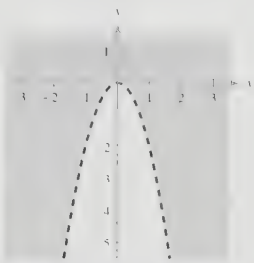
1. Line **2.** Parabola **3.** Circle **4.** Parabola
5. Line **6.** Circle **7.** $(1, 1)$ **8.** $(2, 0)$
9. $(2, 1), \left(-\frac{5}{2}, -\frac{5}{4}\right)$ **10.** $(2, 3), (3, 2)$

1. d **2.** b **3.** a **4.** c **5.** f **6.** e

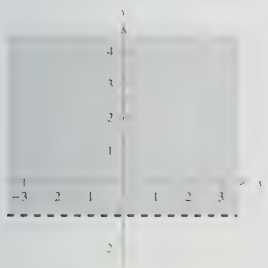
7.



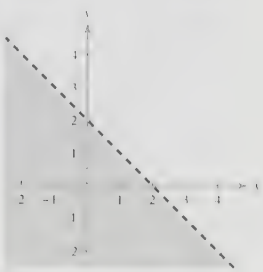
9.



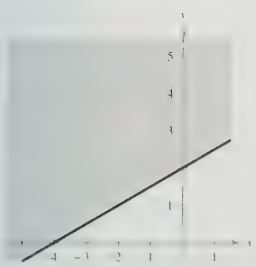
11.



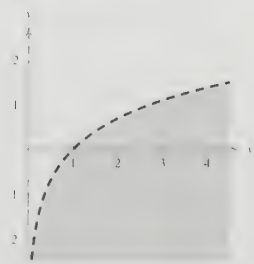
13.



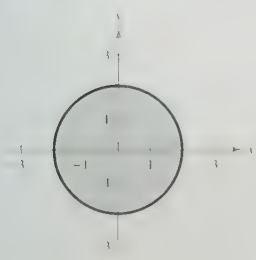
15.



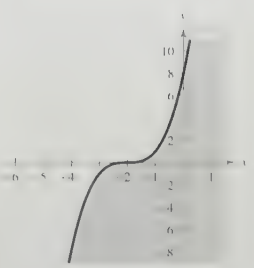
17.



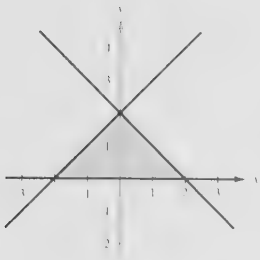
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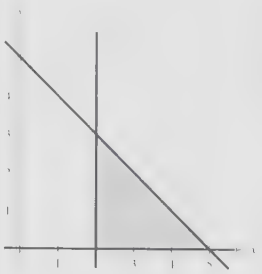
21.



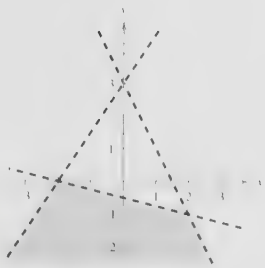
23.



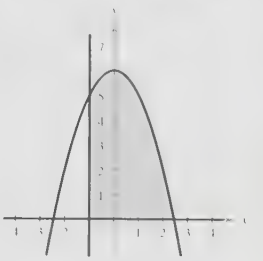
25.



27.



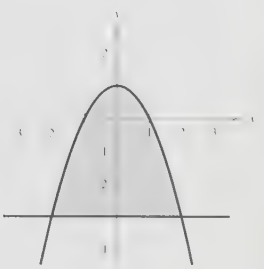
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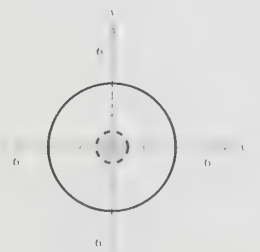
31.



33.



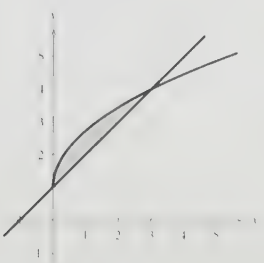
35.



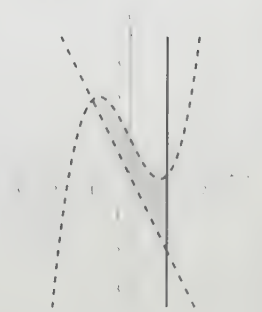
37.



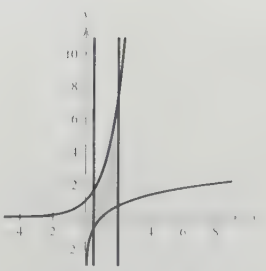
39.



41.



43.



45. $y \leq \frac{3}{2}x, y \geq \frac{3}{2}x - \frac{3}{2}, 0 \leq y \leq 3$

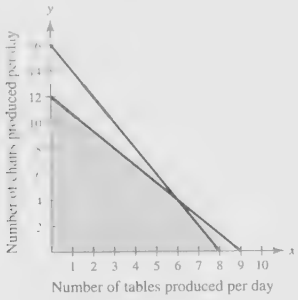
CHAPTER 5

47. $y \le \frac{4}{3}x, y \ge 0, y \le -4x + 16$

49. $x^2 + y^2 \le 16, x \ge 0, y \ge 0$

51. (a) $2x + \frac{3}{2}y \le 18, \frac{3}{2}x + \frac{3}{4}y \le 12, x \ge 0, y \ge 0$

(b)



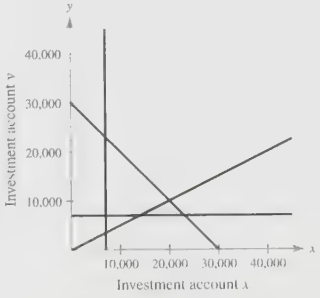
53. Consumer surplus: \$4,777,001.41
Producer surplus: \$477,545.60

55. Consumer surplus: \$40,000,000
Producer surplus: \$20,000,000

57. The consumer surplus and producer surplus are equal when the slope of the demand equation is the negative of the slope of the supply equation.

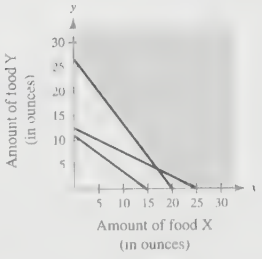
59. (a) $x + y \le 30{,}000$
$x \ge 7000$
$y \ge 7000$
$x - 2y \ge 0$

(b)

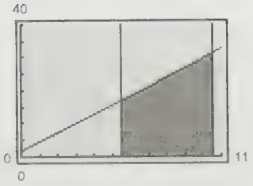


61. (a) $20x + 15y \ge 400$
$10x + 20y \ge 250$
$15x + 20y \ge 220$
$x \ge 0$
$y \ge 0$

(b)



63. (a)



(b) 121 nesting pairs

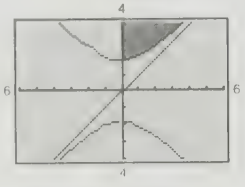
65. The graph is a half-line on the real number line; on the rectangular coordinate system, the graph is a half-plane.

67. Answers will vary. Sample answer:

$$\begin{cases} x \ge 1 \\ x \le 4 \\ y \ge 1 \\ y \le 4 \end{cases}$$

69. (a) $\begin{cases} \pi y^2 - \pi x^2 \ge 10 \\ y > x \\ x > 0 \end{cases}$

(b)



(c) The line is an asymptote to the boundary. The larger the circles, the closer the radii can be with the constraint still satisfied.

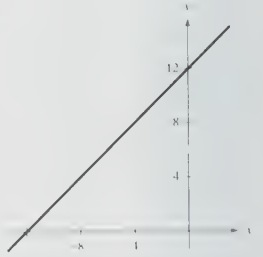
Section 5.5 *(page 460)*

Skills Warm Up *(page 460)*

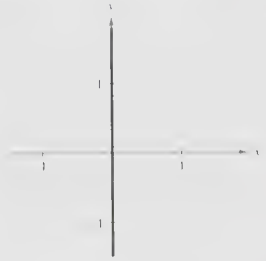
1.



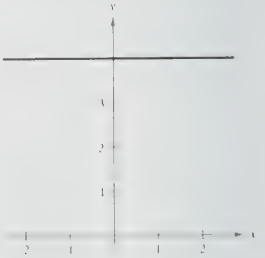
2.



3.

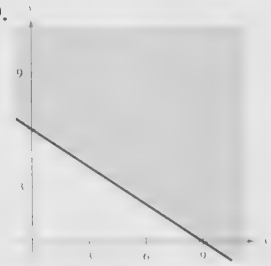


4.

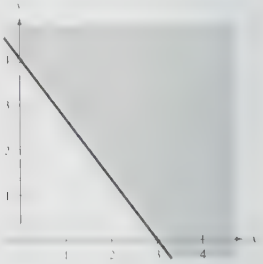


5. (0, 4) **6.** (12, 0) **7.** (3, 1) **8.** (2, 5)

9.



10.



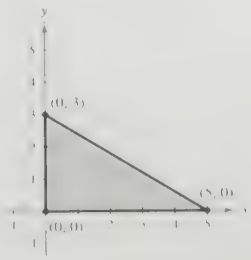
1. Minimum value at (0, 0): 0
Maximum value at (0, 5): 20

3. Minimum value at (0, 0): 0
Maximum value at (5, 0): 45

5. Minimum value at (0, 0): 0
Maximum value at (3, 4): 26

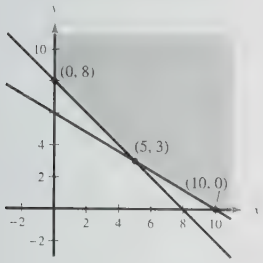
7. Minimum value at (0, 0): 0
Maximum value at (4, 0): 20

9. Minimum value at (0, 0): 0
Maximum value at any point on the line segment connecting the points (5, 0) and (0, 3): 30



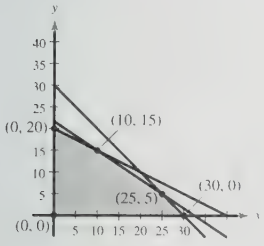
11. Minimum value at (0, 0): 0
Maximum value at (5, 0): 40
Same graph as in Answer 9

13. Minimum value at (5, 3): 35
No maximum value



15. Minimum value at (10, 0): 20
No maximum value
Same graph as in Answer 13

17. Minimum value at (0, 0): 0
Maximum value at any point on the line segment connecting the points (0, 20) and (10, 15): 120



19. Minimum value at (0, 0): 0
Maximum value at any point on the line segment connecting the points (25, 5) and (30, 0): 30
Same graph as in Answer 17

21. Maximum value at (3, 6): 12

23. Maximum value at any point on the line segment connecting the points (0, 10) and (3, 6): 30

25. Maximum value at (5, 4): 58

27. Maximum value at (8, 0): 120

29. Answers will vary.
Sample answer:
$z = 2x + 15y$

31. Answers will vary.
Sample answer:
$z = -x$

33. Answers will vary.
Sample answer:
$z = x + 7y$

35. Answers will vary.
Sample answer:
$z = 6x + 5y$

37. Crop A: 60 acres
Crop B: 90 acres
\$33,150

39. Brand X: 3 bags
Brand Y: 6 bags
\$240

41. Model A: 0 bicycles
Model B: 1600 bicycles
\$120,000

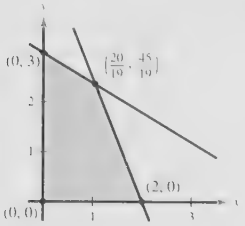
43. 12 audits and 0 tax returns

45. Television: None
Newspaper: \$4,800,000
300 million people

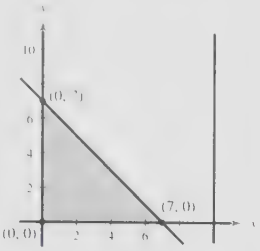
47. Type A: \$62,500
Type B: \$187,500
\$26,875

49. Model A: 929 units
Model B: 77 units
\$99,445

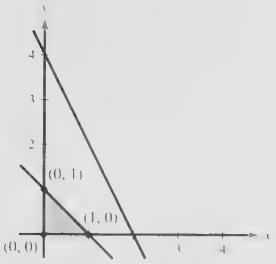
51. z has a maximum value (of 5) at any point on the line segment connecting the vertices $(2, 0)$ and $\left(\frac{20}{19}, \frac{45}{19}\right)$.



53. The constraint $x \leq 10$ is extraneous. The maximum value of z, $z = 14$, occurs at (0, 7).



55. The constraint $2x + y \leq 4$ is extraneous. The maximum value of z, $z = 4$, occurs at (0, 1).



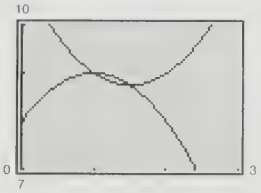
57. No. You need more information about the feasible region to determine whether the minimum also occurs at (0, 0).

59. Yes; The objective function also has the maximum value at any point on the line segment connecting the two vertices, so there are an infinite number of points that produce the maximum value.

Review Exercises *(page 467)*

1. $(-2, 4)$ **3.** $(8, -10)$ **5.** (8, 6), (0, 10)

7. (1, 9), (1.5, 8.75)



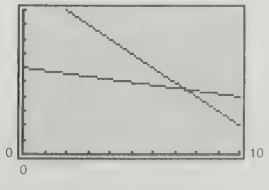
9. 800 plants **11.** During the fourth month of the new format

13. $(3, -5)$ **15.** $\left(a, \frac{4}{3}a - \frac{10}{3}\right)$, where a is any real number

17. Inconsistent **19.** (8, 9)

21. The graph is a point. Solution: $(-1, 1)$

23. (a) $\begin{cases} 0.1x + 0.5y = 0.25(12) \\ x + y = 12 \end{cases}$



(b) 7.5 gallons of 10% solution
4.5 gallons of 50% solution

25. $y = 1.01x + 1.54$ **27.** $x = 71{,}429$ units, $p = \$22.71$

CHAPTER 5

29. (12, 5, 2) **31.** (2, −1, 3)

33. $\left(\frac{1}{5}a + \frac{8}{5}, -\frac{6}{5}a + \frac{42}{5}, a\right)$, a is any real number

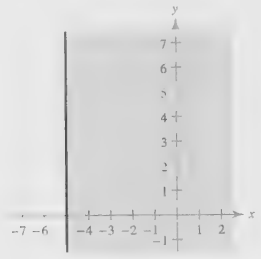
35. Inconsistent **37.** $y = 2x^2 + x - 6$

39. $x^2 + y^2 - 4x + 2y - 4 = 0$

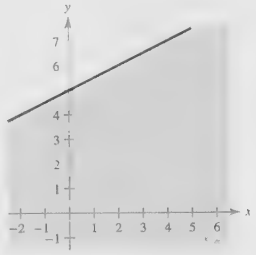
41. $200,000 in certificates of deposit
$100,000 in municipal bonds
$15,000 in blue-chip stocks
$185,000 in growth or speculative stocks

43. $y = 0.114x^2 + 0.80x + 1.6$

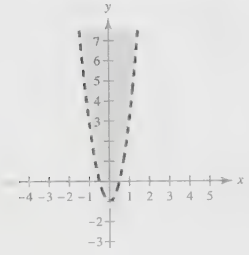
45.



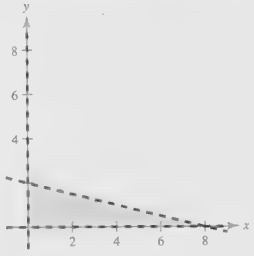
47.



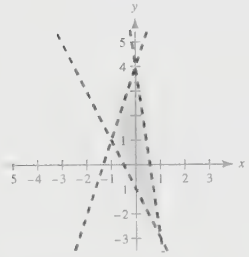
49.



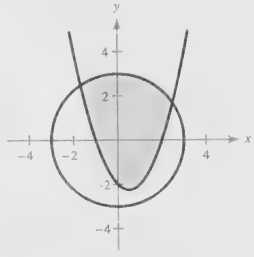
51.



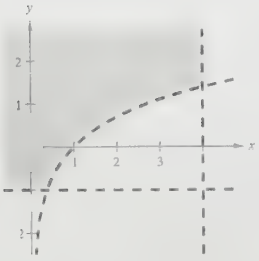
53.



55.



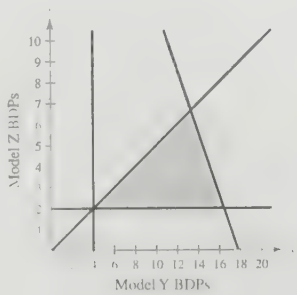
57.



59.
$$\begin{cases} y \le \frac{5}{2}x - \frac{3}{2} \\ y \le \frac{1}{5}x + \frac{27}{5} \\ y \ge \frac{5}{2}x - 13 \\ y \ge \frac{1}{5}x + \frac{4}{5} \end{cases}$$

61. Consumer surplus: $4,500,000
Producer surplus: $9,000,000

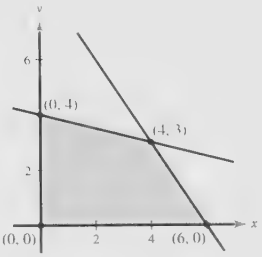
63.
$$\begin{cases} x \ge 2y \\ 100x + 150y \le 4000 \\ x \ge 4 \\ y \ge 2 \end{cases}$$



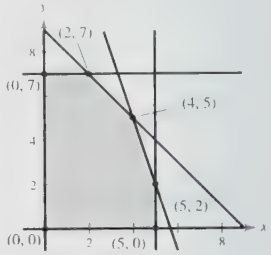
65. Minimum value at (0, 0): 0
Maximum value at (0, 8): 48

67. Minimum value at (0, 0): 0
Maximum value at any point on the line segment connecting (25, 35) and (50, 15): 550

69. Minimum value at (0, 0): 0
Maximum value at (4, 3): 48



71. Minimum value at (0, 0): 0
Maximum value at (4, 5): 47



73. 369 units of model A
538 units of model B
$117,600

75. $525 model: 0 units
$675 model: 305 units
$38,125

77. 8 audits
0 tax returns
$20,000

Test Yourself *(page 472)*

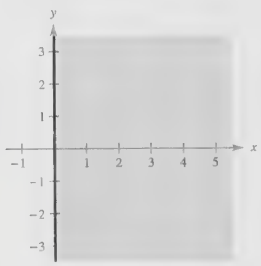
1. (2, 4) **2.** (−2, 5), (3, 0) **3.** (−1, 6), (2, 0)

4. (3.36, −1.32) **5.** (2, −1, 3) **6.** (2, −3, 4)

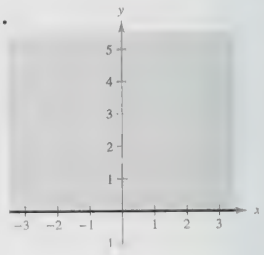
7. $60,000 at 2.3%
$20,000 at 3.1% **8.** (50,000, 34)

9. (a) $y = 0.593x^2 + 4.95x + 46.7$ (b) ≈66.9 million adults

10.



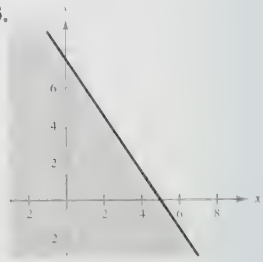
11.



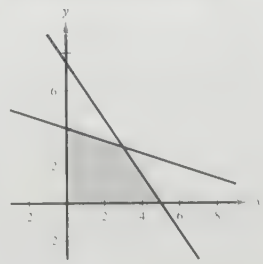
12.



13.



14.



15. Minimum value at (0, 0): 0
Maximum value at (3, 3): 39

16. Model A: 297 units; model B: 570 units
$216,150; The profit model is $P = 200x + 275y$ with the constraints

$$\begin{cases} 3.5x + 8y \le 5600 \\ 2.5x + 2y \le 2000 \\ 1.3x + 0.9y \le 900 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

and the maximum profit occurs at (297, 570).

Chapter 6

Section 6.1 *(page 483)*

Skills Warm Up *(page 483)*

1. -3 **2.** 30 **3.** 6 **4.** $-\frac{1}{6}$ **5.** Solution
6. Not a solution **7.** $(5, 2)$ **8.** $\left(\frac{12}{5}, -3\right)$
9. (40, 14, 2) **10.** $\left(\frac{15}{2}, 4, 1\right)$

1. 2×3 **3.** 4×3 **5.** 4×2 **7.** 2×4

9. $\left[\begin{array}{cc|c} 4 & -3 & 5 \\ -1 & 3 & 12 \end{array}\right]$ **11.** $\left[\begin{array}{ccc|c} 1 & 10 & -2 & 2 \\ 5 & 3 & 4 & 0 \\ 2 & 1 & 0 & 6 \end{array}\right]$

2×3 3×4

13. $\left[\begin{array}{ccc|c} 7 & -5 & 1 & 13 \\ 19 & 0 & -8 & 10 \end{array}\right]$ **15.** $\left[\begin{array}{ccc|c} 1 & 1 & 1 & \\ 0 & 7 & -1 & \end{array}\right]$

2×4

17. $\begin{bmatrix} 1 & 0 & 14 & -11 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$ **19.** Add 5 times R_2 to R_1.

21. Interchange R_1 and R_2. Add 4 times R_1 to R_3.

23. $\begin{cases} 2x + 4y = 6 \\ -x + 3y = -8 \end{cases}$ **25.** $\begin{cases} x + 2z = -10 \\ 3y - z = 5 \\ 4x + 2y = 3 \end{cases}$

27. $\begin{cases} x - 5y = 6 \\ y = -2 \end{cases}$
$(-4, -2)$

29. $\begin{cases} x + 3y - z = 15 \\ y + 4z = -12 \\ z = -5 \end{cases}$
$(-14, 8, -5)$

31. (a) (i) $\left[\begin{array}{cc|c} 3 & 0 & -6 \\ 6 & -4 & -28 \end{array}\right]$ (ii) $\left[\begin{array}{cc|c} 3 & 0 & -6 \\ 0 & -4 & -16 \end{array}\right]$

(iii) $\left[\begin{array}{cc|c} 3 & 0 & -6 \\ 0 & 1 & 4 \end{array}\right]$ (iv) $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 4 \end{array}\right]$

$(-2, 4)$

(b) $\begin{cases} -3x + 4y = 22 \\ 6x - 4y = -28 \end{cases}$
$(-2, 4)$

(c) Answers will vary.

33. (i)

```
row÷(A,2,1)→B
]
      [[3 0 -6]
       [6 -4 -28]]
```

(ii)

```
*row+(-2,B,1,2
)→C]
      [[3 0 -6]
       [0 -4 -16]]
```

(iii)

```
*row(-1/4,C,2)
→D]
      [[3 0 -6]
       [0 1 4]]
```

(iv)

```
*row(1/3,D,1)→
E]
      [[1 0 -2]
       [0 1 4]]
```

35. Reduced row-echelon form **37.** Not in row-echelon form
39. Not in row-echelon form

41. $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **43.** $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

45. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **47.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

49. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ **51.** $(-4, 6)$ **53.** $(-4, -8, 2)$

55. $(-2a - 4, -a + 6, a)$ **57.** $(3, 2)$ **59.** $(4, -2)$
61. $\left(\frac{1}{2}, -\frac{3}{4}\right)$ **63.** Inconsistent **65.** $(5a + 4, -3a + 2, a)$
67. $(0, 2 - 4a, a)$ **69.** $(-3b + 96a + 100, b, 52a + 54, a)$
71. $(0, 0)$ **73.** $(-2a, a, a)$ **75.** $(4, -3, 2)$
77. $(2a + 1, 3a + 2, a)$ **79.** $(7, -3, 4)$
81. Yes; $(-1, 1, -3)$ **83.** No
85. $1,200,000 was borrowed at 8%, $200,000 was borrowed at 9%, and $600,000 was borrowed at 12%.
87. Both are correct. Because there are infinitely many ordered triples that are solutions to this system, a solution can be written in many different ways. If $a = 3$, the ordered triple is $(3, -3, 5)$. You obtain the same triple when $b = 5$.
89. $y = 7.5t + 28$
133 new cases; Because the data values increased in a linear pattern, this estimate seems reasonable.

Section 6.2 *(page 497)*

Skills Warm Up *(page 497)*

1. -5 **2.** -7
3. Not in reduced row-echelon form
4. Not in reduced row-echelon form
5. $\left[\begin{array}{cc|c} -5 & 10 & 12 \\ 7 & -3 & 0 \end{array}\right]$
6. $\left[\begin{array}{ccc|c} 10 & 15 & -9 & 42 \\ 6 & -5 & 0 & 0 \end{array}\right]$
7. $(0, 2)$ **8.** $(2 + a, 3 - a, a)$
9. $(1 - 2a, a, -1)$ **10.** $(2, -1, 1)$

1. $x = -3, y = 2$ **3.** $x = -2, y = 5$
5. (a) $[3 \quad 1]$ (b) $[11 \quad 5]$
(c) $[42 \quad -18]$ (d) $[40 \quad -18]$

7. (a) $\begin{bmatrix} 8 & -1 \\ 1 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 5 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 30 & -12 \\ 18 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 11 & -11 \\ 18 & -14 \end{bmatrix}$

9. (a) $\begin{bmatrix} 7 & 3 \\ 1 & 9 \\ -2 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -5 \\ 3 & -1 \\ -4 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 36 & -6 \\ 12 & 24 \\ -18 & 30 \end{bmatrix}$ (d) $\begin{bmatrix} 21 & -16 \\ 11 & 1 \\ -15 & -10 \end{bmatrix}$

11. (a) $\begin{bmatrix} 3 & 3 & -2 \\ -2 & 5 & 7 \\ 1 & -8 & 11 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 4 & -3 & -11 \\ 1 & 6 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 12 & 12 & -6 \\ 6 & 6 & -12 \\ 6 & -6 & 18 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 5 & -1 \\ 13 & -8 & -35 \\ 4 & 17 & -12 \end{bmatrix}$

13. (a) $[2 \quad 40 \quad 30 \quad -23]$ (b) $[-28 \quad 8 \quad 10 \quad 1]$

(c) $[-78 \quad 144 \quad 120 \quad -66]$ (d) $[-97 \quad 48 \quad 50 \quad -8]$

15. $\begin{bmatrix} -8 & -7 \\ 15 & -1 \end{bmatrix}$ **17.** $[12 \quad -6]$ **19.** $\begin{bmatrix} 55 & -50 \\ 15 & -20 \end{bmatrix}$

21. $\begin{bmatrix} -48 & -8 & 24 \\ -24 & 64 & 24 \end{bmatrix}$ **23.** $\begin{bmatrix} 10 & 8 \\ -59 & 9 \end{bmatrix}$

25. $\begin{bmatrix} -17.143 & 2.143 \\ 11.571 & 10.286 \end{bmatrix}$ **27.** $\begin{bmatrix} -1.581 & -3.739 \\ -4.252 & -13.249 \\ 9.713 & -0.362 \end{bmatrix}$

29. $\begin{bmatrix} -6 & -9 \\ -1 & 0 \\ 17 & -10 \end{bmatrix}$ **31.** $\begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & 0 \\ -\frac{13}{2} & \frac{11}{2} \end{bmatrix}$ **33.** $[-32]$

35. $\begin{bmatrix} -5 \\ 14 \end{bmatrix}$ **37.** $\begin{bmatrix} -7 & 10 & -12 & 17 \\ 6 & 21 & 7 & 15 \\ -3 & 3 & -5 & 6 \end{bmatrix}$ **39.** $\begin{bmatrix} -1 & 19 \\ 4 & -27 \\ 0 & 14 \end{bmatrix}$

41. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}$ **43.** Not possible

45. $\begin{bmatrix} 41 & 7 & 7 \\ 42 & 5 & 25 \\ -10 & -25 & 45 \end{bmatrix}$ **47.** Not possible

49. (a) $\begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}$ (c) $\begin{bmatrix} 9 & 6 \\ 12 & 12 \end{bmatrix}$

51. (a) $\begin{bmatrix} 3 & 5 \\ 1 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 11 & 5 & 0 \\ -7 & -4 & -1 \\ 14 & 8 & 2 \end{bmatrix}$ (c) Not possible

53. (a) $[11]$ (b) $\begin{bmatrix} -4 & 2 & 3 \\ 0 & 0 & 0 \\ -20 & 10 & 15 \end{bmatrix}$ (c) Not possible

55. Cannot perform operation. **57.** Cannot perform operation.

59. Cannot perform operation. **61.** 2×2 **63.** 2×3

65. (a) $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

67. (a) $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

69. (a) $\begin{bmatrix} 1 & -4 & 5 \\ 2 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -24 \\ 8 \end{bmatrix}$ (b) $\begin{bmatrix} -4 \\ -3 \\ -1 \end{bmatrix}$

71. (a) $\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & -1 \\ 2 & -5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 17 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

73. $\begin{bmatrix} 110 & 132 & 66 & 44 \\ 154 & 176 & 220 & 88 \end{bmatrix}$

75.
$$\begin{matrix} \text{Hotel } w & \text{Hotel } x & \text{Hotel } y & \text{Hotel } z \end{matrix}$$
$$\begin{bmatrix} 688.80 & 750.40 & 828.80 & 1108.80 \\ 1114.40 & 1153.60 & 1321.60 & 1237.60 \end{bmatrix} \begin{matrix} \text{Double} \\ \text{Family} \end{matrix} \Big\} \text{Occupancy}$$

77. (a) \$7375 (b) \$5845

(c)
$$\begin{matrix} & \text{Wholesale} & \text{Retail} \end{matrix}$$
$$ST = \begin{bmatrix} \$4650 & \$7375 \\ \$7405 & \$11{,}350 \\ \$5845 & \$9400 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \Big\} \text{Outlet}$$

ST represents the wholesale and retail prices of the laptop computer inventories at the three outlets.

79. (a) $B = [2 \quad 0.25 \quad 0.5]$

(b)
$$\overbrace{\begin{matrix} \text{130-lb} \\ \text{person} \end{matrix} \quad \begin{matrix} \text{155-lb} \\ \text{person} \end{matrix}}^{\text{Calories burned}}$$
$$BA = [1180 \quad 1407.5]$$

BA represents the total calories burned by each person.

81. (a)
$$\begin{matrix} & \text{X} & \text{Y} & \text{Z} \end{matrix}$$
$$D = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 3 \\ 4 & 5 & 3 \end{bmatrix} \begin{matrix} \text{Sacks} \\ \text{Interceptions} \\ \text{Key tackles} \end{matrix}$$

Each entry d_{ij} represents the number of each type of defensive play made by each player.

(b)
$$\begin{matrix} & \text{Sacks} & \text{Interceptions} & \text{Tackles} \end{matrix}$$
$$B = [\$2000 \quad \$1000 \quad \$800]$$

Each entry b_{ij} represents the bonus received for each type of play.

(c)
$$\begin{matrix} & \text{X} & \text{Y} & \text{Z} \end{matrix}$$
$$BD = [\$9200 \quad \$8000 \quad \$9400]$$

Each entry represents the bonus each player will receive.

(d) Player Z

83. $\begin{bmatrix} 0.40 & 0.15 & 0.15 \\ 0.28 & 0.53 & 0.17 \\ 0.32 & 0.32 & 0.68 \end{bmatrix}$ **85.** $\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$; only one correct matrix

87. $AC = \begin{bmatrix} 12 & -6 & 9 \\ 16 & -8 & 12 \\ 4 & -2 & 3 \end{bmatrix} = BC$, but $A \neq B$.

89. One year:

Gold subscribers: 28,750

Galaxy subscribers: 35,750

Nonsubscribers: 35,500

Multiply the original matrix by the 3×1 matrix

$$\begin{bmatrix} 25{,}000 \\ 30{,}000 \\ 45{,}000 \end{bmatrix}$$

which represents the current numbers of subscribers for each company and the number of nonsubscribers.

Two years:
Gold subscribers: 30,813
Galaxy subscribers: 39,675
Nonsubscribers: 29,513
Multiply the original matrix by the 3×1 matrix

$$\begin{bmatrix} 28{,}750 \\ 35{,}750 \\ 35{,}500 \end{bmatrix}$$

which represents the numbers of subscribers for each company and the number of nonsubscribers after 1 year.
Three years:
Gold subscribers: 31,947
Galaxy subscribers: 42,330
Nonsubscribers: 25,724
Multiply the original matrix by the 3×1 matrix

$$\begin{bmatrix} 30{,}813 \\ 39{,}675 \\ 29{,}513 \end{bmatrix}$$

which represents the numbers of subscribers for each company and the number of nonsubscribers after 2 years.
The number of subscribers to each company is increasing each year. The number of nonsubscribers is decreasing each year.

Section 6.3 *(page 508)*

Skills Warm Up *(page 508)*

1. $\begin{bmatrix} 4 & 24 \\ 0 & -16 \\ 48 & 8 \end{bmatrix}$ **2.** $\begin{bmatrix} \frac{11}{2} & 5 & 24 \\ \frac{1}{2} & 0 & 8 \\ 0 & 1 & 4 \end{bmatrix}$

3. $\begin{bmatrix} -5 & -2 & -13 \\ 4 & -13 & -2 \end{bmatrix}$ **4.** $\begin{bmatrix} -13 & 11 \\ -19 & 21 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **6.** $\begin{bmatrix} 6 & 5 \\ 3 & -2 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **9.** $\left[\begin{array}{ccc:cc} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 4 & -3 \end{array}\right]$

10. $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -6 & -4 & 3 \\ 0 & 1 & 0 & 11 & 6 & -5 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$

1–9. $AB = I$ and $BA = I$ **11.** $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

13. $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -1 \end{bmatrix}$ **15.** $\begin{bmatrix} 0 & -1 \\ 1 & 11 \end{bmatrix}$ **17.** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{4} & 0 \end{bmatrix}$

19. Does not exist **21.** Does not exist

23. $\begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$ **25.** $\frac{1}{2}\begin{bmatrix} -3 & 3 & 2 \\ 9 & -7 & -6 \\ -2 & 2 & 2 \end{bmatrix}$

27. $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ **29.** $\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 \\ \frac{7}{20} & -\frac{1}{4} & \frac{1}{5} \end{bmatrix}$

31. Does not exist

33. $\begin{bmatrix} -175 & 37 & -13 \\ 95 & -20 & 7 \\ 14 & -3 & 1 \end{bmatrix}$ **35.** $\frac{1}{11}\begin{bmatrix} 0 & -20 & 10 \\ -110 & 55 & 55 \\ 110 & -30 & -40 \end{bmatrix}$

37. $\begin{bmatrix} -12 & -5 & -9 \\ -4 & -2 & -4 \\ -8 & -4 & -6 \end{bmatrix}$ **39.** Does not exist

41. $\frac{1}{19}\begin{bmatrix} 3 & 2 \\ -2 & 5 \end{bmatrix}$ **43.** Does not exist

45. $\frac{1}{59}\begin{bmatrix} 16 & 15 \\ -4 & 70 \end{bmatrix}$ **47.** $(-2, 1)$ **49.** $(4, 2)$

51. $(-2, 3)$ **53.** $(2, 0)$ **55.** $(3, 8, -11)$
57. $(2, 1, 0, 0)$ **59.** $(2, -2)$ **61.** Inconsistent
63. $(-4, -8)$ **65.** $(-1, 3, 2)$ **67.** $(5, 0, -2, 3)$

69. $\begin{cases} 2x + y + 3z = 16 \\ 4x - 2z = -2 \\ 3y + 2z = 1 \end{cases}$

71. AAA bonds: \$20,000
A bonds: \$5000
B bonds: \$10,000

73. AAA bonds: \$21,000
A bonds: \$5000
B bonds: \$10,000

75. $I_1 = 4$ amperes, $I_2 = 1$ ampere, $I_3 = 5$ amperes

77. 100 bags of potting soil for seedlings
100 bags of potting soil for general potting
100 bags of potting soil for hardwood plants

79. 5 bags of potting soil for seedlings
100 bags of potting soil for general potting
120 bags of potting soil for hardwood plants

81. (a) $\begin{cases} 2.5x + 4y + 2z = 300 \\ x + y + z = 120 \\ -x + 2y + 2z = 0 \end{cases}$

(b) $\begin{bmatrix} 2.50 & 4 & 2 \\ 1 & 1 & 1 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300 \\ 120 \\ 0 \end{bmatrix}$

(c) 80 roses, 10 lilies, 30 irises

83. (a) $AB = \begin{bmatrix} 13 & 4 \\ 1 & 8 \end{bmatrix}$, $BA = \begin{bmatrix} 8 & 1 \\ 4 & 13 \end{bmatrix}$

$AB = C$ and in the product BA, the diagonal entries are the reverse of those in C.

(b) $C^{-1} = \begin{bmatrix} \frac{2}{25} & -\frac{1}{25} \\ -\frac{1}{100} & \frac{13}{100} \end{bmatrix}$

$A^{-1} \cdot B^{-1} = \begin{bmatrix} \frac{13}{100} & -\frac{1}{100} \\ -\frac{1}{25} & \frac{2}{25} \end{bmatrix}$

$B^{-1} \cdot A^{-1} = \begin{bmatrix} \frac{2}{25} & -\frac{1}{25} \\ -\frac{1}{100} & \frac{13}{100} \end{bmatrix}$

$C^{-1} = B^{-1} \cdot A^{-1}$

(c) The inverse of a product AB is equal to the inverse of B times the inverse of A.

85. Answers will vary. Sample answer:
If $k = 3$, then

$$\begin{bmatrix} 4 & 3 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}.$$

If $k = -\frac{3}{2}$, the matrix is singular.

87. True. The inverse of I_n is I_n. **89.** Answers will vary.

Quiz Yourself *(page 512)*

1. Any matrix with four rows and three columns

2. Any matrix with three rows and one column

3. $\left[\begin{array}{ccc|c} 3 & 2 & 0 & -2 \\ 5 & -1 & 0 & 19 \end{array}\right]$ **4.** $\left[\begin{array}{ccc|c} 1 & 0 & 3 & -5 \\ 1 & 2 & -1 & 3 \\ 3 & 0 & 4 & 0 \end{array}\right]$

5. $(2.769, -5.154)$ **6.** $(4, -2, -3)$ **7.** $\begin{bmatrix} -2 & 5 \\ 5 & 15 \end{bmatrix}$

8. $\begin{bmatrix} 9 & -2 & 23 \\ 14 & -4 & 36 \end{bmatrix}$ **9.** $\begin{bmatrix} -1 & 7 \\ -11 & 3 \end{bmatrix}$ **10.** $\begin{bmatrix} -3 & -1 \\ 3 & -2 \end{bmatrix}$

11. $\begin{bmatrix} 3 & -2 \\ 1 & -\frac{1}{2} \end{bmatrix}$ **12.** Not possible **13.** $\begin{bmatrix} -3 & 14 \\ -12 & 16 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -9 \\ 7 & -11 \end{bmatrix}$ **15.** \$22.80 **16.** \$41.40 **17.** \$59.60

18.

$$LW = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \end{array} \overset{\begin{array}{cc} \text{Plant 1} & \text{Plant 2} \end{array}}{\begin{bmatrix} \$22.80 & \$20.20 \\ \$47.80 & \$41.40 \\ \$66.50 & \$59.60 \end{bmatrix}} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{Model}$$

LW represents the total labor costs for each model at each plant.

19. $(4, -2)$ **20.** $(4, 2, -3)$

Section 6.4 *(page 518)*

Skills Warm Up *(page 518)*

1. $\begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} -2 & 8 \\ 2 & -4 \end{bmatrix}$

3. $\begin{bmatrix} 9 & 12 & 6 \\ 3 & 0 & -3 \\ 0 & 3 & 6 \end{bmatrix}$ **4.** $\begin{bmatrix} 0 & 8 & 12 \\ -4 & 8 & 12 \\ -8 & 4 & -8 \end{bmatrix}$

5. -22 **6.** 35 **7.** -15 **8.** $-\frac{1}{8}$

9. -45 **10.** -16

1. -5 **3.** 1 **5.** 3 **7.** 0 **9.** 5 **11.** 4

13. $\frac{11}{6}$ **15.** -2.6 **17.** 248 **19.** 0.14 **21.** -0.838

23. (a) $M_{11} = -5, M_{12} = 2, M_{21} = 4, M_{22} = 3$
(b) $C_{11} = -5, C_{12} = -2, C_{21} = -4, C_{22} = 3$

25. (a) $M_{11} = -4, M_{12} = -2, M_{21} = 1, M_{22} = 3$
(b) $C_{11} = -4, C_{12} = 2, C_{21} = -1, C_{22} = 3$

27. (a) $M_{11} = 3, M_{12} = -4, M_{13} = 1, M_{21} = 2, M_{22} = 2,$
$M_{23} = -4, M_{31} = -4, M_{32} = 10, M_{33} = 8$
(b) $C_{11} = 3, C_{12} = 4, C_{13} = 1, C_{21} = -2, C_{22} = 2,$
$C_{23} = 4, C_{31} = -4, C_{32} = -10, C_{33} = 8$

29. (a) $M_{11} = 30, M_{12} = 12, M_{13} = 11, M_{21} = -36,$
$M_{22} = 26, M_{23} = 7, M_{31} = -4,$
$M_{32} = -42, M_{33} = 12$
(b) $C_{11} = 30, C_{12} = -12, C_{13} = 11, C_{21} = 36,$
$C_{22} = 26, C_{23} = -7, C_{31} = -4,$
$C_{32} = 42, C_{33} = 12$

31. (a) -99 (b) -99 **33.** (a) -145 (b) -145

35. (a) 170 (b) 170 **37.** -58 **39.** -30 **41.** 0

43. 0 **45.** -0.002 **47.** 0 **49.** -108 **51.** 412

53. -126 **55.** -336

57. (a) -3 (b) -2 (c) $\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ (d) 6

59. (a) -8 (b) 0 (c) $\begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}$ (d) 0

61. (a) -21 (b) -19 (c) $\begin{bmatrix} 7 & 1 & 4 \\ -8 & 9 & -3 \\ 7 & -3 & 9 \end{bmatrix}$ (d) 399

63. (a) 2 (b) -6 (c) $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ (d) -12

65. 45; The determinant is the product of the matrix's diagonal entries.

67. 75; The determinant is the product of the matrix's diagonal entries.

69. Lower triangular; -6 **71.** Upper triangular; -16

73. Upper triangular; -168

75. Matrices will vary. The determinant of each matrix is the product of the entries on the main diagonal, which in this case equals 6.

77. Matrices will vary. The determinant of each matrix is the product of the entries on the main diagonal, which in this case equals -18.

79. Matrices will vary. The determinant of each matrix is the product of the entries on the main diagonal, which in this case equals 28.

81. -20 **83.** -18 **85.** 80

87–89. Answers will vary. **91.** ± 2 **93.** $\pm\frac{3}{2}$

95. $1 \pm \sqrt{2}$ **97.** $\frac{1}{2}, 1$ **99.** 3

101. Answers will vary. For an $n \times n$ matrix $(n > 2)$ with consecutive integer entries, the determinant appears to be 0.

103. True. If an entire row is zeros, then each cofactor in the expansion is multiplied by zero.

Section 6.5 *(page 527)*

Skills Warm Up *(page 527)*

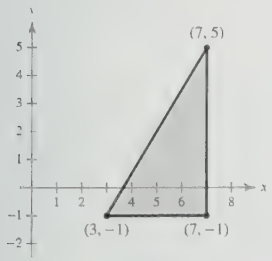
1. 1 **2.** 0 **3.** -8 **4.** x^2 **5.** 8 **6.** 60

7. $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 2 \\ 2 & 5 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 0.14 \\ 0.31 \end{bmatrix}$ **10.** $[7 \;\; 14]$

1. 11 **3.** 28 **5.** $\frac{33}{8}$ **7.** $\frac{5}{2}$ **9.** 28

11. $y = \frac{16}{5}$ or $y = 0$ **13.** $y = -3$ or $y = -11$
15. 307.5 square miles **17.** Collinear **19.** Not collinear
21. Collinear **23.** Not collinear **25.** Collinear
27. $y = -4$ **29.** $x = 3$ **31.** $x - 6y + 13 = 0$
33. $x + 3y - 5 = 0$ **35.** $y = 5$ **37.** $x = -4$
39. $2x + 3y - 8 = 0$
41. [3 15][13 5][0 8][15 13][5 0][19 15][15 14]
48, 81, 28, 51, 24, 40, 54, 95, 5, 10, 64, 113, 57, 100
43. [20 5 24][20 0 13][5 0 1][20 0 23] [15 18 11]
−119, 28, 67, −58, 6, 39, −1, −3, 3, −118, 26, 69, −33, 7, 15
45. 1, −25, −65, 17, 15, −9, −12, −62, −119, 27, 51, 48, 43, 67, 48, 57, 111, 117
47. 29, 53, 48, 60, 115, 120, 20, 38, 35, 7, −12, −51, 72, 148, 165, 62, 143, 181
49. 34, 55, 43, 60, 136, 169, 71, 139, 145, 52, 106, 119, 38, 52, 23
51. 40, 83, 92, −2, −30, −69, −16, −74, −137, 15, 28, 25, 6, −16, −58, 46, 79, 67
53. HAPPY NEW YEAR **55.** SOUND ALL CLEAR
57. SEND MORE MONEY **59.** MEET ME TONIGHT RON
61. Because $[45 \ -35]\begin{bmatrix} w & x \\ y & z \end{bmatrix} = [10 \ 15]$
and $[38 \ -30]\begin{bmatrix} w & x \\ y & z \end{bmatrix} = [8 \ 14]$,
you can solve $\begin{bmatrix} 45 & -35 \\ 38 & -30 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 8 & 14 \end{bmatrix}$;
JOHN RETURN TO BASE
63. 12 square units **65.** Answers will vary.



$A = \frac{1}{2}(4)(6) = 12$ square units

Review Exercises *(page 534)*

1. 1×2 **3.** 2×4
5. $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **9.** $(3, -2)$
11. $(3, 2, -1)$ **13.** $(10 - 4a, a, a)$ **15.** Inconsistent
17. \$40,000 was borrowed at 8%, \$120,000 was borrowed at 10%, and \$40,000 was borrowed at 12%.
19. $x = -6, y = 2$
21. (a) $\begin{bmatrix} 3 & 7 \\ -4 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & 3 \\ 8 & -2 \end{bmatrix}$
(c) $\begin{bmatrix} -4 & 20 \\ 8 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -13 & 21 \\ 22 & -1 \end{bmatrix}$
23. (a) $\begin{bmatrix} 7 & 1 \\ 3 & 6 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & -9 \\ -3 & -8 \\ 5 & -2 \end{bmatrix}$
(c) $\begin{bmatrix} 4 & -16 \\ 0 & -4 \\ 16 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} -7 & -30 \\ -6 & -19 \\ 22 & 2 \end{bmatrix}$
25. (a) $\begin{bmatrix} 3 & 4 & 2 & 1 \\ 3 & -5 & 6 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & -6 & 11 \\ -3 & 7 & 0 & 4 \end{bmatrix}$
(c) $\begin{bmatrix} 4 & 12 & -8 & 24 \\ 0 & 4 & 12 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 13 & -18 & 40 \\ -6 & 17 & 9 & 14 \end{bmatrix}$
27. $\begin{bmatrix} -12 & -3 \\ 5 & -5 \end{bmatrix}$ **29.** $[2 \ -1 \ -3 \ 5]$ **31.** $\begin{bmatrix} 4 & -11 \\ -3 & 1 \\ -1 & -3 \end{bmatrix}$
33. $\frac{1}{2}\begin{bmatrix} 3 & -5 \\ 1 & 4 \\ 9 & 14 \end{bmatrix}$ **35.** $\begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$ **37.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
39. (a) $[4]$ (b) $\begin{bmatrix} 2 & -6 & 8 \\ -2 & 6 & -8 \\ -1 & 3 & -4 \end{bmatrix}$ (c) Not possible
41. $\begin{bmatrix} 96 & 144 & 24 & 48 \\ 48 & 72 & 96 & 24 \\ 168 & 72 & 120 & 96 \end{bmatrix}$
43. (a) \$8325 (b) \$5200
(c) Wholesale Retail

$$ST = \begin{bmatrix} \$5200 & \$8265 \\ \$5075 & \$7985 \\ \$5125 & \$8325 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \} \text{Outlet}$$

ST represents the wholesale and retail values of the car sound system inventory at each outlet.
45. $AB = I$ and $BA = I$
47. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ **49.** $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ **51.** $(3, 4)$
53. $\left(2, \frac{1}{2}, 3\right)$ **55.** $(-6, -1)$
57. 10 units of fluid X, 8 units of fluid Y, 5 units of fluid Z
59. (a) $\begin{bmatrix} \frac{31}{35} & -\frac{27}{35} & \frac{1}{7} \\ -\frac{27}{35} & \frac{87}{70} & -\frac{2}{7} \\ \frac{1}{7} & -\frac{2}{7} & \frac{1}{14} \end{bmatrix}$
$y = -6.14t^2 + 109.6t + 818$
(b) 1213 stores (c) The estimate is too high.
61. 4 **63.** 0
65. (a) $M_{11} = 4, M_{12} = 7, M_{21} = -1, M_{22} = 2$
(b) $C_{11} = 4, C_{12} = -7, C_{21} = 1, C_{22} = 2$
67. (a) $M_{11} = 30, M_{12} = -12, M_{13} = -21,$
$M_{21} = 20, M_{22} = 19, M_{23} = 22,$
$M_{31} = 5, M_{32} = -2, M_{33} = 19$
(b) $C_{11} = 30, C_{12} = 12, C_{13} = -21,$
$C_{21} = -20, C_{22} = 19, C_{23} = -22,$
$C_{31} = 5, C_{32} = 2, C_{33} = 19$
69. 44; Answers will vary. **71.** −12; Answers will vary.
73. −39; Answers will vary.
75. (a) 8 (b) −12 (c) $\begin{bmatrix} -2 & -6 \\ -14 & 6 \end{bmatrix}$ (d) −96
77. 10 **79.** 10 **81.** Not collinear **83.** Collinear
85. $x + 15y - 38 = 0$ **87.** $x = 2$
89. [20 18][1 14][19 13][9 20][0 14][15 23]
94, 132, 44, 59, 77, 109, 78, 107, 42, 56, 99, 137

91. SEIZE THE DAY

93. (a) Because $\begin{bmatrix} -57 & -13 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 23 & 5 \end{bmatrix}$ and

$\begin{bmatrix} 91 & 26 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 13 \end{bmatrix}$, you can solve

$$\begin{bmatrix} -57 & -13 \\ 91 & 26 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 23 & 5 \\ 0 & 13 \end{bmatrix}.$$

(b) WE MISS YOU BIG BOB

Test Yourself *(page 538)*

1. $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 2 \\ 1 & 4 & -1 & 0 \\ 1 & 3 & 3 & -1 \end{array}\right]$; 3×4

2. $\left[\begin{array}{ccc|c} 3 & 4 & 2 & 4 \\ 2 & 3 & 0 & -2 \\ 0 & 2 & -3 & -13 \end{array}\right]$; 3×4 **3.** $\left(\frac{7}{3}a - \frac{10}{3}, -\frac{8}{3}a + \frac{29}{3}, a\right)$

4. $(-12, -16, -6)$ **5.** $(2, -3, 1)$ **6.** $\begin{bmatrix} 2 & 4 \\ 7 & 13 \end{bmatrix}$

7. $\begin{bmatrix} -4 & -8 \\ 13 & 29 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$ **9.** $\begin{bmatrix} -6 & -10 \\ 15 & 19 \end{bmatrix}$

10. $\begin{bmatrix} 8 & -7 \\ 11 & 19 \end{bmatrix}$ **11.** $\begin{bmatrix} -32 & 20 \\ -32 & -56 \end{bmatrix}$ **12.** $\begin{bmatrix} -5 & -25 \\ 5 & 5 \end{bmatrix}$

13. $\begin{bmatrix} -12 & 16 \\ 1 & 3 \end{bmatrix}$ **14.** $\frac{1}{5}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ **15.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

16. $\frac{1}{5}\begin{bmatrix} -9 & 16 & -6 \\ 6 & -9 & 4 \\ -4 & 6 & 1 \end{bmatrix}$ **17.** -17 **18.** -23 **19.** -20

20. $(2, -2, 3)$

21. Matrices will vary. Sample answer:

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

22. $\frac{21}{2}$ square units **23.** Collinear **24.** $-x + y + 3 = 0$

25. $BA = \begin{bmatrix} \$384{,}000 & \$631{,}000 \end{bmatrix}$

BA represents the total value of each product at each warehouse.

Chapter 7

Section 7.1 *(page 547)*

Skills Warm Up *(page 547)*

1. $\frac{1}{5}$ **2.** $\frac{1}{5}$ **3.** -36 **4.** 8 **5.** -5
6. 10 **7.** $\frac{1}{3}$ **8.** 24 **9.** $\frac{13}{24}$ **10.** $\frac{3}{4}$

1. $-5, -3, -1, 1, 3$ **3.** $2, 4, 8, 16, 32$

5. $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}$ **7.** $\frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}$

9. $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}$ **11.** $\frac{5}{3}, \frac{17}{9}, \frac{53}{27}, \frac{161}{81}, \frac{485}{243}$ **13.** $0, 1, 2, 3, 4$

15. $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$ **17.** $5, \frac{19}{4}, \frac{43}{9}, \frac{77}{16}, \frac{121}{25}$

19. $a_n = 3n - 2$ **21.** $a_n = (-1)^{n+1}(2n)$ **23.** $a_n = \frac{2}{n^2}$

25. $a_n = \frac{n+1}{n+2}$ **27.** $a_n = \frac{(-1)^n 2^{n-1}}{3^n}$ **29.** $a_n = 1 + \frac{1}{n}$

31. $3, \frac{9}{2}, \frac{9}{2}, \frac{27}{8}, \frac{81}{40}$ **33.** $3, 6, 9, 12, 15$ **35.** -73

37. $\frac{44}{239}$ **39.** $\frac{4}{14{,}175}$ **41.** a **42.** d **43.** c **44.** b

45. 8 **47.** 19,656 **49.** $(n + 1)(n + 2)$

51. $n + 1$; 1, 2 **53.** $\frac{1}{2n+1}$; $1, \frac{1}{3}$ **55.** $a_n = n!$

57. $a_n = \frac{2^{n-1}}{(n-1)!}$ **59.** 48 **61.** -84 **63.** 72

65. 30 **67.** $\frac{29}{40}$ **69.** 14 **71.** 4 **73.** $\frac{65}{24}$ **75.** 504

77. $\sum_{i=1}^{9} \frac{1}{3i}$ **79.** $\sum_{i=1}^{8} \left[2\left(\frac{i}{8}\right) + 3\right]$ **81.** $\sum_{i=1}^{6} (-1)^{i+1} 3^i$

83. $\sum_{i=1}^{20} \frac{(-1)^{i+1}}{i^2}$ **85.** $\sum_{i=1}^{5} \frac{2^i - 1}{2^{i+1}}$

87. (a) $A_1 = \$25{,}145.83$
$A_2 = \$25{,}292.52$
$A_3 = \$25{,}440.06$
$A_4 = \$25{,}588.46$
$A_5 = \$25{,}737.72$
$A_6 = \$25{,}887.86$

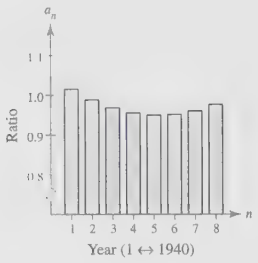
(b) $\$35{,}440.63$

(c) No; $A_{120} = \$50{,}241.53$

89. (a) $a_1 = 1.0156$
$a_2 = 0.9884$
$a_3 = 0.9684$
$a_4 = 0.9556$
$a_5 = 0.95$
$a_6 = 0.9516$
$a_7 = 0.9604$
$a_8 = 0.9764$

Each term represents the number of males divided by the number of females in the United States for the corresponding year.

(b)



In 1940, there were more males than females. The ratio decreased until 1980, and increased thereafter.

(c) 143.3 million females, 137.7 million males

91. \$230.51 million **93.** True by the properties of sums

95. $x, \frac{x^2}{2}, \frac{x^3}{6}, \frac{x^4}{24}, \frac{x^5}{120}$

97. $-\frac{x^2}{2}, \frac{x^4}{24}, -\frac{x^6}{720}, \frac{x^8}{40{,}320}, -\frac{x^{10}}{3{,}628{,}800}$

Section 7.2 *(page 555)*

Skills Warm Up *(page 555)*

1. 36 **2.** 240 **3.** $\frac{11}{2}$ **4.** $\frac{10}{3}$ **5.** 18
6. 4 **7.** 143 **8.** 160 **9.** 430 **10.** 256

1. Arithmetic sequence; $d = 4$

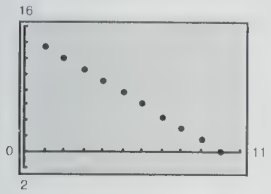
3. Arithmetic sequence; $d = 4$

5. Not an arithmetic sequence

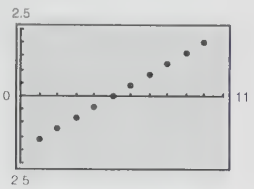
7. Arithmetic sequence; $d = -\frac{1}{2}$

9. Arithmetic sequence; $d = 1$

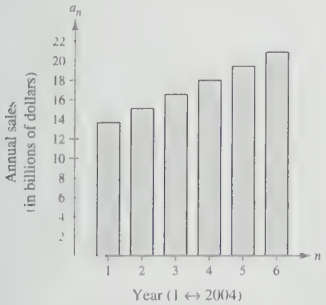
11. Not an arithmetic sequence
13. 8, 11, 14, 17, 20; Arithmetic sequence, $d = 3$
15. 2, 8, 24, 64, 160; Not an arithmetic sequence
17. 1, 1, 1, 1, 1; Arithmetic sequence, $d = 0$
19. Undefined, 2, 1, $\frac{2}{3}$, $\frac{1}{2}$; Not an arithmetic sequence
21. $-1, 1, -1, 1, -1$; Not an arithmetic sequence
23. d **24.** a **25.** b **26.** c
27.



29.



31. $a_n = 5n - 1$ **33.** $a_n = -12n + 62$
35. $a_n = 8n + 1$ **37.** $a_n = -4n - 3$ **39.** $a_n = 6n - 4$
41. $a_n = \frac{3}{4}n + \frac{5}{4}$ **43.** (a) $a_8 = 33$ (b) $a_1 = 5$
45. (a) $a_{13} = -16$ (b) $a_1 = 20$
47. (a) $a_{15} = 15\frac{2}{3}$ (b) $a_1 = 11$ **49.** 2, 6, 10, 14, 18
51. 54, 50, 46, 42, 38 **53.** 0.1, 0.7, 1.3, 1.9, 2.5
55. 110 **57.** -25 **59.** 5050 **61.** 3030 **63.** 216
65. 12 **67.** 23 **69.** 4000 **71.** 1275 **73.** 22,770
75. 44,625 **77.** 670 **79.** 2725
81. Answers will vary. Sample answer: $\sum_{n=1}^{100} 3n$
83. Answers will vary. Sample answer: $\sum_{n=1}^{10} \frac{n+1}{4}$
85. (a) \$43,750 (b) \$236,250
87. 780 seats **89.** 405 bricks
91. (a) (b) \$103.8 billion



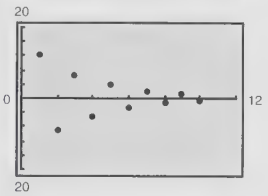
93. \$350,000; Answers will vary. **95.** 490 meters

Section 7.3 *(page 564)*

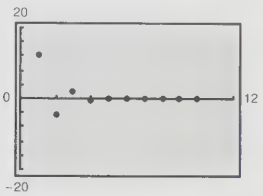
Skills Warm Up *(page 564)*

1. $\frac{64}{125}$ **2.** $\frac{9}{16}$ **3.** $\frac{1}{16}$ **4.** 324 **5.** $6n^3$
6. $27n^4$ **7.** $4n^3$ **8.** n^2
9. $\sum_{n=1}^{4} 2(3)^{n-1}$ **10.** $\sum_{n=1}^{4} 3(2)^{n-1}$

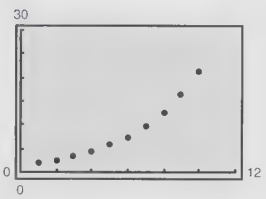
1. 5 **3.** -2 **5.** $\frac{1}{6}$ **7.** 3
9. Geometric sequence; $r = 4$; $a_n = 4^{n-1}$
11. Geometric sequence; $r = -3$; $a_n = 4(-3)^{n-1}$
13. Geometric sequence; $r = -\frac{1}{2}$; $a_n = \left(-\frac{1}{2}\right)^{n-1}$
15. Not a geometric sequence **17.** Not a geometric sequence
19. Geometric sequence; $r = \frac{3}{2}$; $a_n = 4\left(\frac{3}{2}\right)^{n-1}$
21. 4, 8, 16, 32, 64 **23.** $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$
25. $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}, -\frac{1}{54}, \frac{1}{162}$ **27.** 200, 100, 50, 25, 12.5
29. $1, e, e^2, e^3, e^4$ **31.** $16\left(\frac{1}{4}\right)^4 = \frac{1}{16}$
33. $6\left(-\frac{1}{3}\right)^{11} = -\frac{2}{3^{10}}$ **35.** $100e^8$
37. $500(1.02)^{39} \approx 1082.37$ **39.** $a_n = 7(2)^{n-1}$; 448
41. $a_n = 4(3)^{n-1}$; 19,131,876
43. $a_n = 6(5)^{n-1}$; 1.14×10^{14}
45. $a_n = 1\left(\frac{1}{2}\right)^{n-1}$; $\frac{1}{128}$
47. $-40\left(\frac{1}{2}\right) = -20$ or $-40\left(-\frac{1}{2}\right) = 20$
49. $54\left(-\frac{1}{3}\right)^5 = -\frac{2}{9}$ **51.** a **52.** c **53.** b **54.** d
55.



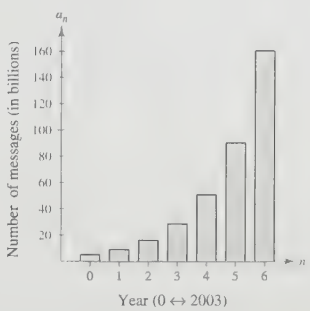
57.



59.



61. 16,368 **63.** 511
65. $\frac{349,525}{32,768} \approx 10.67$ **67.** 43 **69.** $\frac{8}{5}$ **71.** $\approx 29,921.31$
73. $\approx -9.5 \times 10^{15}$ **75.** $\sum_{n=1}^{7} 5(3)^{n-1}$ **77.** $\sum_{n=1}^{7} 2\left(-\frac{1}{4}\right)^{n-1}$
79. $\sum_{n=1}^{6} 0.1(4)^{n-1}$ **81.** 2 **83.** $\frac{2}{3}$
85. Sum does not exist; $|r| \geq 1$ **87.** $\frac{10}{3}$ **89.** $\frac{4}{11}$ **91.** $\frac{7}{22}$
93. (a) No, $S_{48} = \$16,310.50$. (b) Yes, $S_{48} = \$17,385.63$.
95. \$18,416.57; \$18,444.46 **97.** \$74.1 billion
99. (a)



(b) 360.1 billion text messages
101. \$5,368,709.11 for 29 days
\$10,737,418.23 for 30 days
\$21,474,836.47 for 31 days
The job for which the wage doubles daily pays much more.

Quiz Yourself *(page 567)*

1. $-1, 1, 3, 5, 7$ **2.** $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}$ **3.** 20 **4.** 24,360
5. $a_n = \frac{3}{n^2}$ **6.** $a_n = \frac{1}{n+1}$ **7.** $a_n = 5n - 3$
8. $a_n = 3n + 4$ **9.** $a_n = 10(3)^{n-1}$ **10.** $a_n = 4\left(-\frac{1}{2}\right)^{n-1}$
11. 99 **12.** $a_8 = 203$; $a_1 = 7$ **13.** 33.71 **14.** $\frac{100}{2187}$
15. Arithmetic; Common difference is 3.
16. Neither; No common ratio, no common difference

CHAPTER 7

17. Geometric: Common ratio is $\frac{1}{5}$.
18. Geometric: Common ratio is 2. **19.** 45 **20.** 60
21. 1225 **22.** 126 **23.** $\frac{5}{4}$ **24.** $\frac{25}{4}$
25. (a) \$58,200 (b) \$320,100 **26.** \$73,599.44

Section 7.4 *(page 573)*

Skills Warm Up *(page 573)*

1. $5x^3 + 15x^2$ 2. $x^3 + 5x^2 - 3x - 15$
3. $x^2 + 8x + 16$ 4. $4x^2 - 12x + 9$ 5. $\frac{3x^3}{x}$
6. $32x^5$ 7. 120 8. 336 9. 720 10. 20

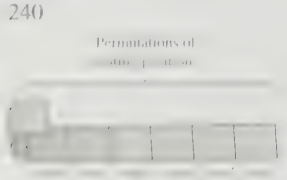
1. 15 **3.** 210 **5.** 1 **7.** 15,504 **9.** 1
11. 4950 **13.** 6 **15.** 35 **17.** 56 **19.** 45
21. $x^4 + 4x^3 + 6x^2 + 4x + 1$
23. $x^4 + 24x^3 + 216x^2 + 864x + 1296$
25. $x^4 - 8x^3 + 24x^2 - 32x + 16$
27. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
29. $81s^4 + 216s^3 + 216s^2 + 96s + 16$
31. $64s^6 - 576s^5 + 2160s^4 - 4320s^3 + 4860s^2 - 2916s + 729$
33. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$
35. $1 - 6x + 12x^2 - 8x^3$
37. $x^{10} - 5x^8y^2 + 10x^6y^4 - 10x^4y^6 + 5x^2y^8 - y^{10}$
39. $\frac{1}{x^5} + \frac{5y}{x^4} + \frac{10y^2}{x^3} + \frac{10y^3}{x^2} + \frac{5y^4}{x} + y^5$ **41.** -4
43. $-2035 + 828i$ **45.** 1
47. $32t^5 - 80t^4 + 80t^3 - 40t^2 + 10t - 1$
49. $81 + 216z + 216z^2 + 96z^3 + 16z^4$ **51.** $120x^7y^3$
53. $360x^3y^2$ **55.** $1,259,712x^2y^7$ **57.** $-4,330,260,000x^3y^9$
59. 8064 **61.** 180 **63.** 489,888 **65.** -56
67. $\frac{1}{128} + \frac{7}{128} + \frac{21}{128} + \frac{35}{128} + \frac{35}{128} + \frac{21}{128} + \frac{7}{128} + \frac{1}{128}$
69. $0.07776 + 0.2592 + 0.3456 + 0.2304 + 0.0768 + 0.01024$
71. Fibonacci sequence
73. The determinants are 1, 3, 6, 10, 15, . . . , which constitute one of the diagonals of Pascal's Triangle.
75. Answers will vary.

Section 7.5 *(page 582)*

Skills Warm Up *(page 582)*

1. 6656 2. 291,600 3. 7920 4. 13,800
5. 792 6. 2300 7. $n(n-1)(n-2)(n-3)$
8. $n(n-1)(2n-1)$ 9. $n!$ 10. $n!$

1. 8 **3.** 6 **5.** 3 **7.** 11 **9.** 10 **11.** 30
13. 240 **15.** 6,760,000 **17.** 1024



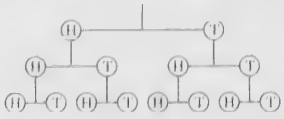
19. (a) 900 (b) 648 (c) 180
21. 64,000 **23.** 120
25. (a) 40,320 (b) 384 **27.** 24 **29.** 24 **31.** 30
33. 1 **35.** 11,880 **37.** 27,907,200 **39.** 197,149,680
41. 60 **43.** 120 **45.** 1260 **47.** 2520 **49.** 4
51. 1 **53.** 15 **55.** 1 **57.** 495 **59.** 15,504
61. 850,668 **63.** $n(n-1)(n-2)$ **65.** $n!$
67. Because order matters, use permutations.
69. Because order does not matter, use combinations.
71. 32,760 **73.** 142,506 **75.** 575,757 **77.** 3,921,225
79. 70 **81.** 3744 **83.** 292,600
85. For some calculators, the answer is too large.
87. Answers will vary.

Section 7.6 *(page 592)*

Skills Warm Up *(page 592)*

1. $\frac{9}{16}$ 2. $\frac{8}{15}$ 3. $\frac{1}{6}$ 4. $\frac{1}{80,730}$
5. $\frac{1}{804,722,688,000}$ 6. $\frac{1}{24}$ 7. $\frac{1}{12}$ 8. $\frac{135}{323}$
9. 0.366 **10.** 0.997

1. $S = \{H1, T1, H2, T2, H3, T3, H4, T4, H5, T5, H6, T6\}$
3. $S = \{ABC, ACB, BAC, BCA, CAB, CBA\}$
5. $S = \{AB, AC, AD, AE, BC, BD, BE, CD, CE, DE\}$
7.



$\{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT\}$

9.



$\{AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC\}$

11. $\frac{1}{2}$ **13.** $\frac{3}{10}$ **15.** $\frac{3}{8}$ **17.** $\frac{7}{8}$ **19.** $\frac{1}{9}$ **21.** $\frac{7}{12}$
23. $\frac{1}{3}$ **25.** $\frac{1}{2}$ **27.** $\frac{10}{13}$ **29.** $\frac{6}{13}$ **31.** $\frac{5}{9}$ **33.** $\frac{1}{36}$
35. $\frac{5}{12}$ **37.** $\frac{3}{100}$ **39.** $\frac{9}{25}$ **41.** $\frac{4}{13}$ **43.** $\frac{4}{13}$ **45.** $\frac{9}{16}$
47. $\frac{3}{32}$ **49.** $\frac{1}{8}$ **51.** $\frac{2}{125}$ **53.** 0.25 **55.** $\frac{1}{3}$ **57.** 0.88
59. $\frac{7}{20}$ **61.** 0.22 **63.** (a) $\frac{139}{198}$ (b) $\frac{59}{198}$ (c) $\frac{1}{33}$
65. (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{9}{100}$ (d) $\frac{1}{30}$ **67.** (a) $\frac{1}{16}$ (b) $\frac{1}{17}$
69. (a) $\frac{1}{720}$ (b) $\frac{1}{120}$ (c) $\frac{1}{24}$ **71.** (a) $\frac{1}{2}$ (b) $\frac{2}{3}$
73. $\frac{1}{64,974}$ **75.** (a) $\frac{14}{55}$ (b) $\frac{12}{55}$ (c) $\frac{54}{55}$
77. (a) $\frac{1}{81}$ (b) $\frac{16}{81}$ (c) $\frac{65}{81}$ **79.** $\frac{7}{32}$
81. $\frac{76,545}{524,288} \approx 0.1459980011$
83. Mutually exclusive events have no outcomes in common. Two events are independent when the occurrence of one event has no effect on the occurrence of the other event.
85. (a) 0.1089 (b) 0.2601

Review Exercises *(page 600)*

1. 5, 9, 13, 17, 21 **3.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ **5.** $1, -\frac{4}{3}, 2, -\frac{16}{5}, \frac{16}{3}$
7. $a_n = 2^{n-1}$ **9.** $a_n = \frac{1}{n}$ **11.** $a_n = (-1)^{n+1}3^n$
13. 43,680 **15.** $n + 5$ **17.** 51 **19.** 100

21. $\sum_{k=1}^{12} \frac{3}{1+k}$ **23.** $\sum_{k=1}^{13} (-1)^{k-1}2^k$

25. (a) $A_1 = \$4060.00, A_2 = \$4120.90, A_3 = \$4182.71,$
$A_4 = \$4245.45, A_5 = \$4309.14, A_6 = \$4373.77,$
$A_7 = \$4439.38, A_8 = \$4505.97$
(b) $A_{40} = \$7256.07$
(c) $A_{120} = \$23,877.29$

27. Arithmetic; $d = 5$ **29.** Arithmetic; $d = -\frac{1}{2}$

31. 9, 14, 19, 24, 29 **33.** $-6, -1, 4, 9, 14$

35. 0, 2, 4, 6, 8 **37.** $a_n = 3n - 1$ **39.** $a_n = 5n - 3$

41. $a_n = 5n + 2$ **43.** 192 **45.** 765

47. (a) Company A: \$40,500
Company B: \$40,500
(b) \$219,000 (c) \$205,500

49. Geometric; $r = -3; a_n = (-3)^{n-1}$

51. Geometric; $r = \frac{1}{2}; a_n = 16\left(\frac{1}{2}\right)^{n-1}$ **53.** 3, 6, 12, 24, 48

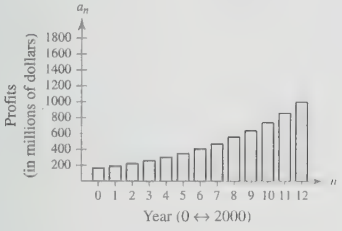
55. $10, -2, \frac{2}{5}, -\frac{2}{25}, \frac{2}{125}$ **57.** $a_{40} = \frac{1}{2^{36}}$ **59.** $a_{10} = -\frac{25}{64}$

61. 8184 **63.** ≈ 22.2 **65.** 1.5

67. The sum does not exist; $\left|\frac{5}{4}\right| > 1$.

69. (a) \$16,469.87
(b) No; it will be more because the interest is compounded.

71. (a)



(b) \$6157.79 million

73. 35 **75.** 1 **77.** 792

79. $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

81. $243x^5 + 405x^4y + 270x^3y^2 + 90x^2y^3 + 15xy^4 + y^5$

83. $x^8 - 16x^7y + 112x^6y^2 - 448x^5y^3 + 1120x^4y^4$
$- 1792x^3y^5 + 1792x^2y^6 - 1024xy^7 + 256y^8$

85. $90a^3b^2$ **87.** 25,344 **89.** 210

91. (a) 7000 (b) 3528 (c) 4500 **93.** 120

95. 504 **97.** 40,320 **99.** 1,663,200

101. (a) 42,504 (b) 1,961,256 (c) 134,596

103. $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **105.** $\frac{3}{13}$ **107.** $\frac{8}{85}$

109. $\frac{73}{85}$ **111.** $\frac{7}{13}$ **113.** 0.43 **115.** $\frac{1}{8}$

117. (a) $\frac{5}{12}$ (b) $\frac{7}{12}$ (c) $\frac{5}{36}$ **119.** $\frac{198}{4165}$

121. (a) $\frac{1}{3}$ (b) $\frac{5}{8}$ **123.** (a) $\frac{1}{32}$ (b) $\frac{1}{16}$ (c) $\frac{31}{32}$

Test Yourself *(page 604)*

1. 3, 5, 7, 9, 11 **2.** $-1, 4, -9, 16, -25$

3. 2, 6, 24, 120, 720 **4.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

5. Arithmetic; $d = 5$ **6.** Neither **7.** Geometric; $r = 2$

8. Neither **9.** $a_n = 8n - 3; a_{20} = 157$ **10.** 10

11. $a_n = 4(7)^{n-1}; a_8 = 3,294,172$ **12.** $a_{18} \approx 20.22$

13. 2600 **14.** $\approx 29,918.311$ **15.** $\frac{1}{2}$ **16.** \$13,176.08

17. $32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$

18. $-192,456$ **19.** (a) 72 (b) 328,440

20. (a) 330 (b) 8,936,928 **21.** 60 **22.** 5040

23. 640 **24.** 16,777,216 **25.** 45,000 **26.** 40,320

27. 220 **28.** $\frac{1}{216}$

Appendix B

Appendix B.1 *(page A16)*

Skills Warm Up *(page A16)*

1. $9x^2 + 16y^2 = 144$ **2.** $x^2 + 4y^2 = 32$

3. $16x^2 - y^2 = 4$ **4.** $243x^2 + 4y^2 = 9$

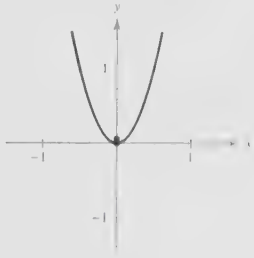
5. $c = 2\sqrt{2}$ **6.** $c = \sqrt{13}$ **7.** $c = 2\sqrt{3}$

8. $c = \sqrt{5}$ **9.** 4 **10.** 2

1. a **2.** c **3.** d **4.** h

5. g **6.** f **7.** e **8.** b

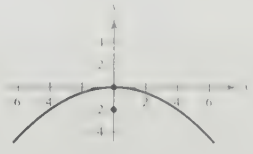
9. Vertex: (0, 0)
Focus: $\left(0, \frac{1}{16}\right)$



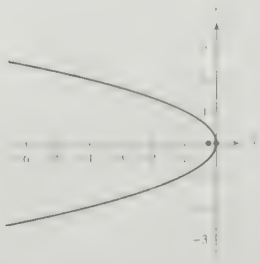
11. Vertex: (0, 0)
Focus: $\left(-\frac{3}{2}, 0\right)$



13. Vertex: (0, 0)
Focus: (0, −2)



15. Vertex: (0, 0)
Focus: $\left(-\frac{1}{4}, 0\right)$

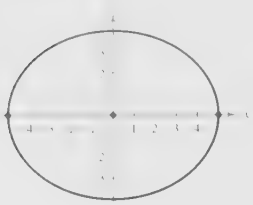


17. $x^2 = -6y$ **19.** $y^2 = -8x$ **21.** $x^2 = 4y$

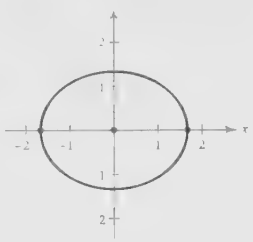
23. $y^2 = -12x$ **25.** $y^2 = 9x$ **27.** $\frac{x^2}{1} + \frac{y^2}{4} = 1$

29. $\frac{x^2}{25} + \frac{y^2}{21} = 1$ **31.** $\frac{x^2}{36} + \frac{y^2}{11} = 1$ **33.** $\frac{21x^2}{400} + \frac{y^2}{25} = 1$

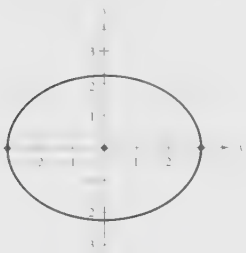
35. Center: $(0, 0)$
Vertices: $(\pm 5, 0)$



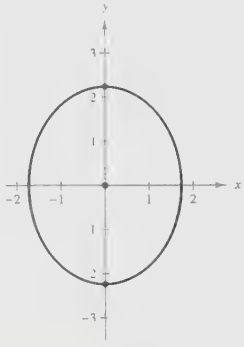
37. Center: $(0, 0)$
Vertices: $\left(\pm \frac{5}{3}, 0\right)$



39. Center: $(0, 0)$
Vertices: $(\pm 3, 0)$

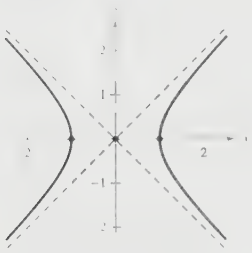


41. Center: $(0, 0)$
Vertices: $(0, \pm\sqrt{5})$

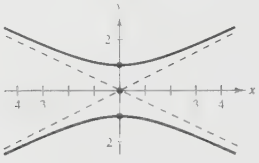


43. $\frac{y^2}{4} - \frac{x^2}{12} = 1$ **45.** $\frac{x^2}{36} - \frac{y^2}{45} = 1$

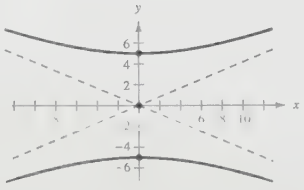
47. Center: $(0, 0)$
Vertices: $(\pm 1, 0)$



49. Center: $(0, 0)$
Vertices: $(0, \pm 1)$



51. Center: $(0, 0)$
Vertices: $(0, \pm 5)$



53. Center: $(0, 0)$
Vertices: $(\pm\sqrt{3}, 0)$

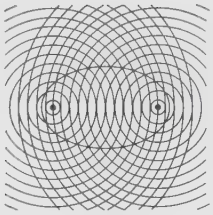


55. $\frac{x^2}{1} - \frac{y^2}{9} = 1$ **57.** $\frac{y^2}{16/5} - \frac{x^2}{64/5} = 1$

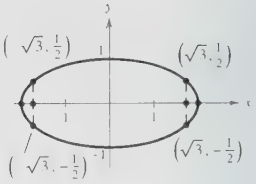
59. $\frac{y^2}{9} - \frac{x^2}{9/4} = 1$ **61.** $x^2 = 12y$

63. The tacks should be placed $\frac{3}{2}$ feet to the left and right of the center along the base. The string should be 5 feet long.

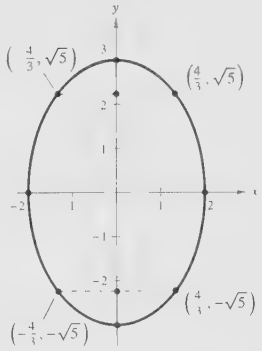
65.



67.



69.



71. The ellipse is nearly circular because the foci are closer to the center than to the vertices.

73. $\left(\sqrt{216 - 72\sqrt{5}}, 0\right) \approx (7.42, 0)$

Appendix B.2 *(page A24)*

Skills Warm Up *(page A24)*

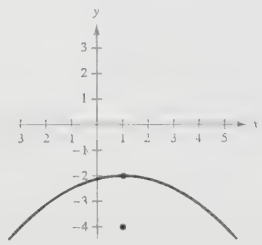
1. Hyperbola **2.** Ellipse **3.** Parabola
4. Ellipse **5.** Circle **6.** Hyperbola
7. Parabola **8.** Parabola **9.** Ellipse
10. Hyperbola

1. Shifted one unit up and two units to the left

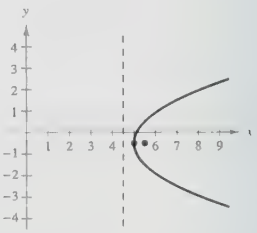
3. Shifted three units down and one unit to the right

5. Shifted two units down and one unit to the right

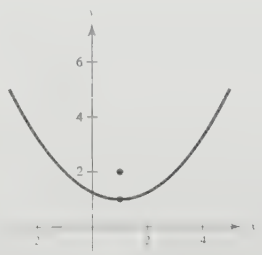
7. Vertex: $(1, -2)$
Focus: $(1, -4)$
Directrix: $y = 0$



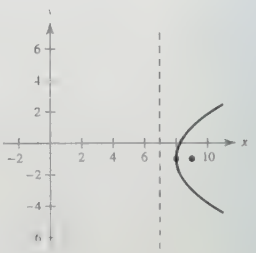
9. Vertex: $\left(5, -\frac{1}{2}\right)$
Focus: $\left(\frac{11}{2}, -\frac{1}{2}\right)$
Directrix: $x = \frac{9}{2}$



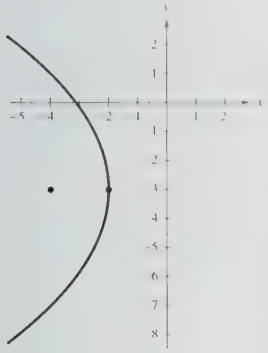
11. Vertex: $(1, 1)$
Focus: $(1, 2)$
Directrix: $y = 0$



13. Vertex: $(8, -1)$
Focus: $(9, -1)$
Directrix: $x = 7$

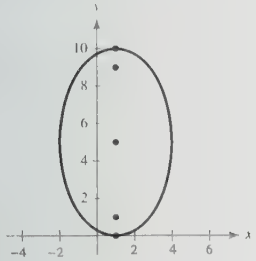


15. Vertex: $(-2, -3)$
Focus: $(-4, -3)$
Directrix: $x = 0$

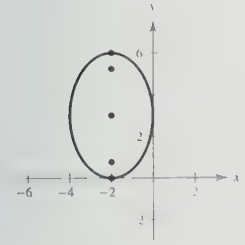


17. $(y - 2)^2 = 4(-2)(x - 3)$ **19.** $x^2 = 4(2)(y - 4)$
21. $(y - 2)^2 = 4(2)x$ **23.** $x^2 = 4\left(-\frac{1}{4}\right)(y - 4)$

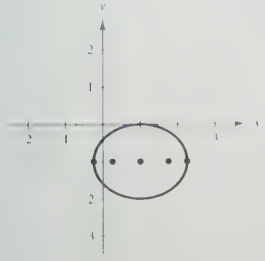
25. Center: $(1, 5)$
Foci: $(1, 9), (1, 1)$
Vertices: $(1, 10), (1, 0)$



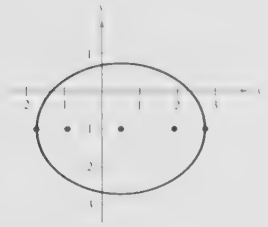
27. Center: $(-2, 3)$
Foci: $\left(-2, 3 - \sqrt{5}\right), \left(-2, 3 + \sqrt{5}\right)$
Vertices: $(-2, 0), (-2, 6)$



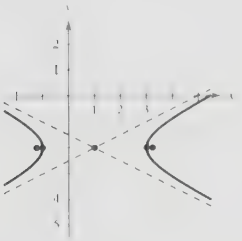
29. Center: $(1, -1)$
Foci: $\left(\frac{1}{4}, -1\right), \left(\frac{7}{4}, -1\right)$
Vertices: $\left(-\frac{1}{4}, -1\right), \left(\frac{9}{4}, -1\right)$



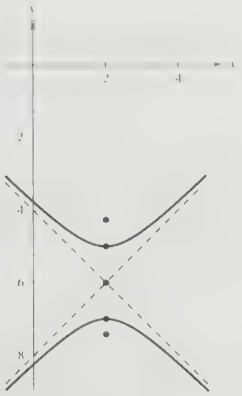
31. Center: $\left(\frac{1}{2}, -1\right)$
Foci: $\left(\frac{1}{2} - \sqrt{2}, -1\right), \left(\frac{1}{2} + \sqrt{2}, -1\right)$
Vertices: $\left(\frac{1}{2} - \sqrt{5}, -1\right), \left(\frac{1}{2} + \sqrt{5}, -1\right)$



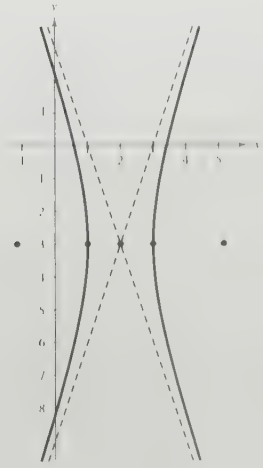
33. Center: $(1, -2)$
Vertices: $(-1, -2), (3, -2)$
Foci: $\left(1 - \sqrt{5}, -2\right), \left(1 + \sqrt{5}, -2\right)$



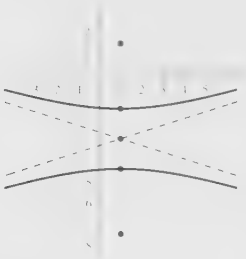
35. Center: $(2, -6)$
Vertices: $(2, -7), (2, -5)$
Foci: $\left(2, -6 - \sqrt{2}\right), \left(2, -6 + \sqrt{2}\right)$



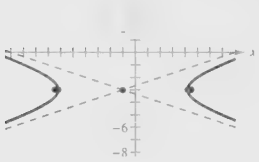
37. Center: $(2, -3)$
Vertices: $(3, -3), (1, -3)$
Foci: $\left(2 - \sqrt{10}, -3\right), \left(2 + \sqrt{10}, -3\right)$



39. Center: $(1, -3)$
Vertices: $(1, -3 - \sqrt{2}), (1, -3 + \sqrt{2})$
Foci: $(1, -3 - 2\sqrt{5}), (1, -3 + 2\sqrt{5})$



41. Center: $(-1, -3)$
Vertices: $(-1 - 3\sqrt{3}, -3), (-1 + 3\sqrt{3}, -3)$
Foci: $(-1 - \sqrt{30}, -3), (-1 + \sqrt{30}, -3)$



43. $\dfrac{(x-2)^2}{4} + \dfrac{(y-2)^2}{1} = 1$ **45.** $\dfrac{(x-2)^2}{16} + \dfrac{y^2}{12} = 1$

47. $(x \quad 2)^2 + \dfrac{4(y+1)^2}{9} \quad 1$

49. $\dfrac{(x-3)^2}{36} + \dfrac{(y-2)^2}{32} = 1$ **51.** $\dfrac{(x-5)^2}{25} + \dfrac{(y-6)^2}{36} \quad 1$

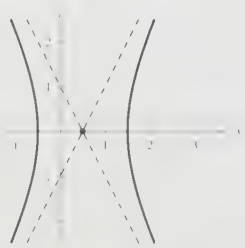
53. $\dfrac{(x-4)^2}{4} - \dfrac{y^2}{12} = 1$ **55.** $\dfrac{(y-5)^2}{16} - \dfrac{(x \quad 4)^2}{9} \quad 1$

57. $\dfrac{y^2}{9} - \dfrac{(x-2)^2}{9/4} = 1$ **59.** $\dfrac{(x-3)^2}{9} - \dfrac{(y-2)^2}{4} = 1$

61. $\dfrac{(x-3)^2}{4} + \dfrac{(y+2)^2}{4} = 1$ **63.** $\left(x \quad \dfrac{1}{2}\right)^2 - \dfrac{y^2}{4} = 1$

Circle Hyperbola

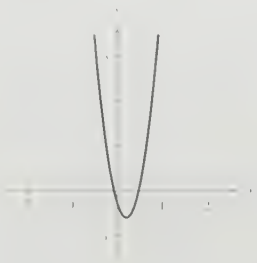




65. $\dfrac{(x+1)^2}{1/4} + \dfrac{(y \quad 4)^2}{1/3} = 1$ **67.** $\left(x - \dfrac{1}{5}\right)^2 \quad 8\left(y + \dfrac{3}{5}\right)$

Ellipse Parabola





69. (a) $\approx 24{,}748.74$ miles per hour
(b) $x^2 = -16{,}400(y - 4100)$

71. $\dfrac{x^2}{25} + \dfrac{y^2}{16} \quad 1$

73. Perihelion: $\approx 9.138 \times 10^7$ miles
Aphelion: $\approx 9.454 \times 10^7$ miles

75. $e \approx 5.414 \times 10^{-2}$

77. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{a^2 - c^2} = 1$

$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{(a^2/a^2)(a^2 - c^2)} = 1$

$\dfrac{(x \quad h)^2}{a^2} + \dfrac{(y-k)^2}{a^2(1 - c^2/a^2)} = 1$

$\dfrac{(x \quad h)^2}{a^2} + \dfrac{(y-k)^2}{a^2[1 - (c/a)^2]} \quad 1$

$\dfrac{(x \quad h)^2}{a^2} + \dfrac{(y \quad k)^2}{a^2(1 \quad e^2)} \quad 1$

79. $\dfrac{x^2}{4.88} + \dfrac{y^2}{1.39} = 1$

Answers to Checkpoints

Chapter 0

Checkpoints for Section 0.1

1 $x \geq 7$ denotes all real numbers greater than or equal to 7.
2 (a) $x \geq 5$ (b) $4 < y \leq 11$ (c) $-1 \leq z \leq 10$
3 12 **4** (a) $-|-6| = -|6|$ (b) $-|5| < |-5|$
5 8 **6** No

Checkpoints for Section 0.2

1 (a) $8, -15x$ (b) $4x^2, -3y, -7$ (c) $z^4, -10z^3, 1$
2 29 **3** (a) 33 (b) 1 (c) -21
4 (a) Multiplicative Identity Property
(b) Commutative Property of Addition
5 $\frac{11x}{12}$ **6** $\frac{11}{12}$
7 Scientific calculator: 6 [×] (8 [y^x] 3 [−] 481) [=]
Graphing calculator: 6 (8 [^] 3 [−] 481) [ENTER]
8 5.87

Checkpoints for Section 0.3

1 (a) 256 (b) 128 (c) $\frac{4}{3}$ **2** $\frac{z^{12}}{27x^6}$ **3** $\frac{7}{5}$
4 (a) 6.318×10^6 (b) 3.45×10^{-3}
5 (a) 428,000 (b) 0.0025 **6** 12.72 min
7 (a) -0.0545 (b) 5 **8** \$1110.12 **9** 2.9%

Checkpoints for Section 0.4

1 (a) $\frac{1}{7}$ (b) 5 (c) 2 **2** $3x^2\sqrt{2x}$ **3** $3x\sqrt[3]{2x}$
4 $\frac{\sqrt[3]{2}}{2}$ **5** $8 + 2\sqrt{11}$ **6** 9 **7** $\sqrt{2}$ **8** $6\sqrt{x}$
9 2.621 **10** 4.164 **11** No

Checkpoints for Section 0.5

1 $-9x^2 + 3x + 7$; degree 2 **2** Not a polynomial
3 $2x^2 + 5x + 4$
4 (a) $x^2 + 14x + 45$ (b) $3x^2 - 2x - 1$
5 $x^4 - 2x^3 + 2x^2 - 11x + 4$ **6** $-x^2 + 9$
7 (a) $x^2 + 8x + 16$ (b) $x^3 - 9x^2 + 27x - 27$
8 $x^2 + 10x - y^2 + 25$ **9** \$12,282.98
10 Volume $= 4x^3 - 44x^2 + 120x$
$x = 2$ inches: $V = 96$ cubic inches
$x = 3$ inches: $V = 72$ cubic inches

Checkpoints for Section 0.6

1 $(x + 1)(3x + 1)$ **2** $x(x + 1)(x - 1)$
3 $4(5 + y)(5 - y)$ **4** (a) $(2x - 1)^2$ (b) $(x - 6)^2$
5 (a) $(y + 1)(y^2 - y + 1)$ (b) $5(z - 2)(z^2 + 2z + 4)$
6 $(x + 3)(x - 2)$ **7** $(2x - 3)(x - 1)$
8 $(x + 1)(x^2 + 5)$ **9** $(2x - 3)(x + 4)$

Checkpoints for Section 0.7

1 (a) All real numbers except $x = 5$ (b) All real numbers
(c) All nonnegative real numbers
2 $\frac{2(x + 1)}{3}, x \neq 1$ **3** $-\frac{3 + x}{2(x + 1)}, x \neq 1$
4 $\frac{1}{x + 1}, x \neq 2$ **5** $\frac{8}{5}, x \neq -y$ **6** $\frac{2(6 - x^2)}{3x}$
7 $\frac{9}{x}, x \neq 1$ **8** $\frac{1}{3}, x \neq 3$

Chapter 1

Checkpoints for Section 1.1

1 Identity **2** (a) 2 (b) -1 **3** (a) -11 (b) $\frac{11}{3}$
4 Infinitely many solutions **5** No solution **6** 30
7 No solution **8** 6 **9** 0.794 **10** 2003 ($t \approx 3$)

Checkpoints for Section 1.2

1 \$950 **2** $p = 6800 + 0.05(6800)$ **3** 20%
4 1.5% **5** 15 feet × 45 feet
6 1.1 hours **7** 32 feet
8 \$1200 was invested at 1% and \$1300 was invested at 2%.
9 About 1.27 feet

Checkpoints for Section 1.3

1 (a) $-3, 4$ (b) $-6, -4$ **2** $0, 2$ **3** -2
4 ± 2 **5** $-3, 5$ **6** 6 feet × 14 feet
7 3 seconds **8** ≈ 8.5 feet **9** 2010 ($t \approx 10.80$)

Checkpoints for Section 1.4

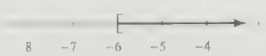
1 One real solution **2** $-1 \pm \sqrt{3}$ **3** $\frac{1}{3}$
4 $-0.831, 1.511$ **5** 2:00 P.M. **6** ≈ 5.1 seconds

Checkpoints for Section 1.5

1 $0, \pm 1$ **2** $1, \pm\sqrt{2}$ **3** $\pm 2, \pm 1$ **4** 12
5 81 **6** $1, 3$ **7** $1, 5$ **8** 5 friends
9 ≈ 4.5% **10** 4,010,025 copies

Checkpoints for Section 1.6

1 (a) $2 \leq x < 7$; bounded (b) $-\infty < x < 6$; unbounded
2 $x < 1$
3 $x \geq -6$



4 $-1 \leq x < 2$



5 $-9 \leq x \leq 5$



6 $x < -4$ or $x > 2$



7 More than 3.5 miles **8** At most $12\frac{1}{2}$ weeks
9 Undercharged by as much as \$0.08 or overcharged by as much as \$0.08

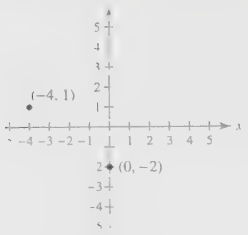
Checkpoints for Section 1.7

1 $(-2, 1)$ **2** $(-\infty, 1) \cup (2, \infty)$
3 (a) The solution set is empty.
(b) The solution set consists of all real numbers except 1.
4 $(-\infty, 2] \cup (3, \infty)$ **5** At least \$35 and at most \$75
6 (a) $[-2, 2]$ (b) $(-\infty, \infty)$

Chapter 2

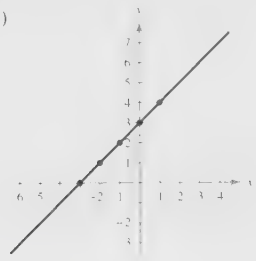
Checkpoints for Section 2.1

1

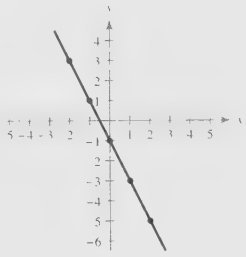


2 (a) 10 (b) $(-3, 2)$ **3** Yes

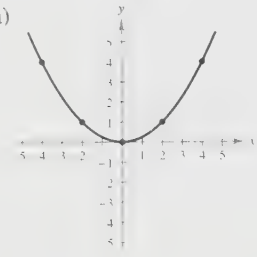
4 (a)



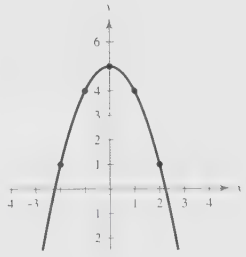
(b)



5 (a)



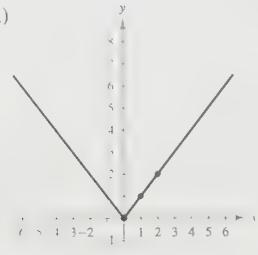
(b)



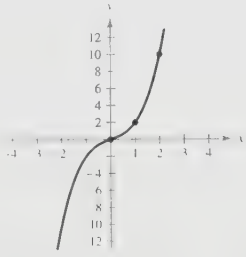
6 (a) $\left(-\frac{7}{4}, 0\right), (0, 7)$ (b) $(-1, 0), (1, 0), (0, -1)$

7 y-axis symmetry

8 (a)

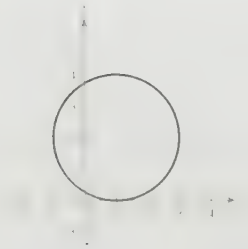


(b)



9 $x^2 + (y - 1)^2 = 25$

10



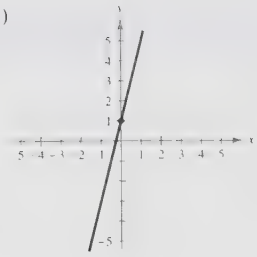
Checkpoints for Section 2.2

1 $\frac{1}{2}$

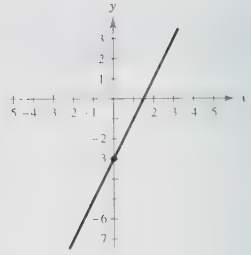
2 (a) $y = -2x + 8$ (b) $y = \frac{3}{2}x + 9$ (c) $y = 1$

3 $y = 0.2x + 1$

4 (a)



(b)



5 (a) $y = 3x - 2$ (b) $y = 3x + 6$ **6** $y = -4x + 4$

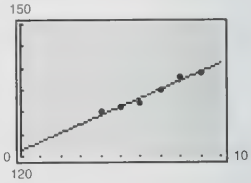
Checkpoints for Section 2.3

1 The model approximates the weight of the puppy best for $t = 2$ months and worst for $t = 10$ months.

2 $y = 0.06x$ **3** $y = 33.84615x$ **4** 6 P.M.

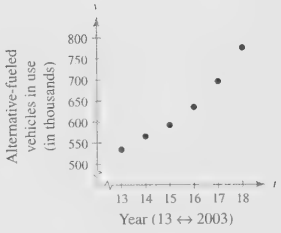
5 836,821 people **6** $V = -195t + 2300$

7 (a) $y = 1.97x + 121.4$ (b)



(c) $\approx$ 143,000,000 tax returns

8



Answers will vary. Sample answer: $v = 50t - 150$

Checkpoints for Section 2.4

1 (a) No (b) Yes (c) No

2 (a) Yes (b) No **3** -3 **4** 10, 2

5 All real numbers **6** $V = 4\pi h^3$ **7** No

8

t	9	10	11	12	13
p	160.2	168.9	177.6	186.2	194.9

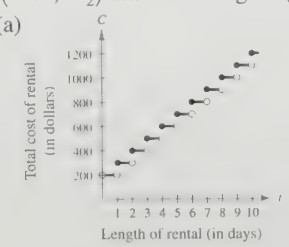
t	14	15	16	17	18	19
p	221.9	239.3	247.3	246.0	235.3	215.3

Checkpoints for Section 2.5

1 Domain: $(-\infty, \infty)$ **2** (a) Yes (b) Yes
Range: $[-3, \infty)$

3 Decreasing on $\left(-\infty, -\frac{3}{2}\right)$ and increasing on $\left(-\frac{3}{2}, \infty\right)$

4 (2, 2) **5** (a)

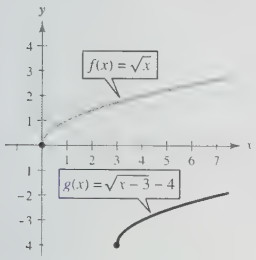


(b) 8 days or less

6 Neither

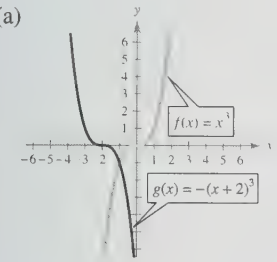
Checkpoints for Section 2.6

1

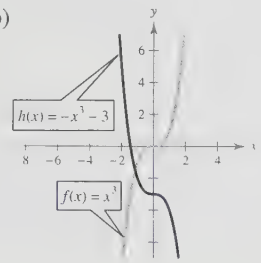


2 (a) The graph of g is a reflection of the graph of f in the x-axis.
(b) The graph of h is a reflection of the graph of f in the y-axis.

3 (a)



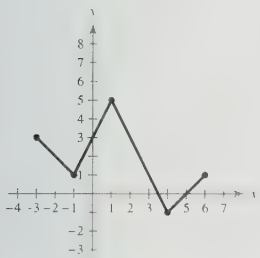
(b)



4 $h(x) = -(x-2)^2 - 1$

5 (a) The graph of g is a vertical stretch of the graph of f by a factor of 4.
(b) The graph of h is a vertical shrink of the graph of f by a factor of $\frac{1}{4}$.

6



Checkpoints for Section 2.7

1 $2x^2 + x - 1$ **2** $-x - 7; -10$

3 Domain of f/h: All real numbers except $x = 3$
Domain of h/f: All real numbers except $x = 1$

4 (a) $x^2 + 2x - 1$ (b) $x^2 - 1$

5 $\sqrt{3 - x^2}; [-\sqrt{3}, \sqrt{3}]$

6 Answers will vary.
Sample answer: $f(x) = x^2 + 2,\ g(x) = x - 1,$
$h(x) = (x-1)^2 + 2 = f(x-1) = f(g(x))$

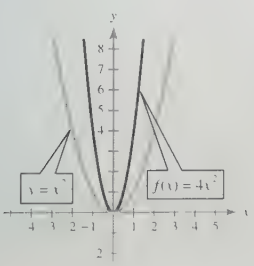
7 f represents the number of Independent senators.

8 About 1 hour and 21 minutes

Chapter 3

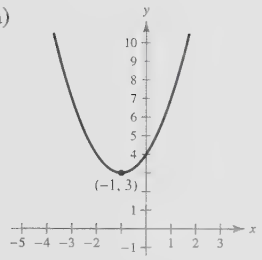
Checkpoints for Section 3.1

1



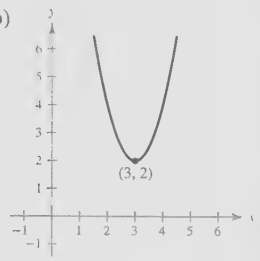
Compared with the graph of $y = x^2$, each output of $f(x) = 4x^2$ vertically stretches the graph by a factor of 4.

2 (a)



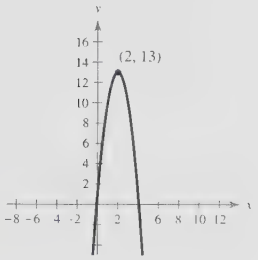
Vertex: $(-1, 3)$

(b)



Vertex: $(3, 2)$

3



Vertex: $(2, 13)$

4 $f(x) = x^2 - 6x + 13$

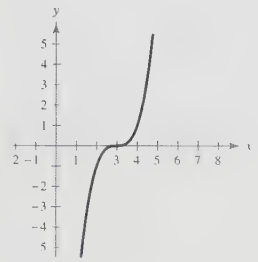
5 ≈ 39.7 feet

6 30 moth sites;
Answers will vary.

7 30 units

Checkpoints for Section 3.2

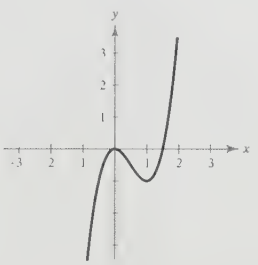
1



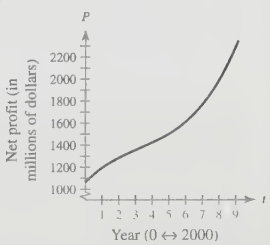
2 (a) Rises to the left; Rises to the right
(b) Falls to the left; Falls to the right

3 Rises to the left; Falls to the right **4** ± 2 **5** $0, \pm 1$

6



7



2006 ($t \approx 5.9$)

Checkpoints for Section 3.3

1 $(x-4)(x-3)(x+1)$ **2** $x^2 - 2x + 4$

3 $5x^2 + 13 + \dfrac{2x + 13}{x^2 + 2x - 4}$ **4** $2x^2 + 3x + 15 - \dfrac{5}{x - 5}$

5 3

6

$$\begin{array}{r|rrrrr} -2 & 1 & 6 & 7 & -6 & -8 \\ & & -2 & -8 & 2 & 8 \\ \hline & 1 & 4 & -1 & -4 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -4 & 1 & 4 & -1 & -4 \\ & & -4 & 0 & 4 \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

$f(x) = (x+2)(x+4)(x^2 - 1)$
$= (x+2)(x+4)(x+1)(x-1)$

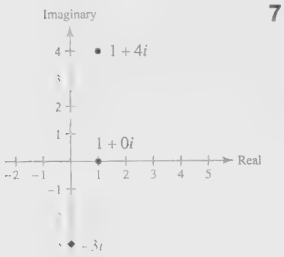
7 (a) $\approx 10.6\%$ (b) $\approx 15.9\%$

Checkpoints for Section 3.4

1 No rational zeros **2** $-2, 1$ **3** $-2, -1, \frac{1}{2}$ **4** -2
5 The function has a zero between 1.3 and 1.4. **6** -1.290
7 $-0.247, 1.445, 2.802$ **8** $x \approx 1.89$ **9** $\approx \$289{,}000$

Checkpoints for Section 3.5

1 (a) $5 + i$ (b) i **2** (a) $-8 + 12i$ (b) $16 - 30i$
3 $4 + i$ **4** $4 + 3i$ **5** $-\frac{3}{2} \pm \frac{\sqrt{7}}{2}i$
6



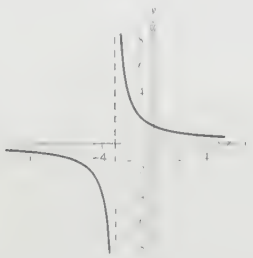
7 The complex number -3 is not in the Mandelbrot Set because for $c = -3$, the corresponding Mandelbrot sequence is $-3, 6, 33, 1086, 1{,}179{,}393, \ldots$, which is unbounded.

Checkpoints for Section 3.6

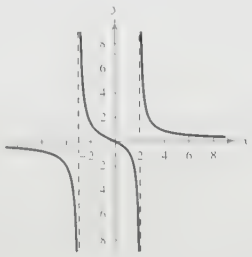
1 Four zeros: $\pm\sqrt{6}, \pm\sqrt{6}i$
2 (a) $\pm 1, \pm 3i$; $f(x) = (x - 1)(x + 1)(x - 3i)(x + 3i)$
(b) $3, 3, -1, i, -i$;
$g(x) = (x - 3)(x - 3)(x + 1)(x - i)(x + i)$
3 (a) Answers will vary. Sample answer: $f(x) = x^4 - 5x^2 - 36$
(b) Answers will vary.
Sample answer: $f(x) = x^3 - 11x^2 + 36x - 26$
4 (a) $(x^2 + 4)(x^2 - 3)$ (b) $(x^2 + 4)(x - \sqrt{3})(x + \sqrt{3})$
(c) $(x + 2i)(x - 2i)(x - \sqrt{3})(x + \sqrt{3})$
(d) Two irrational zeros and two imaginary zeros
5 $\frac{5}{3}, \pm 4i$

Checkpoints for Section 3.7

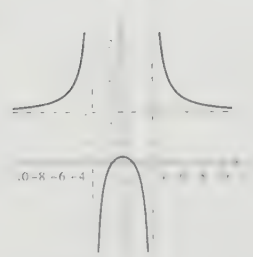
1 The domain of f is all real numbers except $x = 1$. As x approaches 1 from the left, $f(x)$ decreases without bound. As x approaches 1 from the right, $f(x)$ increases without bound.
2 Horizontal asymptote: $y = 1$; No vertical asymptotes
3



4



5



6

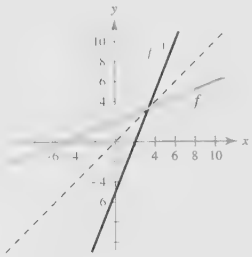


7 \$1,066.667 **8** ≈ 6.8 acres per person

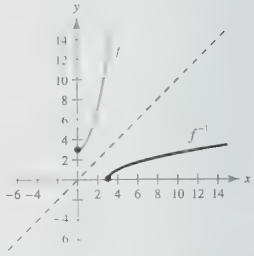
Chapter 4

Checkpoints for Section 4.1

1 $f^{-1}(x) = 6x$ **2** $f^{-1}(x) = x - 10$
3 $(\sqrt[3]{x - 6})^3 + 6 = x$; $\sqrt[3]{(x^3 + 6) - 6} = x$
4 $g(x)$ **5** $f^{-1}(x) = \dfrac{x - 5}{4}$
6



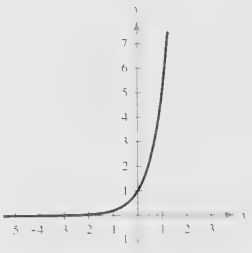
7



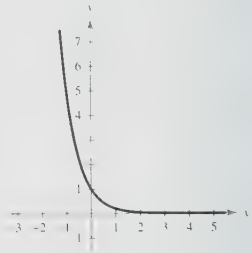
8 (a) f does not have an inverse function.
(b) f has an inverse function.

Checkpoints for Section 4.2

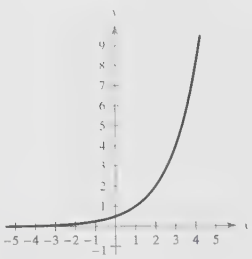
1 3.918
2



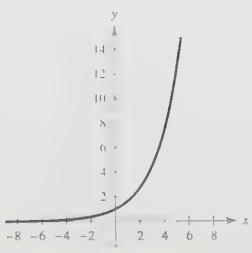
3



4



5 403.4287935; 0.0301973834
6

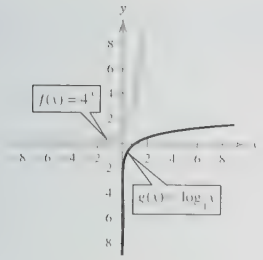


7 (a) \$7927.75 (b) \$7935.08 (c) \$7938.78
8 ≈ 9.971 pounds

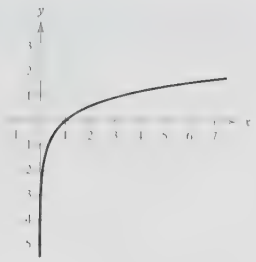
Checkpoints for Section 4.3

1 (a) 2 (b) -2 **2** 2.301 **3** 0

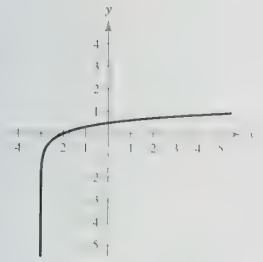
4



5



6



7 (a) 7 (b) 0 (c) $\frac{3}{4}$ **8** -2.303 **9** $(-5, \infty)$
10 103.0 in., 114.0 in., 128.2 in., 132.8 in., 141.7 in.

Checkpoints for Section 4.4

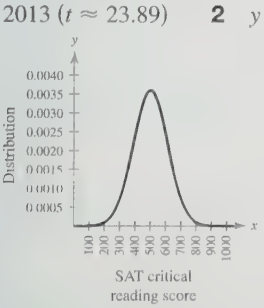
1 1.936 **2** 1.936 **3** $2 \log_{10} 5 - \log_{10} 3$
4 $-\ln \frac{2}{e} = -(\ln 2 - \ln e)$ **5** $\ln 2 + \ln m + 2 \ln n$
$= -\ln 2 + \ln e$
$= 1 - \ln 2$
6 $\log_{10} \frac{(x+1)^2}{(x-1)^3}$ **7** $\ln y = \frac{1}{2} \ln x$ **8** 30 decibels

Checkpoints for Section 4.5

1 (a) 4 (b) 216 (c) 5
2 (a) $\log_6 84 \approx 2.473$ (b) $\log_{10} 38 \approx 1.580$
3 $\ln 15 \approx 2.708$ **4** $\frac{1}{2}(7 + \log_4 24) \approx 4.646$
5 $\ln 3 \approx 1.099$, $\ln 4 \approx 1.386$ **6** (a) 5 (b) $\frac{81}{2}$
7 3 **8** $e^4 \approx 54.598$ **9** 4 **10** ≈ 2.70 years
11 2001 ($t \approx 11.01$)

Checkpoints for Section 4.6

1 2013 ($t \approx 23.89$) **2** $y = 3e^{0.19617x}$ **3** ≈ 7671 years
4



Average score: 501
5 9 days **6** $\approx 31{,}622{,}777$; 100 times greater
7 1×10^{-5} mole of hydrogen per liter
8 Logistic growth model

Chapter 5

Checkpoints for Section 5.1

1 (5, 1)
2 \$4000 is invested at 3% and \$11,000 is invested at 5%.
3 $(-4, -7), (2, 5)$ **4** No solution **5** (4, 0)
6 ≈ 4762 pairs of shoes **7** 2006 ($t \approx 6.7$)

Checkpoints for Section 5.2

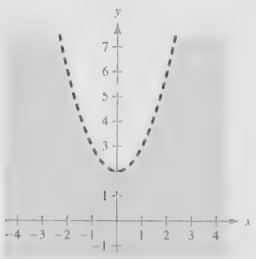
1 $\left(2, \frac{1}{3}\right)$ **2** (3, 2) **3** $(-3, -3)$
4 $(1, -4)$ **5** No solution **6** $(a, -a - 5)$
7 Speed of plane: ≈ 471.18 miles per hour
Speed of wind: ≈ 16.63 miles per hour
8 (2,000,000, 290)

Checkpoints for Section 5.3

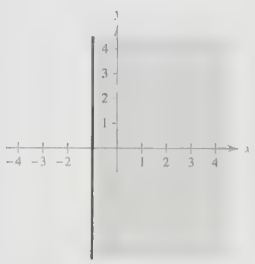
1 $(-1, -3, 4)$ **2** $(1, 2, 3)$ **3** No solution
4 $\left(-\frac{3}{5}a + \frac{3}{10}, -\frac{9}{5}a + \frac{2}{5}, a\right)$ **5** $\left(\frac{1}{4}a, \frac{11}{4}a - 1, a\right)$
6 Answers will vary. Sample answer: \$650,000 at 5%, \$150,000 at 7%, \$200,000 at 8%, \$200,000 at 10%
7 $y = 2x^2 - 2x + 3$

Checkpoints for Section 5.4

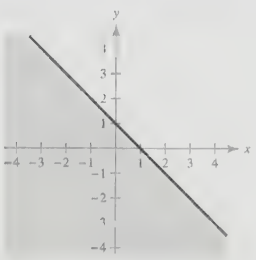
1



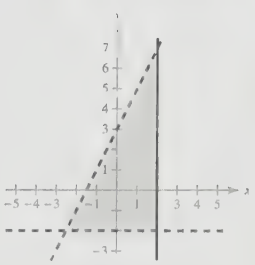
2



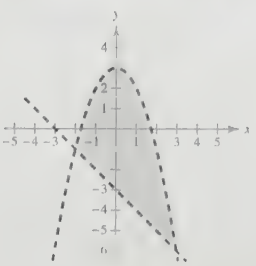
3



4



5



6 Consumer surplus: \$60,000,000
Producer surplus: \$60,000,000
7 No. The combination of 4 cups of dietary drink X and 1 cup of dietary drink Y does not meet all the minimum daily requirements.

Checkpoints for Section 5.5

1 Maximum value at (0, 3): 9
2 (a) Maximum value at (27, 0): 135
(b) Minimum value at (0, 0): 0
3 No maximum value

4 The maximum profit would be \$2925, and it would occur at monthly production levels of 1050 units of product I and 150 units of product II.

5 The minimum cost would be \$0.56 per day, and it would occur when 1 cup of drink X and 4 cups of drink Y were consumed each day.

Chapter 6

Checkpoints for Section 6.1

1 2×3

2 $\left[\begin{array}{ccc|c} 4 & 3 & \vdots & -5 \\ -1 & 3 & \vdots & 12 \end{array}\right]$ **3** Multiply the second row by $\frac{1}{3}$.

2×3

4 $\begin{cases} x - 2y + 5z = 3 \\ y + 4z = -3 \\ z = 2 \end{cases}$ **5** Row-echelon form

$(-29, -11, 2)$

6 $(-1, 0, -1, 3)$ **7** No solution **8** $(6, 5, 5)$

9 $(-9a - 10, -5a - 2, a)$

Checkpoints for Section 6.2

1 $a_{11} = 5, a_{12} = 2, a_{21} = -1, a_{22} = 3$ **2** $\begin{bmatrix} 6 & -2 \\ -2 & -3 \end{bmatrix}$

3 (a) $\begin{bmatrix} 4 & 8 & -2 \\ 0 & 2 & 6 \\ 6 & 4 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & -5 \\ -7 & 6 & 5 \\ -8 & 4 & 12 \end{bmatrix}$

4 $\begin{bmatrix} 1 & 1 \\ 12 & -4 \end{bmatrix}$ **5** $\begin{bmatrix} -24 & -4 \\ -12 & 32 \end{bmatrix}$

6 $\begin{bmatrix} 5 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{bmatrix}$ **7** $\begin{bmatrix} 8 & -8 \\ 4 & -16 \\ -10 & 4 \end{bmatrix}$ **8** $\begin{bmatrix} -3 & -22 \\ 3 & 10 \\ -5 & 10 \end{bmatrix}$

9 Not possible **10** $AB = [6]$, $BA = \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}$

11 $\begin{bmatrix} -2 & -3 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -36 \end{bmatrix}$, $X = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$

12 Women's team: \$2490; Men's team: \$2871

Checkpoints for Section 6.3

1 $AB = I$ and $BA = I$ **2** $\begin{bmatrix} 1 & 1 \\ -5 & -4 \end{bmatrix}$

3 $\begin{bmatrix} -4 & 2 & 5 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **4** $\begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$ **5** $(-1, 2, 1)$

Checkpoints for Section 6.4

1 (a) -7 (b) 10 (c) 0

2 Minors: $M_{11} = -9$, $M_{12} = -10$, $M_{13} = 2$, $M_{21} = 5$, $M_{22} = -2$, $M_{23} = -3$, $M_{31} = 13$, $M_{32} = 5$, $M_{33} = -1$

Cofactors: $C_{11} = -9$, $C_{12} = 10$, $C_{13} = 2$, $C_{21} = -5$, $C_{22} = -2$, $C_{23} = 3$, $C_{31} = 13$, $C_{32} = -5$, $C_{33} = -1$

3 [illegible] **4** [illegible]

Checkpoints for Section 6.5

1 14 **2** Not collinear **3** $x - y + 2 = 0$

4 $[15 \quad 23 \quad 12][19 \quad 0 \quad 1][18 \quad 5 \quad 0]$
$[14 \quad 15 \quad 3][20 \quad 21 \quad 18][14 \quad 1 \quad 12]$

5 110, -39, -59, 25, -21, -3, 23, -18, -5, 47, -20, -24, 149, -56, -75, 87, -38, -37

6 OWLS ARE NOCTURNAL

Chapter 7

Checkpoints for Section 7.1

1 3, 5, 7, 9 **2** $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}$ **3** $a_n = n^2 - 1$

4 6, 3, 1, $\frac{1}{4}$, $\frac{1}{20}$ **5** $\frac{1}{2}, \frac{3}{4}, 2, \frac{15}{2}, 36$ **6** $4(n + 1)$

7 44 **8** $\sum_{i=1}^{4} \frac{i - 1}{2}$; Not unique

9 (a) \$1007.50, \$1015.06, \$1022.67 (b) \$1126.99

Checkpoints for Section 7.2

1 (a) Arithmetic sequence; $d = 2$
(b) Not an arithmetic sequence
(c) Arithmetic sequence; $d = -\frac{1}{2}$

2 $a_n = 2n + 3$ **3** 28 **4** (a) 43 (b) -23

5 6, 9, 12, 15, 18, . . . **6** 175

7 30,100 **8** \$2,500,000

Checkpoints for Section 7.3

1 Geometric; 4 **2** 2, 6, 18, 54, 162

3 ≈ 104.021 **4** $a_n = 4(5)^{n-1}$; 195,312,500

5 $\frac{2187}{32}$ **6** 59,048 **7** $\frac{25}{3}$ **8** \$1168.81

Checkpoints for Section 7.4

1 (a) 10 (b) 1 **2** (a) 36 (b) 15 **3** 84, 126, 36

4 $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

5 $243x^5 + 405x^4 + 270x^3 + 90x^2 + 15x + 1$

6 (a) $8064a^5b^5$ (b) $67{,}500a^3b^3$

Checkpoints for Section 7.5

1 3 **2** 2 **3** 17,576 **4** 2,600,000

5 24 **6** 120 **7** 1260 **8** 10

9 22,100 **10** Permutations

Checkpoints for Section 7.6

1 $S = \{1, 2, 3, 4, 5, 6\}$ **2** $\frac{13}{52} = \frac{1}{4}$ **3** $\frac{1}{12}$ **4** $\frac{1}{6}$

5 $\frac{1}{962{,}598}$ **6** ≈ 0.135 **7** $\frac{4}{13} \approx 0.308$

8 $\frac{66}{529} \approx 0.125$ **9** 0.4225

Answers to Tech Tutors

Section 0.2 *(page 11)*

11; 23

Section 1.2 *(page 85)*

$143.75

Section 1.4 *(page 106)*

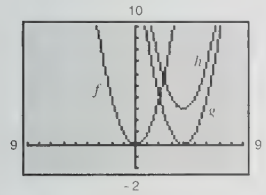
1.854, −4.854

Section 2.2 *(page 175)*

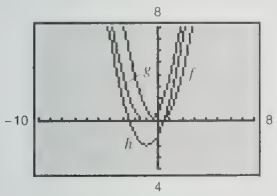
$y = 2.2x + 11.3$

Section 2.6 *(page 221)*

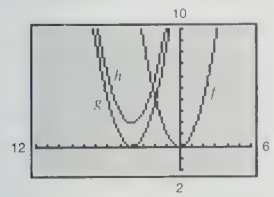
(a) $g(x)$ is shifted four units to the right.
$h(x)$ is shifted four units to the right and three units upward.



(b) $g(x)$ is shifted one unit to the left.
$h(x)$ is shifted one unit to the left and two units downward.



(c) $g(x)$ is shifted four units to the left.
$h(x)$ is shifted four units to the left and two units upward.



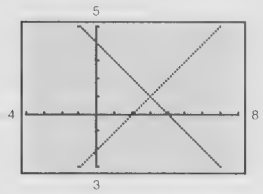
Section 3.6 *(page 307)*

(a) $x = 2$ (b) $x = 3$ (c) $x = 0$ (d) $x = \pm 1$

Section 4.6 *(page 388)*

$C = 138.78e^{0.167t}$; the models are the same.

Section 5.1 *(page 411)*



Solution: (3, 1)

Section 5.2 *(page 423)*

(16, 14)

Section 6.2 *(page 489)*

(a) $\begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & -2 & -4 \\ 3 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$

Section 7.1 *(page 542)*

For increasingly large values of n, the calculator displays the values in scientific notation.

Index

A

Absolute value, 6
  function, graph of, 215
  inequality, solving, 131
  properties of, 6
Adding fractions
  with like denominators, 14
  with unlike denominators, 14
    least common denominator method, 15
Addition, 12
  of complex numbers, 298
  of a constant in an inequality, 128
  of inequalities, 128
  of matrices, 488
    properties of, 490
  of polynomials, 41
Additive identity
  for complex numbers, 298
  for matrices, 491
  for real numbers, 12
Additive inverse
  for complex numbers, 298
  for real numbers, 12
Algebra, basic rules of, 12
Algebraic equation, 80
Algebraic expression(s), 10
  constant terms of, 10
  constants, 10
  domain of, 55
  equivalent, 55
  evaluating, 11
  terms of, 10
  variable terms of, 10
  variables, 10
Algebraic function, 346
Approximately equal to, 3
Area
  common formulas for, 86
  of a triangle, 521
Arithmetic sequence, 550
  common difference of, 550
  finite, sum of, 553
  nth partial sum of, 553
  nth term of, 551
Associative Property of Addition
  for complex numbers, 299
  for matrices, 490
  for real numbers, 12
Associative Property of Multiplication
  for complex numbers, 299
  for matrices, 494
  for real numbers, 12
Associative Property of Scalar Multiplication, 490, 494
Asymptote(s)
  horizontal, 315, 316
  of a hyperbola, A14
  oblique, 319
  slant, 319
  vertical, 315, 316
Augmented matrix, 475
Average value of a population, 391
Axis
  imaginary, 302
  of a parabola, 251, A9
  real, 302
  of symmetry, 251

B

Back-substitution, 411
Balance in an account, 25
Base
  of an exponential expression, 20
  of an exponential function, 346
  natural, 350
Basic rules of algebra, 12
  Additive Identity Property, 12
  Additive Inverse Property, 12
  Associative Property of Addition, 12
  Associative Property of Multiplication, 12
  Commutative Property of Addition, 12
  Commutative Property of Multiplication, 12
  Distributive Property, 12
  Multiplicative Identity Property, 12
  Multiplicative Inverse Property, 12
Bell-shaped curve, 391
Binomial, 40
  cube of, 43
  expanding, 571
  square of, 43
Binomial coefficient, 568
Binomial Theorem, 568
Bounded, 302
  interval(s), 126
    on the real number line, 126
  sequence, 302
Branches of a hyperbola, A13
Break-even point, 415

C

Cartesian plane, 158
Center
  of a circle, 166
  of an ellipse, A11
  of a hyperbola, A13
Certain event, 586
Change-of-base formula, 369
Characteristics of exponential functions, 348
Characteristics of logarithmic functions, 361
Checking a solution, 72
Circle, 166
  area of, 86
  center of, 166
  circumference of, 86
  general form of the equation of, 167
  radius of, 166
  standard form of the equation of, 166, A19
Circular cylinder, volume of, 86
Closed interval, 126
Coded row matrices, 524
Coefficient
  binomial, 568
  leading, 40
  of a term of a polynomial, 40
  of a variable term, 10
Coefficient matrix, 475
Cofactors of a square matrix, 515
  expanding by, 516
Collinear points, test for, 522
Column matrix, 475
Column of a matrix, 474
Combinations of n elements taken r at a time, 580
Common difference, 550
Common formulas, 86
  for area, 86
  miscellaneous, 86
  for perimeter, 86
  for volume, 86
Common graphs of functions, 215
Common logarithmic function, 359
Common ratio, 558
Commutative Property of Addition
  for complex numbers, 299
  for matrices, 490
  for real numbers, 12
Commutative Property of Multiplication
  for complex numbers, 299
  for real numbers, 12
Complement of an event, 593
  probability of, 593
Completely factored, 48
Completing the square, 104, 167
Complex conjugate, 300
Complex fractions, 59
Complex number(s), 297
  addition of, 298
  additive identity, 298
  additive inverse, 298
  Associative Property of Addition, 299
  Associative Property of Multiplication, 299
  Commutative Property of Addition, 299
  Commutative Property of Multiplication, 299
  Distributive Property, 299

equality of, 297
imaginary number, 297
pure imaginary number, 297
standard form of, 297
subtraction of, 298
Complex plane, 302
imaginary axis, 302
real axis, 302
Complex zeros occur in conjugate pairs, 309
Composite number, 15
Composition of two functions, 232
Compound interest, 121, 351
formulas for, 25, 121, 352
Condensing a logarithmic expression, 371
Conditional equation, 70
Conic(s) (conic section(s)), A8
degenerate, A8
ellipse, A11
hyperbola, A13
parabola, A9
standard forms of the equations of, A19
standard position, A19
Conjugate axis of a hyperbola, A14
Conjugate(s), 309
complex, 300
of a radical expression, 32
Consistent system, 424
Constant(s), 10
function, 211, 215, 250
of proportionality, 184
term, 10
of a polynomial, 40
of variation, 184
Constraints, 454
Consumer surplus, 449
Continuous, 262
Continuous compounding, 351
Continuously compounded interest, 351
Coordinate(s), 158
axes, reflection in, 222
of a point on the real number line, 4, 158
x-coordinate, 158
y-coordinate, 158
Correlation coefficient, A6
Co-vertices of an ellipse, A11
Critical numbers
of a polynomial inequality, 138
of a rational inequality, 142
Cross-multiplying, 75
Cryptogram, 524
decode, 524
encode, 524
Cube(s)
of a binomial, 43
difference of two, 49
perfect, 30
root, 29
sum of two, 49
volume of, 86
Cubing function, graph of, 215
Curve
bell-shaped, 391
logistic, 392
sigmoidal, 392
Cylinder, volume of, 86

D

Decimal
repeating, 3
rounded, 3
terminating, 3
Decision digit, 17
Decoding messages, 524
Decreasing function, 211
Defined function, 202
Degenerate conic, A8
Degree
of a polynomial, 40, 41
of a term, 40, 41
Denominator, 12
least common, 15
rationalizing the, 31, 32
Dependent system, 424
Dependent variable, 196, 202
Determinant
of a 2×2 matrix, 506, 513
area of a triangle, 521
of a matrix, 516
of a square matrix, 513, 516
test for collinear points, 522
two-point form of the equation of a line, 523
Diagonal matrix, 519
Difference
of two cubes, factoring, 49
of two functions, 230
of two squares, factoring, 49
Dimension of a matrix, 474
Direct variation, 184
Directly proportional, 184
Directrix of a parabola, A9
Discriminant, 105
Distance
between two numbers, 6
traveled, formula for, 86
Distance Formula, 159
Distinguishable permutations, 579
Distributive Property
for complex numbers, 299
for matrices, 494
for real numbers, 12
Divides evenly, 273
Dividing fractions, 14
Division, 12
Algorithm, 273
Factor Theorem, 277
long, 272
Remainder Theorem, 276
synthetic, 275
uses of the remainder in, 277
Divisors, 15
Domain
of an algebraic expression, 55
of a function, 195, 199, 202
implied, 199, 202
of a rational function, 314
Double inequality, 5, 130
Double solution, 94
Double subscript notation, 474

E

e, natural base, 350
Eccentricity of an ellipse, A26
Elementary row operations, 476
for systems of equations, 433
Elimination
Gaussian, 433
with back-substitution, 480
Gauss-Jordan, 481
method of, 420, 422
Ellipse, A11
center of, A11
co-vertices of, A11
eccentricity of, A26
foci of, A11
latus rectum of, A18
major axis of, A11
minor axis of, A11
standard form of the equation of, A11, A19
vertices of, A11
Encoding messages, 524
Endpoints of an interval, 126
Entry of a matrix, 474
main diagonal, 474
Equal matrices, 487
Equality
of complex numbers, 297
hidden, 80
properties of, 16
Equation(s), 16, 70
algebraic, 80
checking a solution, 72
circle, general form, 167
circle, standard form, 166, A19
conditional, 70
conics, standard form, A19
ellipse, standard form, A11, A19
equivalent, 71
graph of, 161
hyperbola, standard form, A13, A19
identity, 70
of a line
general form, 177
horizontal, 177
intercept form, 181
point-slope form, 174, 177
slope-intercept form, 176, 177
summary of, 177
two-point form, 175, 523
vertical, 177

linear, 71
parabola, standard form, A9, A19
polynomial
second-degree, 93
solution of, 266
position, 97, 109
quadratic, 93
quadratic type, 116
recognition, 166
solution(s) of, 70, 161
solving, 70
system of, 410
solution of, 410
true, 70
Equation editor, A1
Equilibrium, point of, 427, 449
Equivalent
algebraic expressions, 55
equations, 71
forming, 71
fractions, 14
generate, 14
inequalities, 128
systems of equations, 421, 433
operations that produce, 433
Escape velocity, 35
Evaluating an algebraic expression, 11
Even function, 214
test for, 214
Event(s), 585
certain, 586
complement of, 593
impossible, 586
independent, 591
mutually exclusive, 589
probability of, 586
Existence theorem, 306
Expanding
a binomial, 571
by cofactors, 516
a logarithmic expression, 371
Experiment, 585
Exponent(s), 20
properties of, 21, 33
rational, 33
Exponential
decay model, 387
equation
Inverse Property, 377
One-to-One Property, 377
strategies for solving, 377
form, 20
function
with base a, 346
characteristics of, 348
natural, 350
growth model, 387
notation, 20
Exponentiating, 380
Expression(s)
algebraic, 10
domain of, 55
equivalent, 55
fractional, 56
radical, simplest form of, 31
rational, 56
simplified, 63
Extracting square roots, 95
Extraneous solution, 74, 117
Extrapolation, linear, 175

F

f of x, 197
Factor(s), 15
of a polynomial, 266, 310
Factor Theorem, 277
Factorial, 542
Factoring, 48
difference of two cubes, 49
difference of two squares, 49
by grouping, 52
perfect square trinomial, 49
special polynomial forms, 49
sum of two cubes, 49
Falling object
position equation for, 97, 109
Family of functions, 221
Feasible solutions, 454
Finding a determinant, expanding by cofactors, 516
Finding inverse functions, 339
Finding an inverse matrix, 504
Finding test intervals for a polynomial, 138
Finding zeros of a polynomial
Intermediate Value Theorem, 285
Rational Zero Test, 282
zoom-and-trace technique, 287
Finite
sequence, 540
series, 544
Fixed costs, 91
Focus (foci)
of an ellipse, A11
of a hyperbola, A13
of a parabola, A9
FOIL Method, 42
Forming equivalent equations, 71
Formula(s)
change-of-base, 369
common
area, 86
miscellaneous, 86
for distance traveled, 86
for simple interest, 86
for temperature, 86
perimeter, 86
volume, 86
for compound interest, 25, 121, 352
Distance, 159
Midpoint, 160
Pythagorean Theorem, 98
Quadratic, 104
recursion, 552
Fractal, 302
Fractal geometry, 302
Fraction(s), 3
adding
like denominators, 14
unlike denominators, 14
least common denominator method, 15
complex, 59
denominator, 12
dividing, 14
equivalent, 14
generate, 14
least common denominator, 15
multiplying, 14
numerator, 12
properties of, 14
rules of signs, 14
subtracting
like denominators, 14
unlike denominators, 14
least common denominator method, 15
Fractional expression, 56
Function(s), 195, 202
absolute value, graph of, 215
algebraic, 346
common graphs of, 215
composition of, 232
constant, 211, 215, 250
continuous, 262
cubing, graph of, 215
decreasing, 211
defined, 202
dependent variable, 196, 202
difference of, 230
domain of, 195, 199, 202
even, 214
exponential, 346
characteristics of, 348
natural, 350
family of, 221
graph of, 209
greatest integer, 213
identity, graph of, 215
implied domain, 199, 202
increasing, 211
independent variable, 196, 202
inverse, 336, 337
finding, 339
linear, 250
logarithmic, 358
characteristics of, 361
common, 359
natural, 362
properties of, 359, 362, 370
name of, 202
notation, 197
objective, 454

odd, 214
piecewise-defined, 198
polynomial, 250
product of, 230
quadratic, 250
quotient of, 230
range of, 195, 202, 209
rational, 314
guidelines for graphing, 317
relative maximum of, 212
relative minimum of, 212
square root, graph of, 215
squaring, graph of, 215
step, 213
sum of, 230
terminology, summary of, 202
test for even and odd, 214
transcendental, 346
undefined, 202
value at x, 197, 202
Vertical Line Test for, 210
zero of, 209, 266
Fundamental Counting Principle, 576
Fundamental Theorem of Algebra, 306
Fundamental Theorem of Arithmetic, 15

G

Gaussian elimination for a system of linear equations, 433
with back-substitution, 480
Gaussian model, 387, 391
Gauss-Jordan elimination, 481
General form of the equation
of a circle, 167
of a line, 177
Generating equivalent fractions, 14
Geometric sequence, 558
common ratio of, 558
finite, sum of, 561
nth term of, 559
Geometric series, 562
sum of, 562
Graph
of an absolute value function, 215
continuous, 262
of a cubing function, 215
of an equation, 161
intercept of, 163
symmetry of, 164
of an exponential function, 347
of a function, 209
horizontal shift of, 220
of an identity function, 215
of an inequality, 126
in two variables, 445
of an inverse function, 340
of a logarithmic function, 360
nonrigid transformation of, 224
of a rational function, 317
reflection of, 222
rigid transformation of, 224
of a square root function, 215
of a squaring function, 215
turning points of, 266
vertical shift of, 220
vertical shrink of, 224
vertical stretch of, 224
Graphical interpretation of solutions of systems of linear equations, 424
Graphing, point-plotting method, 161
Graphing utility
equation editor, A1
features
intersect, A5
maximum, A7
minimum, A7
regression, A6
root, A4
table, A2
trace, A3
zero, A4
zoom, A3
uses of, A1
viewing window, A2
square setting, A3
Greater than, 4
or equal to, 4
Greatest integer function, 213
Guidelines for graphing rational functions, 317

H

Half-life, 353
Hidden
equality, 80
product, 85
Horizontal asymptote, 315
of a rational function, 315, 316
Horizontal line, 177
Horizontal Line Test, 341
Horizontal shift, 220
Human memory model, 364
Hyperbola, 315, A13
asymptotes of, A14
branches of, A13
center of, A13
conjugate axis of, A14
foci of, A13
standard form of the equation of, A13, A19
transverse axis of, A13
vertices of, A13

I

Identity, 70
function, graph of, 215
matrix, 494
If and only if, 14
Imaginary axis, 302
Imaginary number, 297
pure, 297
Imaginary unit i, 297
Implied domain, 199, 202
Impossible event, 586
Improper rational expression, 273
Inclusive or, 14
Inconsistent system, 424
Increasing annuity, 563
Increasing function, 211
Independent events, 591
probability of, 591
Independent system, 424
Independent variable, 196, 202
Index
of a radical, 29
of summation, 544
Inductive, 516
Inequalities, 5
absolute value, solving, 131
double, 5, 130
equivalent, 128
graph of, 126
linear, 129
in two variables, 445
properties of, 128
rational, 142
satisfy, 126
solution set of, 126
solutions of, 126
solving, 126
symbols, 5
in two variables
graph of, 445
sketching the graph of, 445
solution of, 445
Infinite
geometric series, 562
sum of, 562
sequence, 540
series, 544
wedge, 448
Infinity
negative, 127
positive, 127
Integers, 2
irreducible over, 48
negative, 2
positive, 2
Intercept
of a graph, 163
x-intercept, 163
y-intercept, 163
Intercept form of the equation of a line, 181
Interest
compound, 25, 121, 351, 352
continuously compounded, 351, 352
simple, 25, 86
Intermediate Value Theorem, 285
Interpolation, linear, 175
Intersect feature, A5

Intersection
of A and B, 589
point of, 414
Interval
bounded, 126
closed, 126
endpoints of, 126
open, 126
unbounded, 127
Inverse
of a real number
additive, 12
multiplicative, 12
of a square matrix, 502
finding, 504
Inverse function, 336, 337
finding, 339
Horizontal Line Test for, 341
Inverse Property
of exponential equations, 377
of logarithmic equations, 377
Invertible matrix, 503
Irrational numbers, 3
Irreducible
over the integers, 48
over the rationals, 310
over the reals, 310

L

Latus rectum of an ellipse, A18
Law of Trichotomy, 5
Leading 1, 478
Leading coefficient of a polynomial, 40
Leading Coefficient Test, 264
Least common denominator method for adding or subtracting fractions, 15
Less than, 4
or equal to, 4
Like radicals, 34
Like terms of a polynomial, 41
Line(s)
equation of
general form, 177
intercept form, 181
point-slope form, 174, 177
slope-intercept form, 176, 177
summary of, 177
two-point form, 175, 523
horizontal, 177
parallel, 178
perpendicular, 179
slope of, 172, 173
vertical, 177
Linear equation, 71
Linear extrapolation, 175
Linear Factorization Theorem, 306
Linear function, 250
Linear inequality, 129
in two variables, 445
system of, 447
Linear interpolation, 175
Linear programming, 454
Linear programming problem
optimal solution of, 454
solving, 454
unbounded, 457
Linear regression, 188
Locus, A8
Logarithmic equation
Inverse Property, 377
One-to-One Property, 377
solving, 377
by exponentiating, 380
strategies for solving, 377
Logarithmic expression
condensing, 371
expanding, 371
Logarithmic function
with base a, 358
change-of-base formula, 369
characteristics of, 361
common, 359
natural, 362
properties of, 362, 370
properties of, 359, 370
Logarithmic model, 387, 393
Logistic curve, 392
Logistic growth model, 387, 392
Long division of polynomials, 272
Lower limit of summation, 544
Lower triangular matrix, 520

M

Main diagonal entries, 474
Major axis of an ellipse, A11
Mandelbrot Set, 302
Mathematical model, 80, 183
Mathematical modeling, 80
translating key words and phrases, 81
Matrix (matrices), 474
addition of, 488
properties of, 490
additive identity, 491
augmented, 475
coefficient, 475
column, 475
column of, 474
determinant of, 506, 513, 516
diagonal, 519
dimension of, 474
Distributive Property, 494
elementary row operations, 476
entry of, 474
equal, 487
finding an inverse, 504
identity, 494
invertible, 503
multiplication of, 492
properties of, 494
nonsingular, 503
product of two, 492
row, 475
coded and uncoded, 524
row-echelon form, 478
reduced, 478
row-equivalent, 476
row of, 474
scalar, 488
multiple, 488
multiplication, 488
properties of, 490
singular, 503
square, 474
cofactors of, 515
determinant of, 513, 516
inverse of, 502
main diagonal entries of, 474
minors of, 515
stochastic, 501
triangular, 520
lower and upper, 520
zero, 491
Maximum feature, A7
Method of elimination, 420, 422
Method of substitution, 410
Midpoint Formula, 160
Midpoint of a line segment, 160
Minimum feature, A7
Minor axis of an ellipse, A11
Minors of a square matrix, 515
Miscellaneous common formulas, 86
Mixture problem, 85
Model
mathematical, 80, 183
verbal, 80
Monomial, 40
Multiplication, 12
by a constant in an inequality, 128
matrix, 492
scalar, 488
Multiplicative Identity Property for real numbers, 12
Multiplicative Inverse Property for real numbers, 12
Multiplicity, 267
Multiplying fractions, 14
Mutually exclusive events, 589
probability of, 589

N

n factorial, 542
Name of a function, 202
Natural base, 350
Natural exponential function, 350
Natural logarithmic function, 362
properties of, 362, 370
Natural numbers, 2
Negation, properties of, 13
Negative
infinity, 127

integers, 2
numbers, 4
Nonnegative real numbers, 4
Nonrigid transformation of a graph, 224
Nonsingular matrix, 503
Nonsquare system, 436
Normally distributed population, 391
Not equal to, 3
Notation
exponential, 20
function, 197
scientific, 23
sigma, 544
summation, 544
*n*th partial sum, 544
of an arithmetic sequence, 553
*n*th root of a number, 29
principal, 29
*n*th term
of an arithmetic sequence, 551
of a geometric sequence, 559
Number(s)
of combinations of n elements taken r at a time, 580
complex, 297
composite, 15
imaginary, 297
pure, 297
irrational, 3
natural, 2
negative, 4
nonnegative real, 4
*n*th root of, 29
of permutations of n elements, 577
taken r at a time, 578
positive, 4
prime, 15
principal *n*th root of, 29
rational, 3
real, 2
nonnegative, 4
of solutions of a linear system, 434
whole, 3
Number line, real, 4
Numerator, 12

O

Objective function, 454
Oblique asymptote, 319
Odd function, 214
test for, 214
One-to-One Property
of exponential equations, 377
of logarithmic equations, 377
Open interval, 126
Operations that produce equivalent systems, 433
Optimal solution of a linear programming problem, 454
Optimization, 454
Order of operations, 62
Order on the real number line, 4
Ordered pair, 158
Ordered triple, 432
Origin, 4
of the real number line, 4
of the rectangular coordinate system, 158
symmetry, 164
Outcomes, 585

P

Parabola, 250, A9
axis of, 251, A9
axis of symmetry, 251
directrix of, A9
focus of, A9
standard form of the equation of, A9, A19
vertex of, 251, A9
Parallel lines, 178
Pascal's Triangle, 570
Perfect
cubes, 30
square trinomial, factoring, 49
squares, 30
Perimeter, common formulas for, 86
Permutation(s), 577
distinguishable, 579
of n elements, 577
taken r at a time, 578
Perpendicular lines, 179
Piecewise-defined function, 198
Point of diminishing returns, 271
Point of equilibrium, 427, 449
Point of intersection, 414
Point-plotting method of graphing, 161
Point-slope form of the equation of a line, 174, 177
Polynomial(s), 40
addition of, 41
binomial, 40
completely factored, 48
complex zeros of, 309
constant term of, 40
degree of, 40, 41
division of
long, 272
synthetic, 275
equation
second-degree, 93
solution of, 266
factor(s) of, 266, 310
Factor Theorem, 277
factoring of, 48
finding test intervals for, 138
finding zeros of
Intermediate Value Theorem, 285
Rational Zero Test, 282
zoom-and-trace technique, 287
FOIL method, 42
function, 250
real zeros of, 266
of x with degree n, 250
inequality
critical numbers of, 138
test intervals of, 138
Intermediate Value Theorem, 285
irreducible over the integers, 48
leading coefficient of, 40
like terms of, 41
monomial, 40
prime, 48
product of, 42
Remainder Theorem, 276
special products of, 43
standard form of, 40
subtraction of, 41
term
coefficient of, 40
degree of, 40, 41
trinomial, 40
in x, 40
x-intercept of, 266
zero, 41
zeros of, 138
Position equation, 97, 109
Positive
infinity, 127
integers, 2
numbers, 4
Power, 20
Present value of money, 355
Prime
factorization, 15
number, 15
polynomial, 48
Principal *n*th root of a number, 29
Principal square root of a negative number, 301
Probability, 585
of a complement, 593
of an event, 586
properties of, 586
of independent events, 591
of mutually exclusive events, 589
of the union of two events, 589
Producer surplus, 449
Product
of polynomials, 42
of two functions, 230
of two matrices, 492
Proper rational expression, 273
Properties
of absolute value, 6
of Equality, 16
Reflexive, 16
Symmetric, 16
Transitive, 16
of exponents, 21, 33
of fractions, 14
of inequalities, 128

Inverse, 377
of logarithms, 359, 370
of matrix addition, 490
of matrix multiplication, 494
of natural logarithms, 362, 370
of negation, 13
One-to-One, 377
of the probability of an event, 586
of radicals, 30
of scalar multiplication, 490
of sums, 545
of zero, 14
Zero-Factor, 14, 93
Proportion, 84
Proportionality, constant of, 184
Pure imaginary number, 297
Pythagorean Theorem, 98

Q

Quadrants, 158
Quadratic equation, 93
double solution, 94
repeated solution, 94
solutions of, 105
solving
by extracting square roots, 95
by factoring, 93
using Quadratic Formula, 104
Quadratic Formula, 104
discriminant, 105
Quadratic function, 250
standard form of, 253
Quadratic type, 116
Quotient of two functions, 230

R

Radical(s)
conjugate, 32
index of, 29
like, 34
properties of, 30
simplest form, 31
symbol, 29
Radicand, 29
Radius of a circle, 166
Range of a function, 195, 202, 209
Rate, 172
Rate of change, 172, 186
Ratio, 84, 172
Rational, 3
exponents, 33
properties of, 33
expression, 56
improper, 273
proper, 273
undefined values of, 142
zeros of, 142
function, 314
domain of, 314
guidelines for graphing, 317
horizontal asymptote of, 315, 316
slant asymptote of, 319
vertical asymptote of, 315, 316
inequality, critical numbers of, 142
numbers, 3
irreducible over, 310
Rational Zero Test, 282
Rationalizing the denominator, 31, 32
Real axis, 302
Real number(s), 2
absolute value of, 6
basic rules of algebra for, 12
irreducible over, 310
nonnegative, 4
Real number line, 4
bounded intervals on, 126
coordinate, 4
order on, 4
origin of, 4
unbounded intervals on, 127
Real zeros of polynomial functions, 266
Reciprocal, 12
Rectangle
area of, 86
perimeter of, 86
Rectangular coordinate system, 158
origin of, 158
quadrants, 158
x-axis, 158
y-axis, 158
Rectangular solid
volume of, 86
Recursion formula, 552
Red herring, 92
Reduced row-echelon form, 478
Reflection, 222
in the x-axis, 222
in the y-axis, 222
Reflexive Property of Equality, 16
Regression feature, A6
Relative maximum, 212
Relative minimum, 212
Remainder Theorem, 276
Repeated solution, 94
Repeated zero, 267
Repeating decimal, 3
Rigid transformation of a graph, 224
Root
cube, 29
nth, 29
principal, 29
square, 29
principal, 308
Root feature, A4
Rounded, 3
Rounding decimals, 17
decision digit for, 17
Row-echelon form
of a matrix, 478
reduced, 478
of a system of linear equations, 432
Row-equivalent matrices, 476
Row matrix, 475
coded, 524
uncoded, 524
Row of a matrix, 474
Row operations
elementary
for matrices, 476
for systems of equations, 433
Rules of signs for fractions, 14

S

Sample space, 585
Satisfying an inequality, 126
Scalar, 488
Identity Property, 490
multiple, 488
multiplication, 488
properties of, 490
Scatter plot, 188
Scientific notation, 23
Second-degree polynomial equation in x, 93
Sequence, 540
arithmetic, 550
common difference of, 550
nth term of, 551
bounded, 302
finite, 540
geometric, 558
common ratio of, 558
nth term of, 559
infinite, 540
terms of, 540
unbounded, 302
Series
finite, 544
geometric, 562
infinite, 562
infinite, 544
Set, 2
of real numbers, 2
Shift
horizontal, 220
vertical, 220
Sigma notation, 544
Sigmoidal curve, 392
Simple interest formula, 25, 86
Simplest form of a radical, 31
Singular matrix, 503
Slant asymptote, 319
Slope-intercept form of the equation of a line, 176, 177
Slope of a line, 172, 173
Solution(s)
double, 94
of an equation, 70, 161
checking, 72

extraneous, 74, 117
feasible, 454
of an inequality, 126, 445
of a polynomial equation, 266
of a quadratic equation, 105
repeated, 94
set of an inequality, 126
of a system of equations, 410
of a system of linear inequalities, 447
Solve
an equation in x, 70
an inequality in x, 126
Solving
an absolute value equation, 119
an absolute value inequality, 131
an equation, 70
an equation with fractions, 118
cross-multiplying, 75
an equation of quadratic type, 116
an equation with radicals, 117
an equation with rational exponents, 117
an exponential equation, 377
an inequality, 126
a linear programming problem, 454
a logarithmic equation, 377
exponentiating, 380
a quadratic equation
by extracting square roots, 95
by factoring, 93
using Quadratic Formula, 104
a system of equations, 410
by Gaussian elimination, 433
by Gaussian elimination with back-substitution, 480
by method of elimination, 420, 422
by method of substitution, 410
Special products of polynomials, 43
cube of a binomial, 43
square of a binomial, 43
sum and difference of two terms, 43
Sphere, volume of, 86
Square(s)
area of, 86
of a binomial, 43
completing the, 104
factoring difference of two, 49
perfect, 30
perimeter of, 86
Square matrix, 474
cofactors of, 515
determinant of, 513, 516
inverse of, 502
minors of, 515
Square root(s), 29
extracting, 95
function, graph of, 215
principal, 301
Square setting, A3
Square system, 436
Squaring function, graph of, 215
Standard form
of a complex number, 297
of the equation of a circle, 166, A19
of the equation of an ellipse, A11, A19
of the equation of a hyperbola, A13, A19
of the equation of a parabola, A9, A19
of the equations of conics, A19
of a polynomial, 40
of a quadratic function, 253
Standard position of conics, A19
Step function, 213
Stochastic matrix, 501
Strategies for solving exponential and logarithmic equations, 377
Strategy for solving word problems, 87
Subset, 2
Substitution, method of, 410
Substitution Principle, 11
Subtracting fractions
with like denominators, 14
with unlike denominators, 14
least common denominator method, 15
Subtraction, 12
of complex numbers, 298
of polynomials, 41
Sum(s)
of a finite arithmetic sequence, 553
of a finite geometric sequence, 561
of a geometric series, 562
of an infinite geometric series, 562
of matrices, 488
nth partial, 544
of an arithmetic sequence, 553
properties of, 545
of two cubes, factoring, 49
of two functions, 230
Sum and difference of two terms, 43
Summation
index of, 544
lower limit of, 544
notation, 544
upper limit of, 544
Surplus
consumer, 449
producer, 449
Symmetric Property of Equality, 16
Symmetry, 164
axis of, 251
tests for, 165
with respect to origin, 164
with respect to x-axis, 164
with respect to y-axis, 164
Synthetic division, 275
for a cubic polynomial, 275
uses of the remainder, 277
System of equations, 410
equivalent, 421, 433
point of intersection, 414
solution of, 410
solving, 410
Gaussian elimination with back-substitution, 480
by method of elimination, 420, 422
by method of substitution, 410
with a unique solution, 507
System of linear equations
consistent system, 424
dependent system, 424
graphical interpretation of solutions, 424
inconsistent system, 424
independent system, 424
nonsquare, 436
number of solutions of, 434
row-echelon form, 432
row operations, 433
solving
by Gaussian elimination, 433
square, 436
System of linear inequalities, 447
solution of, 447

T

Table feature, A2
Temperature formula, 86
Term(s)
of an algebraic expression, 10
constant, 10
of a polynomial, 40
of a sequence, 540
sum and difference of two, 43
variable, 10
Terminating decimal, 3
Test(s)
for collinear points, 522
for even and odd functions, 214
for symmetry, 165
Test intervals of a polynomial inequality, 138
Trace feature, A3
Transcendental function, 346
Transformations of graphs
horizontal shift, 220
nonrigid, 224
reflection, 222
rigid, 224
vertical shift, 220
vertical shrink, 224
vertical stretch, 224
Transitive Property of Equality, 16
Transitive Property of Inequalities, 128
Translating key words and phrases, 81
Transverse axis of a hyperbola, A13
Tree diagram, 585
Triangle
area of, 86, 521
perimeter of, 86
Triangular matrix, 520
lower, 520
upper, 520
Trinomial, 40

True equation, 70
Turning points of a graph, 266
Two-point form of the equation of a line, 175, 523

U

Unbounded
  interval, 127
  region, 457
  sequence, 302
Uncoded row matrices, 524
Undefined function, 202
Union of two events, probability of, 589
Upper limit of summation, 544
Upper triangular matrix, 520
Uses of a graphing utility, A1
Uses of the remainder in synthetic division, 277

V

Value of f at x, 197, 202
Variable, 10
  costs, 91
  dependent, 196, 202
  independent, 196, 202
  term(s), 10
    coefficient of, 10
Variation
  constant of, 184
  direct, 184
Vary directly, 184
Verbal model, 80
Vertex (vertices)
  of an ellipse, A11
  of a hyperbola, A13
  of a parabola, 251, A9
Vertical asymptote, 315
  of a rational function, 315, 316
Vertical line, 177
Vertical Line Test, 210
Vertical shift, 220
Vertical shrink, 224
Vertical stretch, 224
Viewing window, A2
  square setting, A3
Volume, common formulas for, 86

W

Whole numbers, 3
With replacement, 575
Without replacement, 575
Word problems, strategy for solving, 87

X

x-axis, 158
  reflection in, 222
  symmetry, 164
x-coordinate, 158
x-intercept, 163

Y

y-axis, 158
  reflection in, 222
  symmetry, 164
y-coordinate, 158
y-intercept, 163

Z

Zero(s), 2
  of a function, 209, 266
  matrix, 491
  multiplicity of, 267
  of a polynomial, 138
  polynomial, 41
  properties of, 14
  of a rational expression, 142
  repeated, 267
Zero-Factor Property, 14, 93
Zero feature, A4
Zoom feature, A3
Zoom-and-trace technique, 287

Applications (continued from front endsheets)

Satellite orbit, A25, A26
Skydiving wind resistance, 101
Sound intensity, 373, 376, 399
Speed
  of light, 24, 28
Stream carrying power, 38
Temperature, 182
Thawing a package of steaks, 399, 407
Throwing an object, 111, 112, 246
  baseball, 111
  on the moon, 109, 111, 112
Work, 366

CONSTRUCTION

Brick pattern, 556, 557
Fireplace arch, A17
Mountain tunnel, A18
Nail length, 375
Power line installation, 125
Suspension bridge, A17
Wheelchair ramp, 182

CONSUMER

Annual salary, 80
Camera packages, 604
Camping outfitters, 419
Charitable contributions, 257, 268
Choice of two jobs, 419
Comparative shopping, 136
Comparing taxi fares, 132
Computer systems, 602
Consumer price index, 190
Cost
  of forklift rental, 213
  of internet access, 207
  of MP3s, 549
  of overnight delivery, 219
Discount, 81, 88, 91, 152, 239
Food portion size increases, 89
Health care plans, 501
Home mortgage
  monthly payment, 367
Job offer, 419, 556, 601
List price, 90, 152
Market value of a home, 207
Markup, 91
Movie theatre admission price, 345
Percent of a raise, 82
Pet spending, 19
Property tax, 191, 245
Salary, 88, 89
  and bonus, 247
  increase, 81, 82, 136, 152, 567
Sales commission, 192
Sales tax, 191, 207
Satellite television, 501
Sharing the cost, 120, 124, 154
Weekly paycheck, 88
Wholesale price, 90

GEOMETRY

Allowable width of a package, 136
Area
  of the base for a tank, 247
  of a building, 281
  optimal, 259, 330
  of a room, 281
  of a square, 137
  of a trapezoid, 91
  of a triangle, 88, 91, 521, 537, 538
Area and circumference of a circle, 208
Cutting across the lawn, 98
Depth
  of an underwater cable, 153
  of water in a trough, 92
  of a whale, 102
  of a whale shark, 102
Diagonals of a polygon, 443
Dimensions
  of a billboard, 101, 153
  of a box, 293
  of a building, 101
  of a child care play area, 111
  of a cube, 37, 39
  of a cylindrical beaker, 200
  of a lot, 101
  of a picture frame, 90
  of a room, 83, 90, 96, 153
  of a square base, 111, 114, 154
  of a square classroom, 37
  of a storage bin, 293
  of a swimming pool, 111
  of a terrarium, 293
  of a volleyball court, 153
Floor plan, 47
Fractals, 574
Geometric modeling, 54
Height
  of a balloon, 205
  of a shipping barrel, 87
Length
  of a field, 146, 155
  of the legs of an equilateral triangle, 102
  of the legs of an isosceles right triangle, 102
  of a propane tank, 92
  of a room, 54
Liters and gallons, 191
Patterns in Pascal's Triangle, 574
Perimeter of a rectangle, 88, 91
Ripples in a pond, 207, 239
Volume
  of a box, 45, 47, 205, 246, 271
  of a cube, 125
  of a rectangular prism, 91
  and surface area of a sphere, 22
Weather balloon, 22
Width
  of a deck, 101
  of a path, 101

INTEREST RATE

Balance in an account, 25, 28, 39, 66, 246, 368, 394, 397, 546, 548, 567, 600, 601, 604
Becoming a millionaire, 28
Bond investment, 510
Borrowing money, 124
Cash advance, 124, 154
Compound interest, 47, 91, 121, 124, 146, 155, 352, 355, 396, 404, 408, 565, 566
Doubling and tripling an investment, 382
Increasing annuity, 563
Investment
  mix, 85, 91, 412, 417, 452, 472
  plan, 404
  portfolio, 430, 441, 463, 469
  time, 367, 405
Loan mix, 437, 440, 486, 534
Monthly payment, 61
Present value of money, 355
Rate of inflation, 26
Savings plan, 44, 47
Simple interest, 90, 91, 137, 153, 191
Stock mix, 91

MISCELLANEOUS

3000-meter speed skating winning times, 325
Accuracy of a measurement, 133, 137, 154
Aircraft boarding, 602
Alumni association, 594
Australian football, A26
Baking as a leisure activity, 472
Baseball player getting a hit, 595
Bike race, 583
Candle sales, 441
Choice of newscasts, 467
Choosing officers, 583
College bound, 593, 603
College class makeup, 595
Combination lock, 583
Comparing calories, 89
Concert seating arrangement, 583
Contest for season ticket holders, 593
Counting card hands, 581
Course grade, 90
Course schedule, 582
Cryptography, 524-526, 528, 529, 537
Dance video game, 194
Digital cinema screens, 356
Drawing a card, 586, 589, 592, 593, 594, 603
Drawing marbles, 592, 593
Drawing pieces of paper, 575
Drug enforcement, 324
Early childhood development, 103
Exercise program, 132
Floral design, 511
Flour production, 125
Flower arrangements, 581
Football player salaries, 136
Football players' weights, 603
Forming a committee, 584
Forming an experimental group, 583
Game show, 584, 594, 603
Hair product sales, 441
Heads or tails?, 585, 586, 592
House painting, 125
Humidity control, 137
IQ scores, 135, 397
Land area of crops, 296
Law enforcement, 584
Letter mix-up, 594
Letter pairs, 576
License plate numbers, 582
Loans, earnings, expenses, and depreciation, 182
Lottery, 584, 588
Mandelbrot Set, 303, 305
Marketing a book, 122
Media usage, 90
Multiple choice quiz, 604
Notes on a musical scale, 38
Offshore drilling attitudes, 595
Olympic diver, 101
Payroll mix-up, 594
Pin numbers, 581
Pizza toppings, 604
Poker hand, 584, 594, 603
Population,
  of a city, 208, 356, 395, 396, 406, 407, 408
  of Japan, 192
  of North America, 171
  of South America, 171
  of a town, 356, 366, 398, 404
  of the world, 147
Preparing for a test, 594, 603
Prescriptions filled, 430
Random number generator, 592, 593, 594, 603
Red herring, 92
Research, 19, 92, 239, 305, 376, 431, 529
Sales tax, hall rental, refrigerator cost, car rental, 194
Seating capacity, 111, 556, 601
Singapore Flyer observation wheel, 171
Single file, 583
Snow removal, 405
Softball team expenses, 496
State income tax, 184
Stone cutting, 182
Sugar production, 125
Taste testing, 592
Telephone numbers, 576, 603
Temperature, 9, 103
Test scores, 391, 407
Texting survey, 595
Time study, 137
Toboggan ride, 582
Tossing
  a coin, 595, 603
  a coin and a die, 592
  a die, 592
  three dice, 604
  two dice, 587, 591
True-false exam, 582
Tutor facility utilization, 397
Van seating arrangement, 604

Voting preference, 501
Wage the doubles daily, 566
Weight loss program, 136
Wimbledon tennis winnings, 102
Winning an election, 593
World internet users, 90

TIME AND DISTANCE

Airplane speed, 429
Distance
  from a dock, 112
  of a hit baseball, 261, 330
  miles and kilometers, 191
  to a star, 91
  traveled by a car, 88, 191
English and metric systems, 185
Feet and meters, 245
Flying distance, 102, 112
Flying speed, 112, 426
Height
  of a building, 91
  of a mountain climber, 186
  of a parachutist, 191
  of Petronas Tower, 84
  of a tree, 91
Navigation, A18
Travel, 9
Travel time, 88, 91, 152

U.S. DEMOGRAPHICS

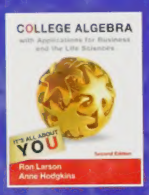
ACT scores, 391
Age at first marriage, 357
Air transportation employees, 385
Alternative-fuel vehicles in use, 189
Autistic children, 192
Cable TV industry revenue, 566
Cell sites, 67
Children's Health Insurance Program, 79
Colleges and universities by region, 588
Comparing populations
  Alabama and Colorado, 419
  Arizona and Indiana, 419
  Kentucky and South Carolina, 467
Corn used to produce ethanol, 468
Cost
  of college, 28
  to stay in college dormitory, 147, 330
Electric power revenues, 99
Electromedical exports, 103
Federal
  farmland, 206
  fast food industry revenues, 125
  government expenses, 19
  income tax returns filed, 188
  payroll for civilian employees, 156
  school breakfast program, 152, 368
  school lunch program, 208
  student aid, 47
Gold prices, 170, 218
Grade level salaries for federal employees, 219
Gross domestic product, 357
Health care expenditures, 194, 325
Health care practitioners, 238
Health services industry employees, 194
Home mortgage debt, 229
Hospital expenses, 383
Hourly earnings in manufacturing, 76
Houses sold, 201
Houses sold by region, 594
Human heights, 386
Hunting and firearms spending, 442
Licensed drivers, 486
Life expectancy, 124, 125
Medicare enrollees, 444
Men's heights, 397
Migrant interdictions, 206
Miles traveled by vehicles, 170
Minimum wage, 79
Mobile home prices, 67
Movie prices, 68
New car dealership advertising expenses, 549
New vehicle sales, 89
Non-wage earners, 190
Oil
  imported, 153
  strategic reserve, 219
Per capita
  land area, 321
  spending on video games and newspapers, 419
Personal income in West Virginia, 79
Political makeup of the U.S. Senate, 234
Population
  foreign-born, 271
  median age of U.S., 367
  projection for children under five, 248
  projection for people 85 and older, 397
  ratio of males to females, 549
  of San Francisco, 187
  of U.S., 47, 103
Prescription drug prices, 135, 136, 444
Prescriptions filled, 357, 385
Public college enrollment, 136
Restaurant sales, 430
Retail sales
  of DVDs, 324
  of food and beverage stores, 419
  of general merchandise stores, 419
Revenue of golf courses, 260
Sailboats sold, 469
Sales of electricity, 325
Sales of petroleum and coal products, 218
Solar energy, 294
Text messages sent annually in U.S., 566
VCR use, 112

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STUDENT COPY ISBN:
978-1-133-10849-8

ISBN-13: 978-1-133-11185-6
ISBN-10: 1-133-11185-8